# Notes on Homological Algebra

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# Contents

# Foreword

These are the notes of a course I taught in Utrecht in the fall of 2003, in the context of the Master Class on Non-Commutative Geometry, a one year special programme for a group of around 15 students from many different countries. The material is a selection of standard results which can be found in many of the text books on the subject, e.g. the one by C. Weibel (Cambridge University Press) or the one by P. Hilton and U. Stammbach (Springer-Verlag). A first draft of the notes was prepared by these students. Subsequently, the notes were polished with the help of Federico De Marchi, who also added an appendix on categorical language. Later, Roald Koudenburg added two sections on the Künneth formula and the Eilenberg-Zilber isomorphism. I'd like to take this opportunity to express my thanks to the students of the Master Class for being an enthusiastic audience, and to Federico, Roald and these students for all their work on the notes.

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# 1. Modules over a ring

By a *ring*, in this course, we intend an abelian group (in additive notation) with a product operation, which distributes over the sum, is associative and has a unit $1 \neq 0$.

The most standard examples of such a structure are the (commutative) ring $\mathbb{Z}$ of integer numbers, the fields $\mathbb{Q}$, $\mathbb{R}$ and $\mathbb{C}$ of rational, real and complex numbers, respectively, and the group ring $\mathbb{Z}[G]$ (see Chapter 2, Definition 2.1.1), which is *not necessarily* commutative.

## 1.1 Modules

Modules play to rings the same role as vector spaces do with respect to fields.

**Definition 1.1.1** A *left $R$-module* is an abelian group $(A, +)$ equipped with an *action* by $R$:

$$R \times A \ni (r, a) \mapsto r \cdot a \in A$$

satisfying the following conditions:

*a)* $r \cdot (a + b) = r \cdot a + r \cdot b$;
*b)* $r \cdot 0 = 0$;
*c)* $(r + s) \cdot a = r \cdot a + s \cdot a$;
*d)* $r \cdot (s \cdot a) = (rs) \cdot a$;
*e)* $1 \cdot a = a$.

Typically, when the action $R \times A \longrightarrow A$ is fixed in the context, we will write $ra$ instead of $r \cdot a$.

**Example 1.1.2** The following is a list of basic examples of modules:

*a)* Every vector space over a field $k$ is a $k$-module;
*b)* Every abelian group is a $\mathbb{Z}$-module;
*c)* Every ring $R$ is a module over itself, by taking as action $R \times R \longrightarrow R$ the multiplication of $R$.

**Definition 1.1.3** Let $A$ and $B$ be left $R$-modules. A mapping $\phi\colon A \longrightarrow B$ is called an *$R$-homomorphism* if, for all $a, a' \in A$ and $r \in R$, we have:

$$\begin{aligned} \phi(a + a') &= \phi(a) + \phi(a'), \\ \phi(ra) &= r \cdot \phi(a). \end{aligned}$$

The composite of two $R$-homomorphisms is again an $R$-homomorphism, and the identity map on a module is always an $R$-homomorphism. So, $R$-modules and $R$-homomorphisms form a category (see Definition A.0.1), which we shall call the *category of left $R$-modules* and will denote by $R\text{-}\mathsf{mod}$.

**Remark 1.1.4** The category $\mathsf{mod}\text{-}R$ of *right $R$-modules* is defined similarly. Only, this time the action associates on the right of the elements.

If $(R, +, \cdot)$ is a ring, its *dual* $R^\circ$ is the ring with the same underlying group, but with reversed multiplication:

$$r * s = s \cdot r,$$

for $r, s \in R$. It is then a trivial observation that the categories $R^\circ\text{-}\mathsf{mod}$ and $\mathsf{mod}\text{-}R$ are isomorphic.

When $R$ is a commutative ring, it coincides with its dual, hence the categories of left and right $R$-modules coincide (up to isomorphism), and their objects are simply called $R$-modules.

