ABELIAN VARIETIES

Exercises for week 38 (September 18)

Exercise 1. Let Y be a variety, and let F and G be vector bundles on X of ranks r and s, respectively. Suppose these are given, with respect to some open cover $Y = \cup V_i$, by cocycles $\{\phi_{ij}\}$ and $\{\psi_{ij}\}$.

(i) For each of the following vector bundles, give the rank and the corresponding cocycle:

$$F \oplus G\,, \quad F \otimes G\,, \quad F^\vee\,, \quad \mathit{Hom}_{O_X}(F, G)\,, \quad \mathrm{Sym}^2(F)\,, \quad \wedge^2 F\,.$$

(ii) If $f\colon X \to Y$ is a morphism, explain how to describe f^*F in terms of a cocycle on X.

(iii) Conclude that f^* is compatible—in the obvious sense—with the operations $\oplus$, $\otimes$, $(\ )^\vee$, *Hom*, etc. (This is in fact true for arbitrary sheaves of modules but for locally free sheaves the method used here gives a simple way to make this explicit.)

Exercise 2. Let C be a complete non-singular curve of genus g over an algebraically closed field. If $D = \sum n_P \cdot P$ is a divisor on C, let $\deg(D) = \sum n_P$. (Caution: this notion of a degree is the right one only over an algebraically closed field, and for varieties of higher dimension, there is no good notion of degree of a divisor.) For a line bundle L we define the degree via the correspondence $\mathit{Cl}(C) \cong \mathrm{Pic}(C)$. Let $\mathrm{Pic}^n(C) \subset \mathrm{Pic}(C)$ be the subset of (isomorphism classes of) line bundles of degree n, so that $\mathrm{Pic}(C) = \oplus_{n\in\mathbb{Z}} \mathrm{Pic}^n(C)$.

(i) If $g = 0$, show that $\mathrm{Pic}(C) \cong \mathbb{Z}$ via the degree map.

In the rest of the exercise we assume that $g > 0$.

(ii) Prove that the map $C \to \mathrm{Pic}^1(C)$ given by $P \mapsto O_C(P)$ is injective.

(iii) Prove that the map $C^g \to \mathrm{Pic}^g(C)$ given by $(P_1, \ldots, P_g) \mapsto O_C(P_1 + \cdots + P_g)$ is surjective.

(iv) Suppose $g = 1$. Choose a point $O \in C$. Conclude from (ii) and (iii) that there is a unique group structure on C such that that map $C \to \mathrm{Pic}^0(C)$ given by $P \mapsto O_C(P - O)$ is a homomorphism. (Remark: As we shall discuss, this group structure makes the curve into a *group variety*. The curve C with this group structure, determined by the choice of an origin, is called an elliptic curve.)

Exercise 3. Let $\Phi\colon E_1 \to E_2$ be a homomorphism of (geometric) vector bundles on a variety X over an algebraically closed field k. Let $\phi\colon \mathscr{E}_1 \to \mathscr{E}_2$ be the corresponding homomorphism between the associated sheaves of sections.

(i) For $x \in X$, explain the relation between the stalk $\mathscr{E}_{i,x}$ of the sheaf $\mathscr{E}_i$ at x (which is a module over $O_{X,x}$), and the fibre $E_i(x)$ of E_i over the point x (which is a k-vector space).

(ii) As discussed in the lecture, if ϕ is injective, this does not mean that Φ is fibrewise injective. Explain how this is related to the fact that the functor $-\otimes_{O_{X,x}} k$ is not left exact (unless X is a point).