# 1. Divisors

Let $X$ be a complete non-singular curve.

**Definition 1.1.** A *divisor* on $X$ is an element of the free abelian group $\mathbb{Z}^{(X)}$ on $X$, i.e., a $\mathbb{Z}$-valued function $D\colon X \to \mathbb{Z}$ such that $D(P) \neq 0$ for at most finitely many $P \in X$.

We often write divisors as a finite formal sum $D = \sum_{P\in X} D(P)\cdot P$ with $D(P) \in \mathbb{Z}$. The *degree* $\deg D$ of a divisor $D$ is the element $\sum_{P\in X} D(P)$ of $\mathbb{Z}$. Clearly the degree induces a homomorphism of abelian groups $\mathbb{Z}^{(X)} \to \mathbb{Z}$. For example, if $P, Q, R$ are points on $X$, then $D = -P + 3Q - 4R$ is a divisor on $X$, of degree $-2$.

Let $f \in k(X)^*$ be a non-zero rational function on $X$. For a point $P \in X$, let $v_P\colon \operatorname{Frac} \mathcal{O}_{X,P} \to \mathbb{Z} \cup \{\infty\}$ be the discrete valuation of $\mathcal{O}_{X,P}$.

**Lemma 1.2.** *We have $v_P(f) \neq 0$ for only finitely many $P \in X$.*

*Proof.* Choose an open affine cover $X = U_1 \cup \ldots \cup U_n$ of $X$. We can write $f|_{U_i} = g_i/h_i$ for suitable $g_i, h_i \in \mathcal{O}(U_i)$. As each $U_i$ has dimension one, the zero loci $Z(g_i)$ and $Z(h_i)$ are finite sets. For $P \in U_i \setminus (Z(g_i) \cup Z(h_i))$ we have $v_P(f) = 0$. □

**Definition 1.3.** Let $f \in k(X)^*$ be a non-zero rational function on $X$. We define the *divisor of* $f$ to be the divisor $\operatorname{div} f = \sum_{P\in X} v_P(f)\cdot P$ on $X$. By the Lemma, this is well-defined.

The map $k(X)^* \to \mathbb{Z}^{(X)}$ given by $f \mapsto \operatorname{div} f$ is a homomorphism of abelian groups. The image is called the group of *principal divisors* on $X$, notation $\operatorname{Princ}(X)$. The quotient group $\mathbb{Z}^{(X)}/\operatorname{Princ}(X)$ is called the *class group* of $X$, notation $\operatorname{Cl}(X)$. Elements of $\operatorname{Cl}(X)$ are called *divisor classes*. We say that two divisors $D, E$ on $X$ are *linearly equivalent*, notation $D \sim E$, if $D$ and $E$ define the same class in $\operatorname{Cl}(X)$.

Let $\varphi\colon X \to Y$ be a morphism of complete non-singular curves. Note that $\varphi$ is either constant or surjective. Assume that $\varphi$ is surjective. Then $X$ is the normalization of $Y$ in the function field of $X$. In particular, the morphism $\varphi$ is finite, hence quasi-finite and proper. Let $P \in X$ and put $Q = \varphi(P)$. The ramification index of $\varphi$ at $P$ can be obtained as follows: let $\pi_Q$ a generator of the maximal ideal of $\mathcal{O}_{Y,Q}$, then $e_P = v_P(\varphi^*(\pi_Q))$. We recall that $\sum_{P\in X, \varphi(P)=Q} e_P = \deg \varphi$.

**Definition 1.4.** View $Q$ as a divisor of degree one on $Y$. We define $\varphi^*(Q)$ to be the divisor $\sum_{P\in X,\varphi(P)=Q} e_P \cdot P$ on $X$. By extending this linearly we obtain a homomorphism of abelian groups $\varphi^*\colon \mathbb{Z}^{(Y)} \to \mathbb{Z}^{(X)}$.

This definition may seem a little ad hoc; it becomes more natural when we choose, instead, to work with 'Cartier' divisors. Anyway, the pullback of divisors defined in this way has some good properties: for $D \in \mathbb{Z}^{(Y)}$ we clearly have

$$\deg \varphi^*(D) = (\deg \varphi) \cdot \deg D\,. \tag{1}$$

Moreover, we have functoriality: if $\psi\colon Y \to Z$ is a surjective morphism, then $(\psi \circ \varphi)^* = \varphi^* \circ \psi^*$ as maps from $\mathbb{Z}^{(Z)}$ to $\mathbb{Z}^{(X)}$.