**Definition 1.1.5** Given two rings $R$ and $S$, we say that $A$ is an $(R, S)$*-bimodule* if $A$ is a left $R$-module and a right $S$-module, and the two actions associate as follows:

$$(ra)s = r(as),$$

for all $r \in R$, $a \in A$, $s \in S$.

Given two $(R, S)$-bimodules, we call $(R, S)$*-homomorphism* a map between them which is simultaneously an $R$-homomorphism and an $S$-homomorphism. $(R, S)$-bimodules and $(R, S)$-homomorphisms form again a category, which we shall denote by $R\text{-}\mathsf{mod}\text{-}S$.

**Example 1.1.6** In Example 1.1.2 *c*) above, $R$ is an $(R, R)$-bimodule.

Let $(A_i)_{i\in I}$ be a family of left $R$-modules and let $\prod_{i\in I} A_i$ be their cartesian product as sets. Then, a typical element of $\prod_{i\in I} A_i$ is a family $(a_i)$ with $a_i \in A_i$ for all $i \in I$ and $(a_i) = (a'_i)$ if and only if $a_i = a'_i$ for each $i \in I$. If $(a_i)$ and $(a'_i)$ are elements of $\prod_{i\in I} A_i$ and $r \in R$ we set:

$$\begin{aligned}(a_i) + (a'_i) &= (a_i + a'_i),\\ r(a_i) &= (ra_i).\end{aligned}$$

With these operations, $\prod_{i\in I} A_i$ becomes a left $R$-module, called the *direct product* of the family $(A_i)_{i\in I}$. We call $\pi_j\colon \prod_{i\in I} A_i \longrightarrow A_j$ the *$j$-th canonical projection*. The direct product of a family $(A_i)_{i\in I}$ is uniquely characterised (up to isomorphism) by the following universal property, which we leave you to check for yourself.

**Proposition 1.1.7** *For any left $R$-module $B$ and any family $(g_i\colon B \longrightarrow A_i)_{i\in I}$ of $R$-homomorphisms, there exists a unique $R$-homomorphism $g\colon B \longrightarrow \prod_{i\in I} A_i$ such that $\pi_i \circ g = g_i$ for all $i \in I$.*

The subset of $\prod_{i\in I} A_i$ consisting of those families $(a_i)$ with finitely many non-zero elements $a_i$ clearly inherits the structure of an $R$-module. It is called the *direct sum* of the family $(A_i)_{i\in I}$, and we shall denote it by $\bigoplus_{i\in I} A_i$. For any $j$

in $I$ there is an obvious map $\sigma_j\colon A_j \longrightarrow \bigoplus_{i\in I} A_i$ taking $a \in A_j$ to the family $(a_i)$, with $a_i = 0$ for $i \neq j$ and $a_j = a$. We call this map the *$j$-th canonical injection.* The direct sum of a family of left $R$-modules is again uniquely characterised (up to homomorphism) by a universal property, which again you should check for yourself.

**Proposition 1.1.8** *For any family $(f_i\colon A_i \longrightarrow B)_{i\in I}$ of $R$-homomorphisms with common codomain $B$, there exists a unique $R$-homomorphism $f\colon \bigoplus_{i\in I} A_i \longrightarrow B$ such that $f \circ \sigma_i = f_i$ for all $i \in I$.*

**Remark 1.1.9** Note that, when $I$ is a finite set $\{1, ..., n\}$, the $R$-modules $\prod_{i\in I} A_i$ and $\bigoplus_{i\in I} A_i$ do coincide. In this case, we shall denote the resulting module by $A_1 \oplus \cdots \oplus A_n$.

**Definition 1.1.10** An $R$-module $F$ is called *free* if it is isomorphic to a direct sum of copies of $R$; that is, if there is a (possible infinite) index set $I$ with:

$$F = \bigoplus_{i\in I} R_i$$

where $R_i \simeq R$ for all $i$.

Note that, for a field $k$, every $k$-module is free.

