ALGEBRAIC GEOMETRY—MIDTERM ASSIGNMENT

Handing in. Hand in by Tuesday April 1 at the latest, preferably at the lecture. Keep a copy of your work. You are strongly encouraged to write your solutions in TEX and hand in a printout. If you write your solutions by hand, write legibly! Parts that we cannot easily decipher will simply be ignored.

The rules of the game. Your solutions to these exercises should convince us that you have a good understanding of what has been discussed so far. This involves more than just getting the correct answers, which, anyway, might be obtained from various internet sources or the brain of a fellow student. You may use everything that has been discussed so far in the course, as well as HAG, Chap. I, Sections 1–3 plus Chap. II, Section 1. Give full details for your arguments, with precise references for the results you use. What you hand in should be your own, individual work.

Exercise 1. In this exercise k is an algebraically closed field of characteristic $\neq 2$. Let $A = (a_{ij}) \in M_3(k)$ be a symmetric 3×3 matrix with coefficients in k. We have an associated homogeneous quadratic polynomial $F(X_0, X_1, X_2) = \sum_{i,j=0}^{2} a_{ij} X_i X_j \in k[X_0, X_1, X_2]$ that we may succinctly write as $F(X) = X \cdot A \cdot X^T$. Here we use X as a shorthand for the row vector (X_0, X_1, X_2). Write $C = \mathscr{Z}(F) \subset \mathbb{P}^2$. We assume that A has maximal rank, i.e., $\det(A) \neq 0$.

(i) Show that C does not contain a line.

For a point $P = (p_0 : p_1 : p_2) \in \mathbb{P}^2$ we denote by $L_P \subset \mathbb{P}^2$ the line given by the equation $X \cdot A \cdot P^T = 0$.

(ii) Show that

$$P \in L_P \quad \Leftrightarrow \quad P \in C \quad \Leftrightarrow \quad L_P \cap C \text{ consists of precisely one point.}$$

(iii) Show that for P running through a line $M \subset \mathbb{P}^2$, the corresponding L_P all pass through one point Q.

Exercise 2. Let X be a topological space. Let $0 \longrightarrow \mathscr{F}' \xrightarrow{\varphi} \mathscr{F} \xrightarrow{\psi} \mathscr{F}'' \longrightarrow 0$ be a short exact sequence of sheaves of abelian groups on X.

(i) Prove that

$$0 \longrightarrow \mathscr{F}'(X) \xrightarrow{\varphi(X)} \mathscr{F}(X) \xrightarrow{\psi(X)} \mathscr{F}''(X) \qquad (*)$$

is a short exact sequence of abelian groups.

(ii) Let $Y = \mathbb{C}\setminus\{0\}$, equipped with the Euclidean topology. Let $\mathscr{A}_Y$ be the sheaf of holomorphic functions on Y; this means that for an open $U \subset Y$ we have

$$\mathscr{A}_Y(U) = \{\text{holomorphic functions } U \to \mathbb{C}\}$$

with group structure given by addition of functions, and with the obvious restriction maps $\mathscr{A}_Y(U) \to \mathscr{A}_Y(V)$ for open $V \subset U$. Define a homomorphism of sheaves $\mathrm{d}\colon \mathscr{A}_Y \to \mathscr{A}_Y$ by $\mathrm{d}f = f'$ (the derivative of f), for $f \in \mathscr{A}_Y(U)$. You don't need to prove that $\mathscr{A}_Y$ is a sheaf and that d is indeed a homomorphism of sheaves. Finally, let $\mathbb{C}_Y$ be the constant sheaf on Y associated with the group $\mathbb{C}$. Prove that we have a short exact sequence of sheaves

$$0 \longrightarrow \mathbb{C}_Y \longrightarrow \mathscr{A}_Y \xrightarrow{\mathrm{d}} \mathscr{A}_Y \longrightarrow 0$$

but that the sequence

$$0 \longrightarrow \mathbb{C}_Y(Y) \longrightarrow \mathscr{A}_Y(Y) \xrightarrow{\mathrm{d}} \mathscr{A}_Y(Y) \longrightarrow 0$$

is *not* exact on the right.

(iii) With notation as in part (i), suppose that for any open $U \subseteq X$ the restriction homomorphism $\mathscr{F}'(X) \to \mathscr{F}'(U)$ is surjective. Prove that in this case the sequence $(*)$ is also exact on the right, i.e.,

$$0 \longrightarrow \mathscr{F}'(X) \xrightarrow{\varphi(X)} \mathscr{F}(X) \xrightarrow{\psi(X)} \mathscr{F}''(X) \longrightarrow 0$$

is a short exact sequence of abelian groups. [*Hint:* Given $s \in \mathscr{F}''(X)$, look at the open sets $U \subset X$ such that $s|_U$ lies in the image of $\psi(U)$. By Zorn's lemma, the collection of such U has a maximal element.]

(iv) Again let $Y = \mathbb{C} \setminus \{0\}$, but now equipped with the Zariski topology. To avoid confusion with part (ii), let us denote this space as Y^{Zar}. Let $\mathbb{C}_Y^{\mathrm{Zar}}$ be the constant sheaf on Y^{Zar} associated with the group $\mathbb{C}$, and let $\mathscr{O}_Y$ be the structure sheaf of Y as a quasi-affine variety (so: the sheaf of regular functions). Again we have a sheaf homomorphism $\mathrm{d}\colon \mathscr{O}_Y \to \mathscr{O}_Y$ given by $\mathrm{d}f = f'$. Prove that the sheaf $\mathbb{C}_Y^{\mathrm{Zar}}$ has the property that for any Zariski-open $U \subset Y$ the restriction map $\mathbb{C}_Y^{\mathrm{Zar}}(Y) \to \mathbb{C}_Y^{\mathrm{Zar}}(U)$ is surjective. Is the sequence

$$0 \longrightarrow \mathbb{C}_Y^{\mathrm{Zar}}(Y) \longrightarrow \mathscr{O}_Y(Y) \xrightarrow{\mathrm{d}} \mathscr{O}_Y(Y) \longrightarrow 0$$

exact? Provide details for your answer.

Exercise 3. In this exercise we work over an algebraically closed field k of characteristic $\neq 2$. Using $(w : x : y : z)$ as homogeneous coordinates on $\mathbb{P}^3$, let $X \subset \mathbb{P}^3$ be the projective variety given by the equations

$$x^2 - xz - yw = 0\,, \quad yz - xw - zw = 0\,.$$

(You don't need to prove that X is a projective variety.) Let $P = (1:0:0:0)$ and denote by φ the projection from P to the hyperplane $\mathscr{Z}(w) \subset \mathbb{P}^3$.

(i) Identifying the hyperplane $\mathscr{Z}(w)$ with $\mathbb{P}^2$ (with homogeneous coordinates $(x : y : z)$), let $Y \subset \mathbb{P}^2$ denote the Zariski closure of the image of $X \setminus \{P\}$ under φ. Prove that Y is a closed subvariety of $\mathbb{P}^2$ and give a homogeneous prime ideal $I \subset k[x, y, z]$ such that $Y = \mathscr{Z}(I)$. (You should give I by a finite number of explicit generators, and you should prove that your I has the desired properties.)

(ii) Prove that $\varphi\colon X \setminus \{P\} \to Y$ extends to a morphism $\hat{\varphi}\colon X \to Y$ and that $\hat{\varphi}$ is an isomorphism. Determine the point $\hat{\varphi}(P)$.