Recall that we can view each element $f \in k(Y)$ as a rational map $f\colon Y \dashrightarrow \mathbb{P}^1$. As $Y$ is non-singular and $\mathbb{P}^1$ is complete, the rational map $f$ extends as a morphism $f\colon Y \to \mathbb{P}^1$. If $f$ is not constant, then $f$ is surjective. For a non-constant $f \in k(Y)$ we have $\operatorname{div} f = f^*(0 - \infty)$. We obtain:

**Proposition 1.5.** *Let $\varphi\colon X \to Y$ be a surjective morphism, and $f \in k(Y)^*$ a non-zero rational function. Then*
*(i)* $\varphi^* \operatorname{div} f = \operatorname{div} \varphi^* f$;
*(ii)* $\deg \operatorname{div} f = 0$.

*Proof.* We obtain (i) for non-constant $f$ by functoriality. For constant $f$ both sides of the equality are zero. We obtain (ii) from equation (1) and the fact that $\deg(0-\infty) = 0$. □

By item (i), the group homomorphism $\varphi^*\colon \mathbb{Z}^{(Y)} \to \mathbb{Z}^{(X)}$ descends to a group homomorphism $\varphi^*\colon \mathrm{Cl}(Y) \to \mathrm{Cl}(X)$. By item (ii), the degree homomorphism $\deg\colon \mathbb{Z}^{(Y)} \to \mathbb{Z}$ factors through a homomorphism $\mathrm{Cl}(Y) \to \mathbb{Z}$. We have a commutative diagram

$$\begin{array}{ccc} \mathrm{Cl}(Y) & \xrightarrow{\deg} & \mathbb{Z} \\ \downarrow{\scriptstyle \varphi^*} & & \downarrow{\scriptstyle \cdot \deg \varphi} \\ \mathrm{Cl}(X) & \xrightarrow{\deg} & \mathbb{Z} \end{array}$$

of abelian groups.

A divisor $D$ on $X$ is called effective, notation $D \geq 0$, if $D(P) \geq 0$ for all $P \in X$. We write $E \geq D$ if $E - D \geq 0$.

**Definition 1.6.** Let $D$ be a divisor on $X$. The *Riemann-Roch space* of $D$ is the subset

$$\begin{aligned} \mathcal{L}(D) &= \{0\} \cup \{f \in k(X)^* | \operatorname{div} f + D \geq 0\} \\ &= \{0\} \cup \{f \in k(X)^* | v_P(f) \geq -D(P) \text{ for all } P \in X\} \end{aligned}$$

of $k(X)$.

**Proposition 1.7.** *(i) The Riemann-Roch space $\mathcal{L}(D)$ is a sub-$k$-vector space of $k(X)$.*
*(ii) If $D \sim E$, say $D = E + \operatorname{div} g$, where $g \in k(X)^*$, then the multiplication $f \mapsto fg$ yields a $k$-linear isomorphism $\mathcal{L}(D) \xrightarrow{\sim} \mathcal{L}(E)$.*
*(iii) If $\deg D < 0$, then $\mathcal{L}(D) = (0)$.*
*(iv) If $\deg D \geq 0$, the dimension $\dim_k \mathcal{L}(D)$ is finite, and bounded above by $\deg D + 1$.*

*Proof.* (i) This follows from the fact that $\operatorname{div} \alpha f = \operatorname{div} f$ for all $\alpha \in k^*$ and the fact that $v_P(f + g) \geq \min\{v_P(f), v_P(g)\}$ for all $f, g \in k(X)$ and all $P \in X$.
(ii) Note that $\operatorname{div} fg = \operatorname{div} f + \operatorname{div} g$. Hence $D + \operatorname{div} f \geq 0 \Leftrightarrow E + \operatorname{div} fg \geq 0$.
(iii) Assume $f$ is a non-zero element of $\mathcal{L}(D)$. Then $\operatorname{div} f + D \geq 0$, in particular we have $\deg D = \deg(\operatorname{div} f + D) \geq 0$.
(iv) By (ii) we may assume that $D$ is effective. Write $D = \sum_{P \in X} D(P) \cdot P$ with all $D(P) \in \mathbb{Z}_{\geq 0}$. Choose a uniformizer $\pi_P$ for each point $P \in X$. Let $f \in \mathcal{L}(D)$. Then $f \in \pi_P^{-D(P)} \mathcal{O}_{X,P}$ and this projects to an element $\overline{f} \in \pi_P^{-D(P)} \mathcal{O}_{X,P}/\mathcal{O}_{X,P}$. By the theory of discrete valuation rings, the latter is an $\mathcal{O}_{X,P}/\mathfrak{m}_P$-module, i.e. a $k$-vector space, of dimension $D(P)$. Collecting all points $P \in X$ together we obtain a $k$-linear 'evaluation' map $\mathrm{ev}\colon \mathcal{L}(D) \to \bigoplus_{P \in X} \pi_P^{-D(P)} \mathcal{O}_{X,P}/\mathcal{O}_{X,P}$. An element in the kernel of ev is regular at all $P \in X$, hence constant. The vector space on the right hand side has dimension $\sum_{P \in X} D(P) = \deg D$. We find that $\dim_k \mathcal{L}(D) \leq \deg D + 1$. □

We write $l(D)$ as a shorthand for $\dim_k \mathcal{L}(D)$. The map $\mathbb{Z}^{(X)} \to \mathbb{Z}_{\geq 0}$ given by $D \mapsto l(D)$ factors over $\mathrm{Cl}(X)$ by (ii).