## 1.2 The Hom Functor

Let $A$ and $B$ be two left $R$-modules. We can define on the set $\mathrm{Hom}_{R\text{-}\mathsf{mod}}(A, B)$ of $R$-homomorphisms from $A$ to $B$ a sum operation, by setting

$$(\phi + \psi)(a) = \phi(a) + \psi(a).$$

This gives the set the structure of an abelian group, which we denote by $\mathrm{Hom}_R(A, B)$.

**Remark 1.2.1** Notice that there is a subtle but essential difference between $\mathrm{Hom}_{R\text{-}\mathsf{mod}}(A, B)$ and $\mathrm{Hom}_R(A, B)$: their underlying set is the same, but the latter has much more structure than the former. In particular, they live in two different categories, namely $\mathsf{Set}$ and $\mathsf{Ab}$ (for definitions, see the Appendix, Example A.0.2).

**Proposition 1.2.2** *Just like the hom-sets give rise to $\mathsf{Set}$-valued functors (see Definition A.0.6 and Example A.0.10), the $\mathrm{Hom}_R$-groups determine $\mathsf{Ab}$-valued functors:*

*a) For a fixed $R$-module $A$, the assignment $B \mapsto \mathrm{Hom}_R(A, B)$ defines a covariant functor $\mathrm{Hom}_R(A, -)\colon R\text{-}\mathsf{mod} \longrightarrow \mathsf{Ab}$;*

*b) Analogously, for a fixed $R$ module $B$, $\mathrm{Hom}_R(-, B)$ is a contravariant functor from $R$-$\mathsf{mod}$ to $\mathsf{Ab}$.*

**Remark 1.2.3** Suppose $R$ is a commutative ring. Then, we define an action of $R$ on the left of $\mathrm{Hom}_R(A, B)$ as follows:

$$(r \cdot \phi)(a) = r \cdot (\phi(a)),$$

for $r \in R$, $\phi \in \mathrm{Hom}_R(A, B)$ and $a \in A$. This determines on $\mathrm{Hom}_R(A, B)$ the structure of a left $R$-module.

Notice that, in this case, $\mathrm{Hom}_R(A, -)$ and $\mathrm{Hom}_R(-, B)$ can be seen as functors from the category $R$-$\mathsf{mod}$ to itself. Indeed, assume that $A$, $A'$, $A''$ and $B$ are left $R$-modules and let $\phi: A' \longrightarrow A$ be an $R$-homomorphism. Then, $\phi$ defines by precomposition an $R$-homomorphism

$$\phi^*: \mathrm{Hom}_R(A, B) \longrightarrow \mathrm{Hom}_R(A', B);$$

moreover, if $\psi: A'' \longrightarrow A'$ is another $R$-homomorphism, then we have the equality $\psi^* \circ \phi^* = (\phi \circ \psi)^*$ of $R$-homomorphisms from $\mathrm{Hom}_R(A, B)$ to $\mathrm{Hom}_R(A'', B)$.

Analogously, for left $R$-modules $A$, $B$, $B'$ and $B''$, any $R$-homomorphism $\phi: B \longrightarrow B'$ determines by composition an $R$-homomorphism

$$\phi_*: \mathrm{Hom}_R(A, B) \longrightarrow \mathrm{Hom}_R(A, B')$$

and, for a second map $\psi: B' \longrightarrow B''$, we have the equality of $R$-homomorphisms $\psi_* \circ \phi_* = (\psi \circ \phi)_*: \mathrm{Hom}_R(A, B) \longrightarrow \mathrm{Hom}_R(A, B'')$.

**Remark 1.2.4** Let $R$ and $S$ be two rings, and $A$ and $B$ two abelian groups. Then, the functor Hom relates different categories, depending on the way $R$ and $S$ act on $A$ and $B$.

