ALGEBRAIC GEOMETRY—SECOND HOMEWORK ASSIGNMENT

Handing in. Hand in by Tuesday May 20 at the latest, preferably at the lecture. Keep a copy of your work and *include an email address + home address to which we can return the graded exercises.* You are strongly encouraged to write your solutions in TEX and hand in a printout. If you write your solutions by hand, write legibly! Parts that we cannot easily decipher will simply be ignored.

The rules of the game. Your solutions to these exercises should convince us that you have a good understanding of what has been discussed so far. This involves more than just getting the correct answers, which, anyway, might be obtained from various internet sources or the brain of a fellow student. You may use everything that has been discussed so far in the course, as well as HAG, Chap. I, Sections 1–6 plus Chap. II, Section 1. Give full details for your arguments, with precise references for the results you use. What you hand in should be your own, individual work.

Exercise 1. In this exercise we work over an algebraically closed field k of characteristic zero. Let X be the affine curve in $\mathbb{A}^2$ given by the equation $y^5 = x^4(x^2-1)$.

(i) Show that $(0,0)$ is the only singular point of X.

(ii) Let $\tilde{X}$ be the strict transform of X inside the blow-up of $\mathbb{A}^2$ at $(0,0)$. Give an explicit cover of $\tilde{X}$ by affine open subsets. Find an element in $k(X)$ that is integral over $\mathscr{O}(X)$ but is not itself in $\mathscr{O}(X)$, and prove that the element you give has the required properties.

(iii) Let $\overline{X}$ be the projective closure of X in $\mathbb{P}^2$. Let $P = (0:1:0)$. Show that $\overline{X} \setminus X = \{P\}$. Prove that P is a nonsingular point of $\overline{X}$.

(iv) Describe a complete nonsingular curve C birationally equivalent with X. The curve C should be described either directly as a closed subset of some $\mathbb{P}^n$, or by giving some nonsingular affine curves and explicit gluing data. If you need to do calculations, make sure that these are presented in a crystal clear way.

Let $\varphi\colon C \dashrightarrow \mathbb{P}^1$ be the rational map given by the projection $(x,y) \mapsto x$ on X.

(v) Show that φ extends to a surjective morphism $\hat{\varphi}\colon C \to \mathbb{P}^1$. Show that there is a unique point $P_\infty \in C$ with $\hat{\varphi}(P_\infty) = (1:0)$, the point at infinity on $\mathbb{P}^1$.

(vi) Compute the divisors of the rational functions x, y and y/x on C.

(vii) Recall that for a divisor D on C we define

$$\mathscr{L}(D) = \{0\} \cup \big\{ f \in k(C)^* \;\big|\; \operatorname{div}(f) + D \geqslant 0 \big\}\,.$$

For each $n \geqslant 6$, give an element of $\mathscr{L}(n \cdot P_\infty) \setminus \mathscr{L}\big((n-1)\cdot P_\infty\big)$.

Exercise 2. Given non-negative integers m and n, let $r = mn+m+n$, so that $(m+1)(n+1) = r+1$. The goal of this exercise is to discuss the *Segre embedding*, which is an embedding of $\mathbb{P}^m \times \mathbb{P}^n$ into $\mathbb{P}^r$. On $\mathbb{P}^r$ we shall use the projective coordinates Z_{ij} with $0 \leqslant i \leqslant m$ and $0 \leqslant j \leqslant n$, sorted alphabetically. The Segre embedding is then the map $s\colon \mathbb{P}^m \times \mathbb{P}^n \to \mathbb{P}^r$ that sends a point $P = \big((a_0 : \cdots : a_m), (b_0 : \cdots : b_n)\big)$ to the point with coordinates $a_i b_j$; so

$$s(P) = \big(a_0b_0 : \cdots : a_0b_n : a_1b_0 : \cdots : a_1b_n : \cdots : a_mb_0 : \cdots : a_mb_n\big)\,.$$

(i) Prove that s is a morphism.

(ii) Consider the homomorphism of k-algebras $k[Z_{ij}] \to k[X_0, \ldots, X_m, Y_0, \ldots, Y_n]$ that sends Z_{ij} to $X_i Y_j$. Let $\mathfrak{p} \subset k[Z_{ij}]$ be the kernel. Prove that $\mathfrak{p}$ is a homogeneous prime ideal and that s gives an isomorphism $\mathbb{P}^m \times \mathbb{P}^n \xrightarrow{\sim} \mathscr{Z}(\mathfrak{p}) \subset \mathbb{P}^r$.

(iii) Prove that the product of two projective varieties is again projective.

Exercise 3. In this exercise we work over an arbitrary algebraically closed field k.

(i) Let Z be a closed subvariety of $\mathbb{P}^N$ for some N. Let $H \subset \mathbb{P}^N$ be a hypersurface, i.e., the zero locus of a single homogeneous irreducible polynomial $F \in k[X_0, \ldots, X_N]$. Prove that $Z \setminus (Z \cap H)$ is an affine variety.

For an integer $n \geqslant 1$, consider the projective space $\mathbb{P}^n$ of 1-dimensional linear subspaces of k^{n+1}. In other words, $\mathbb{P}^n$ is the Grassmannian $G(1, n+1)$. There is also a dual projective space $\check{\mathbb{P}}^n = G(n, n+1)$ of n-dimensional linear subspaces of k^{n+1}. Consider the incidence variety $\mathbb{I} \subset \mathbb{P}^n \times \check{\mathbb{P}}^n$ given by

$$\mathbb{I} = \left\{ (L, W) \in \mathbb{P}^n \times \check{\mathbb{P}}^n \;\middle|\; L \subset W \right\}.$$

You may use without proof that $\mathbb{I}$ is a closed subvariety of $\mathbb{P}^n \times \check{\mathbb{P}}^n$. (Cf. Exercise 7.3.)

(ii) Prove that the complement $Y := (\mathbb{P}^n \times \check{\mathbb{P}}^n) \setminus \mathbb{I}$ is an affine variety. [*Hint:* You will need part (i).]

(iii) Let $\pi\colon Y \to \mathbb{P}^n$ be the projection map, $(L, W) \mapsto L$. Prove that for every $L \in \mathbb{P}^n$ there exists an open $U \subset \mathbb{P}^n$ containing L and an isomorphism $\varphi\colon \pi^{-1}(U) \xrightarrow{\sim} U \times \mathbb{A}^n$ for which the diagram

$$\begin{tikzcd} \pi^{-1}(U) \arrow[rr, "\varphi"] \arrow[dr, "\pi"'] & & U \times \mathbb{A}^n \arrow[dl, "\mathrm{pr}_U"] \\ & U & \end{tikzcd}$$

is commutative. (In this diagram $\mathrm{pr}_U\colon U \times \mathbb{A}^n \to U$ is the first projection.)

(iv) Let X be a projective variety. Prove that there exists an affine variety A and a surjective morphism $p\colon A \to X$ such that all fibers of p are isomorphic to $\mathbb{A}^n$ for some $n \geqslant 1$.