*Example* 1.1. If $D$ is effective, then $\mathcal{L}(D)$ contains $k$, and hence $l(D) \geq 1$. If $D = 0$, then $\mathcal{L}(D) = k$ and $l(D) = 1$.

*Example* 1.2. Let $P \in X$. If $l(P) \geq 2$, then $X \cong \mathbb{P}^1$. Indeed, let $f \in \mathcal{L}(P)$ be non-constant. Then $f$ is a surjective morphism $f\colon X \to \mathbb{P}^1$. From the condition $\operatorname{div} f + P \geq 0$ we infer that $f^*\infty \leq P$. As $f^*\infty > 0$ we find that $f^*\infty = P$ hence $\deg f = 1$. So the inclusion of function fields $k(\mathbb{P}^1) \to k(X)$ induced by $f$ is an equality. It follows that $f$ is an isomorphism by Theorem 8.25.

*Example* 1.3. Consider $X = \mathbb{P}^1$ together with the divisor $D = m_\infty \cdot \infty + \sum_{i=1}^n m_i \cdot a_i$ on $X$. Here $a_i \in \mathbb{A}^1 = k$ for $i = 1, \ldots, n$. We would like to compute $l(D)$. Put $d = \deg D = m_\infty + \sum_{i=1}^n m_i$. By Proposition 1.7(iii) we can assume that $d \geq 0$. For $h = \prod_{i=1}^n (x - a_i)^{m_i}$ we have $\operatorname{div} h = \sum_{i=1}^n m_i \cdot a_i - (\sum_{i=1}^n m_i) \cdot \infty$. Let $D' = D - \operatorname{div} h = d \cdot \infty$. By Proposition 1.7(ii) multiplication by $h^{-1}$ yields a $k$-linear isomorphism $\mathcal{L}(D) \xrightarrow{\sim} \mathcal{L}(D')$. Note that

$$\mathcal{L}(D') = \mathcal{L}(d \cdot \infty) = \{ f \in k[x] | \deg f \leq d \} = k \oplus k \cdot x \oplus \ldots \oplus k \cdot x^d \,.$$

We find $l(D) = l(D') = d + 1 = \deg D + 1$.

## 2. Differentials

Let $A$ be a ring and $B$ an $A$-algebra. Let $M$ be a $B$-module. An $A$-derivation from $B$ to $M$ is an $A$-linear map $\mathrm{d}\colon B \to M$ such that $\mathrm{d}(fg) = f\mathrm{d}g + g\mathrm{d}f$ for all $f, g \in B$. There exists a universal $A$-derivation $\mathrm{d}\colon B \to \Omega_{B/A}$; the $B$-module $\Omega_{B/A}$ is called the module of *Kähler differentials* of $B$ over $A$. An explicit description of $\Omega_{B/A}$ is as the module generated over $B$ by formal symbols $\mathrm{d}f$, where $f$ runs through $B$, together with the relations $\mathrm{d}(fg) = f\mathrm{d}g + g\mathrm{d}f$ for all $f, g \in B$, $\mathrm{d}(f+g) = \mathrm{d}f + \mathrm{d}g$ for all $f, g \in B$, and $\mathrm{d}a = 0$ for all $a \in A$. If $B \to C$ is a ring morphism, note that there is a natural exact sequence of $C$-linear maps $\Omega_{B/A} \otimes_B C \to \Omega_{C/A} \to \Omega_{C/B} \to 0$ (cf. [HAG], Proposition II.8.3A, the 'first exact sequence').

[HAG], Theorem II.8.6A says:

**Theorem 2.1.** *Let $K$ be a finitely generated extension of a field $k$. Then $\dim_K \Omega_{K/k} \geq$ tr.deg$(K/k)$, and equality holds if and only if $K$ is separably generated over $k$.*

Let $X$ be a complete nonsingular curve, and $k(X)$ be its function field. We obtain as corollaries:

**Proposition 2.2.** *The module of Kähler differentials $\Omega_{k(X)/k}$ is a 1-dimensional $k(X)$-vector space.*

*Proof.* Recall that $k(X)$ is finitely separably generated over $k$. □

The elements of $\Omega_{k(X)/k}$ are called *rational differential forms* on $X$.

Let $\varphi\colon X \to Y$ be a surjective morphism of complete nonsingular curves.