*a)* Let $A$ be a left $S$-module and $B$ an $(S, R)$-bimodule. In this situation, the abelian group $\mathrm{Hom}_S(A, B)$ can be endowed the structure of a right $R$-module, the action of $r \in R$ on $\phi \in \mathrm{Hom}_S(A, B)$ being given by $\phi \cdot r: A \longrightarrow B$ defined as:

$$(\phi \cdot r)(a) = \phi(a)r.$$

It follows that $\mathrm{Hom}_S(A, -)$ is a covariant functor from $S$-$\mathsf{mod}$-$R$ to $\mathsf{mod}$-$R$ and $\mathrm{Hom}_S(-, B)$ is a contravariant functor from $S$-$\mathsf{mod}$ to $\mathsf{mod}$-$R$;

*b)* If $A$ is an $(S, R)$-bimodule and $B$ is a right $R$-module, then $\mathrm{Hom}_R(A, B)$ is a right $S$-module with:

$$(\phi \cdot s)(a) = \phi(sa),$$

thus determining the following functors: $\mathrm{Hom}_R(A, -): \mathsf{mod}\text{-}R \longrightarrow \mathsf{mod}\text{-}S$ and $\mathrm{Hom}_R(-, B): S\text{-}\mathsf{mod}\text{-}R \longrightarrow \mathsf{mod}\text{-}S$;

*c)* If $A$ is a right $R$-module and $B$ is an $(S, R)$-bimodule, then $\mathrm{Hom}_R(A, B)$ is a left $S$-module with:

$$(s\phi)(a) = s\phi(a),$$

thus determining the following functors: $\mathrm{Hom}_R(A, -): S\text{-}\mathsf{mod}\text{-}R \longrightarrow S\text{-}\mathsf{mod}$ and $\mathrm{Hom}_R(-, B): \mathsf{mod}\text{-}R \longrightarrow S\text{-}\mathsf{mod}$;

*d)* If $A$ is an $(S, R)$-bimodule and $B$ is a left $S$-module, then $\mathrm{Hom}_S(A, B)$ is a left $R$-module with:

$$(r\phi)(a) = \phi(ar),$$

thus determining the following functors: $\mathrm{Hom}_S(A, -): S\text{-}\mathsf{mod} \longrightarrow R\text{-}\mathsf{mod}$ and $\mathrm{Hom}_S(-, B): S\text{-}\mathsf{mod}\text{-}R \longrightarrow R\text{-}\mathsf{mod}$.

The universal properties of product and sums (Propositions 1.1.7 and 1.1.8) can now be refined as follows.

**Proposition 1.2.5** *Let $A$ and $B = \prod_{j\in J} B_j$ be left $R$-modules. Then, the map*

$$(\pi_j \circ -)\colon \mathrm{Hom}_R(A, B) \longrightarrow \prod_{j\in J} \mathrm{Hom}_R(A, B_j)$$

*taking a map $\phi$ to the family of composites $(\pi_j \circ \phi)$ is an isomorphism of abelian groups. Moreover, if each $B_i$ is an $(R, S)$-bimodule, then $(\pi_j \circ -)$ is an isomorphism of right $S$-modules.*

**Proposition 1.2.6** *Let $A = \bigoplus_{i\in I} A_i$ and $B$ be left $R$-modules. Then, the map*

$$[- \circ \sigma_i]\colon \mathrm{Hom}_R(A, B) \longrightarrow \prod_{i\in I} \mathrm{Hom}_R(A_i, B)$$

*taking a map $\phi$ to the family of composites $(\phi\sigma_i)$ is an isomorphism of abelian groups. Moreover, if each $A_i$ is an $(R, S)$-bimodule, then this is actually an isomorphism of left $S$-modules.*

### Exercises

*a)* Recall from the Appendix Definition A.0.12; then show that the assignment $\phi \mapsto \phi^*$ (resp. $\phi \mapsto \phi_*$) of Remark 1.2.3 defines a natural transformation between the functors $\mathrm{Hom}_R(A, -)$ and $\mathrm{Hom}_R(A', -)$ (resp. $\mathrm{Hom}_R(-, B)$ and $\mathrm{Hom}_R(-, B')$).

*b)* For a commutative ring $R$, consider the category $\mathsf{End}(R\text{-}\mathsf{mod})$, whose objects are functors from $R$-$\mathsf{mod}$ to itself and morphisms are natural transformations between them. Using the previous exercise and Remark 1.2.3, show that the assignments $A \mapsto \mathrm{Hom}_R(A, -)$ and $B \mapsto \mathrm{Hom}_R(-, B)$ define respectively a contravariant and a covariant functor from $R$-$\mathsf{mod}$ to $\mathsf{End}(R\text{-}\mathsf{mod})$.

