ABELIAN VARIETIES

Exercises for week 3 (October 2)

Exercise 1. Let S be a base scheme. (If you prefer, you can take $S = \mathrm{Spec}(k)$, though this does not simplify the exercise.) Let G and H be group schemes over S. As part of the data of a group scheme we have identity sections $e_G\colon S \to G$ and $e_H\colon S \to H$ (these are sections of the structural morphisms $G \to S$ and $H \to S$, respectively). Let $f\colon G \to H$ be a homomorphism of group schemes over S, and consider the scheme $K = S \times_{e_H,H,f} G$; this means that we have a fibre product diagram

$$\begin{array}{ccc} K & \longrightarrow & G \\ \big\downarrow & & \big\downarrow f \\ S & \xrightarrow{e_H} & H \end{array}$$

Prove that K represents the functor $T \mapsto \mathrm{Ker}\big(f(T)\colon G(T) \to H(T)\big)$, and conclude that K itself is again a group scheme. (In fact, a subgroup scheme of G.) We call K the kernel (group scheme) of f; usually it is denoted by $\mathrm{Ker}(f)$.

Exercise 2. Let k be a base field, and consider the group scheme $\mathbb{G}_{\mathrm{m}} = \mathrm{Spec}\big(k[x, x^{-1}]\big)$ over k. Let $f_n\colon \mathbb{G}_{\mathrm{m}} \to \mathbb{G}_{\mathrm{m}}$ be the morphism given by $g \mapsto g \cdot g \cdots g$ (product of n factors g).

(i) Make sure you understand what we mean by this definition of f_n. What if $n < 0$? Give this morphism by an explicit map on coordinate rings. Also describe it as a morphism of functors, i.e., given a scheme T over k, what is the map $f_n(T)\colon \Gamma(T, O_T)^* \to \Gamma(T, O_T)^*$?

(ii) Prove in both descriptions that f is a homomorphism of group schemes.

(iii) Define μ_n to be the kernel of f_n. (See the first exercise.) Show that this is an affine scheme and give its coordinate ring. Also describe μ_n as a functor: what is $\mu_n(T)$ for a k-scheme T ?

(iv) Describe the underlying topological space of μ_8 if we take for k the following fields:

$$\mathbb{Q}, \qquad \mathbb{Q}[i], \qquad \overline{\mathbb{Q}}, \qquad \overline{\mathbb{F}}_3, \qquad \overline{\mathbb{F}}_2 .$$

Exercise 3.

(i) If X is an abelian variety over a field k and α is a global 1-form on X (i.e., a global section of the sheaf of differentials $\Omega^1_{X/k}$), prove that α is translation-invariant, i.e., that for every point $x \in X(k)$ with associated translation $t_x\colon X \to X$, we have $t_x^*(\alpha) = \alpha$.

(ii) Why is the analogous assertion not true if we replace X by an affine group variety such as $\mathbb{G}_{\mathrm{m}}$, even though we still have that $\Omega^1_{X/k} \cong O_X$? E.g., the form $\mathrm{d}x$ on $\mathbb{G}_{\mathrm{m}}$ is not translation-invariant (why?); write down a 1-form that *is* translation invariant.

(iii) Assume, for simplicity, that $\mathrm{char}(k)$ is not 2 or 3. If E is an elliptic curve over k given by a Weierstrass equation $y^2 = x^3 + Ax + B$, write down a global 1-form on E, and prove that the form you give is everywhere regular (and therefore translation invariant).