**Proposition 2.3.** *The natural $k(X)$-linear map $\Omega_{k(Y)/k} \otimes_{k(Y)} k(X) \to \Omega_{k(X)/k}$ is either zero or an isomorphism. It is an isomorphism if and only if the finite field extension $k(X) \supset k(Y)$ is separable.*

*Proof.* Both $\Omega_{k(Y)/k} \otimes_{k(Y)} k(X)$ and $\Omega_{k(X)/k}$ are 1-dimensional $k(X)$-vector spaces, from which the first statement follows. The second follows from the 'first exact sequence'. □

*Example* 2.1. Assume that $k$ has characteristic $p > 0$ and consider the morphism $F\colon \mathbb{P}^1 \to \mathbb{P}^1$ defined by $x \mapsto x^p$. The corresponding extension of function fields is $k(x) \supset k(x^p)$. As $\mathrm{d}(x^p) = 0$ we have that the natural $k(x)$-linear map $\Omega_{k(x^p)/k} \otimes_{k(x^p)} k(x) \to \Omega_{k(x)/k}$ is the zero map.

A finite field extension $K \supset L$ is a tower of finite separable extensions and purely inseparable extensions. We say that $\varphi\colon X \to Y$ is purely inseparable if the corresponding function field extension $k(X) \supset k(Y)$ is purely inseparable. Note that a purely inseparable $\varphi\colon X \to Y$ of degree $p$ is everywhere ramified with ramification index $p$. If $P \in X$ and $\pi_P \in k(X)$ is a generator of the maximal ideal of $\mathcal{O}_{X,P}$ then the surjective morphism $X \to \mathbb{P}^1$ defined by $\pi_P$ is unramified at $P$ and hence the morphism is separable. In particular, the element $\mathrm{d}\pi_P$ is non-zero in $k(X)$.

Let $\omega$ be a non-zero element of $\Omega_{k(X)/k}$, and $P \in X$. Let $\pi_P$ be a generator of the maximal ideal of $\mathcal{O}_{X,P}$. We define the element $\omega(\pi_P) \in k(X)$ to be the unique element $f \in k(X)$ such that $\omega = f\mathrm{d}\pi_P$. We define $v_P(\omega)$ to be the valuation at $P$ of $\omega(\pi_P)$. It is straightforward to verify that this definition is independent of the choice of generator $\pi_P$.

**Lemma 2.4.** *We have $v_P(\omega) \neq 0$ for only finitely many $P \in X$.*

*Proof.* Let $P \in X$ be a point, choose a uniformizer $\pi_P$ at $P$ and write $\omega = f\mathrm{d}\pi_P$. Let $U$ be the dense open subset of points $Q \in X$ where $f$ is non-zero regular, $\pi_P$ is regular, and $\pi_P - \pi_P(Q)$ is a uniformizer (for the latter we have to discard the poles of $\pi_P$ and the finitely many points where $\pi_P$ ramifies). As $\mathrm{d}\pi_P = \mathrm{d}(\pi_P - \pi_P(Q))$ for all $Q \in U$ we have $v_Q(\omega) = 0$ for all $Q \in U$. As $X \setminus U$ is finite, the result follows. □

We put

$$\operatorname{div}\omega = \sum_{P \in X} v_P(\omega) \cdot P\,.$$

By the Lemma $\operatorname{div}\omega$ is a divisor on $X$. We call such a divisor a *canonical divisor* of $X$. For $f \in k(X)$ a non-zero rational function one has

$$\operatorname{div}(f\omega) = \operatorname{div} f + \operatorname{div}\omega\,.$$

Thus, the class of a non-zero differential form on $X$ is a well-defined element of $\mathrm{Cl}(X)$, the *canonical divisor class.*

*Example* 2.2. Take $\omega = \mathrm{d}x$ on $X = \mathbb{P}^1$. Then $\operatorname{div}\omega = -2 \cdot \infty$.

**Definition 2.5.** Let $K$ be a canonical divisor on $X$. The *genus* of $X$ is the dimension $l(K) = \dim_k \mathcal{L}(K)$ of the Riemann-Roch space associated to $K$.

*Example* 2.3. For $X = \mathbb{P}^1$ we have $\deg K = -2 < 0$ hence the genus of $\mathbb{P}^1$ is zero.

## 3. Riemann-Roch

A very powerful result is the Riemann-Roch theorem, which we state without proof.

**Theorem 3.1.** *Let $X$ be a complete nonsingular curve, and $D$ a divisor on $X$. Let $K$ be a canonical divisor on $X$. Let $g$ be the genus of $X$. Then the equality*

$$l(D) - l(K - D) = \deg D - g + 1$$

*holds.*

By taking $D = K$ and recalling that $\mathcal{L}(0) = k$ we find that $\deg K = 2g - 2$. If $\deg D > 2g - 2$ then $K - D$ has negative degree and we find that $l(D) = \deg D - g + 1$.