*c)* Check that any $R$-module $A$ is isomorphic to $\mathrm{Hom}_R(R, A)$.

## 1.3 Tensor Product

When working with modules, one can choose between two different product operations. One is the obvious cartesian product, which we have already examined in Proposition 1.1.7; the other one, which will be essential for us, is called the tensor product.

Let $A$ be a right $R$-module and $B$ a left one. Then, we can build the free abelian group $F$ over their cartesian product $A \times B$. Its elements can be uniquely written as finite sums

$$\sum_{i,j} p_{ij}(a_i, b_j),$$

for $p_{ij} \in \mathbb{Z}$, $a_i \in A$ and $b_j \in B$.

Let now $T$ be the subgroup of $F$ generated by all elements of the form

$$(a + a', b) - (a, b) - (a', b), \qquad (a, b + b') - (a, b) - (a, b'), \qquad (a, rb) - (ar, b)$$

where $a, a' \in A$, $b, b' \in B$ and $r \in R$.

**Definition 1.3.1** The quotient $F/T$ has an obvious abelian group structure. We denote it by

$$A \otimes_R B,$$

and we call it the *tensor product* of $A$ and $B$.

The image of a pair $(a, b)$ under the projection $F \longrightarrow A \otimes_R B$ is denoted by $a \otimes b$. With this notation, $A \otimes_R B$ consists of all finite sums $\sum_i a_i \otimes b_i$, satisfying the relations:

$$\begin{aligned} (a + a') \otimes b &= a \otimes b + a' \otimes b \\ a \otimes (b + b') &= a \otimes b + a \otimes b' \\ a \otimes rb &= ar \otimes b. \end{aligned}$$

The map $f: A \times B \longrightarrow A \otimes_R B$ defined by $f(a, b) = a \otimes b$ is *bilinear*, i.e. it satisfies the following properties:

*a)* $f(a + a', b) = f(a, b) + f(a', b)$;
*b)* $f(a, b + b') = f(a, b) + f(a, b')$;
*c)* $f(ar, b) = f(a, rb)$.

These properties characterise the tensor product uniquely, up to isomorphism. In fact, you should check that the following universal property holds for tensor products:

**Proposition 1.3.2** *If $C$ is an abelian group and $\phi: A \times B \longrightarrow C$ is any bilinear map, then there is a unique group homomorphism $\overline{\phi}: A \otimes_R B \longrightarrow C$ making the following diagram commute:*

$$\begin{tikzcd} A \times B \arrow[r, "f"] \arrow[dr, bend right, "\phi"'] & A \otimes_R B \arrow[d, dotted, "\overline{\phi}"] \\ & C. \end{tikzcd} \tag{1.1}$$

**Example 1.3.3** Let $\phi: A \longrightarrow A'$ and $\psi: B \longrightarrow B'$ be homomorphisms of right and left $R$-modules, respectively. Then, the map $\gamma: A \times B \longrightarrow A' \otimes_R B'$ defined by

$$\gamma(a, b) = \phi(a) \otimes \psi(b)$$

is clearly bilinear, hence it induces a (unique!) homomorphism of abelian groups, $\overline{\gamma}: A \otimes_R B \longrightarrow A' \otimes_R B'$, making (1.1) commute, i.e. such that for all $(a, b) \in A \times B$

$$\overline{\gamma}(a \otimes b) = \phi(a) \otimes \psi(b).$$

We shall call $\overline{\gamma}$ the *tensor product* of $\phi$ and $\psi$, and we shall denote it by $\phi \otimes \psi$.

In particular, when $\phi = \mathsf{id}_A$, an $R$-homomorphism $\psi: B \longrightarrow B'$ induces a homomorphism of abelian groups $\mathsf{id}_A \otimes \psi: A \otimes_R B \longrightarrow A \otimes_R B'$, and similarly, when $\psi = \mathsf{id}_B$, every $\phi: A \longrightarrow A'$ induces a map $\phi \otimes \mathsf{id}_B: A \otimes_R B \longrightarrow A' \otimes_R B$.

**Proposition 1.3.4** *With the action on maps defined above, $A \otimes_R -$ and $- \otimes_R B$ determine covariant functors from $R$-$\mathsf{mod}$ to $\mathsf{Ab}$ and $\mathsf{mod}$-$R$ to $\mathsf{Ab}$, respectively.*

When either $A$ or $B$ have the structure of a bimodule, we can endow the abelian group $A \otimes_R B$ with the structure of a module, as follows.

Let $R$ and $S$ be two rings, and assume that $A$ is an $(S, R)$-bimodule and $B$ is a left $R$-module. For each $s \in S$, the map $\phi_s \colon A \times B \longrightarrow A \otimes_R B$ defined by $\phi_s(a, b) = sa \otimes b$ is easily shown to be bilinear; hence, it determines a homomorphism $\overline{\phi_s} \colon A \otimes_R B \longrightarrow A \otimes_R B$ such that $\overline{\phi_s}(a \otimes b) = sa \otimes b$ for any pair $(a, b) \in A \times B$.

We now define an action of $S$ on $A \otimes_R B$ by setting

$$s(a \otimes b) = \overline{\phi_s}(a \otimes b) = sa \otimes b.$$

It is straightforward to check that, under this action, $A \otimes_R B$ becomes a left $S$-module. Accordingly, $A \otimes_R -$ and $- \otimes_R B$ become covariant functors from $R$-$\mathsf{mod}$ to $S$-$\mathsf{mod}$ and from $S$-$\mathsf{mod}$-$R$ to $S$-$\mathsf{mod}$, respectively.

Analogously, when $B$ is an $(R, S)$-bimodule and $A$ is a right $R$-module, $A \otimes_R B$ can be regarded as a right $S$-module by putting:

$$(a \otimes b)s = a \otimes bs.$$

In this case, $A \otimes_R -$ and $- \otimes_R B$ will become covariant functors from $R$-$\mathsf{mod}$-$S$ to $\mathsf{mod}$-$S$ and from $\mathsf{mod}$-$R$ to $\mathsf{mod}$-$S$, respectively.

Finally, we observe that the universal properties of sum and tensor can be used to check the following:

**Proposition 1.3.5** *Let $A = \bigoplus_{i \in I} A_i$ and $B = \bigoplus_{j \in J} B_j$ be direct sums of right $R$-modules and left $R$-modules, respectively. Then, there exists an isomorphism of abelian groups*

$$A \otimes_R B \simeq \bigoplus_{(i,j) \in I \times J} (A_i \otimes_R B_j).$$

### Exercise

*a)* Show that two integer numbers $n$ and $m$ are relatively prime if and only if $\mathbb{Z}/n \otimes_{\mathbb{Z}} \mathbb{Z}/m = 0$.

## 1.4 Adjoint Isomorphism

Let $R$ and $S$ be two rings, $A$ a right $R$-module, $C$ a right $S$-module and $B$ an $(R, S)$-bimodule. Then, with the notation introduced in the previous sections, we may form the groups $\mathrm{Hom}_S(A \otimes_R B, C)$ and $\mathrm{Hom}_R(A, \mathrm{Hom}_S(B, C))$. These are related as follows.

**Theorem 1.4.1** *The map*

$$\tau_{A,C} \colon \mathrm{Hom}_S(A \otimes_R B, C) \longrightarrow \mathrm{Hom}_R(A, \mathrm{Hom}_S(B, C)) \tag{1.2}$$

*which takes an $S$-homomorphism $\alpha \colon A \otimes_R B \longrightarrow C$ to the $R$-homomorphism $\alpha' \colon A \longrightarrow \mathrm{Hom}_S(B, C)$ defined by*

$$\alpha'(a)(b) = \alpha(a \otimes b)$$

*is an isomorphism of abelian groups.*

**Proof.** It is clear that the given map is a group homomorphism. To define its inverse, let $\beta\colon A \longrightarrow \mathrm{Hom}_S(B, C)$ be an $R$-homomorphism. Then, the map $A \times B \longrightarrow C$ defined by $(a, b) \mapsto \beta(a)(b)$ is clearly bilinear. Hence, there exists a group homomorphism $\psi(\beta)\colon A \otimes_R B \longrightarrow C$ such that

$$\psi(\beta)(a \otimes b) = \beta(a)(b).$$

Since $\beta(a)$ is an $S$-homomorphism, so is $\psi(\beta)$, and the assignment $\beta \mapsto \psi(\beta)$ defines a homomorphism of abelian groups, which is easily checked to be the inverse of $\tau_{A,C}$. $\square$

### Exercises

*a)* Recall from the Appendix Definition A.0.16. Then, using Remark 1.2.1 and Theorem 1.4.1, show that, for an $(R, S)$-bimodule $B$, the functor $- \otimes_R B\colon \textsf{mod-}R \longrightarrow S\textsf{-mod}$ is left adjoint to the functor $\mathrm{Hom}_S(B, -)$.

*b)* State and prove a similar adjunction when Hom is taken to be one of the functors in Remark 1.2.4 *a)* – *d)*.

## 1.5 Change of Ring

Let $\phi\colon R \longrightarrow S$ be a ring homomorphism. Then, for any left $S$-module $A$, the action of $S$ induces *via* $\phi$ an action of $R$ on the left of $A$, defined by

$$r \cdot a = \phi(r)a.$$

If $B$ is another left $S$-module and $\psi \in \mathrm{Hom}_S(A, B)$, then

$$\psi(r \cdot a) = \psi(\phi(r) \cdot a) = \phi(r)\psi(a) = r \cdot \psi(a).$$

So, $\psi$ can also be regarded as a member of $\mathrm{Hom}_R(A, B)$. In this way, $\phi$ defines a functor $\phi^*\colon S\textsf{-mod} \longrightarrow R\textsf{-mod}$, or analogously from $\textsf{mod-}S$ to $\textsf{mod-}R$.

In particular, by Example 1.1.2 *c*), we can regard $S$ as an $S$-module, and $\phi^*$ makes it into a right $R$-module, where the action is defined by $s \cdot r = s\phi(r)$. We have that

$$(ss') \cdot r = (ss')\phi(r) = s(s'\phi(r)) = s(s' \cdot r),$$

thus giving $S$ the structure of an $(S, R)$-bimodule, and we have the functor

$$\phi_! = S \otimes_R -\colon R\textsf{-mod} \longrightarrow S\textsf{-mod}.$$

The action of $\phi_!$ on an $R$-homomorphism $\tau\colon B \longrightarrow C$ gives the $S$-homomorphism $\phi_!(\tau) = \mathsf{id}_S \otimes \tau\colon S \otimes_R B \longrightarrow S \otimes_R C$.

The functors $\phi^*$ and $\phi_!$ are related in the following way.

**Theorem 1.5.1** *There is an isomorphism of abelian groups*

$$\mathrm{Hom}_S(\phi_!(B), A) \simeq \mathrm{Hom}_R(B, \phi^*(A)), \tag{1.3}$$

*which is natural in $A$ and $B$. In particular, the functor $\phi_!$ is left adjoint to $\phi^*$.*

**Proof.** The proof is similar to that of Theorem 1.4.1, and we leave it as an exercise. (In fact, you can try to see it as an instance of one of the adjunctions in Exercise 1.4–*b*), using the observation that $\phi^*(A) \cong \mathrm{Hom}_S(S, A)$ as $R$-modules.) $\square$

**Proposition 1.5.2** *Let $A = \bigoplus_{i \in I} A_i$ be a left $S$-module. Then, there exists an isomorphism of $R$-modules*

$$\phi^*(A) \simeq \bigoplus_{i \in I} \phi^*(A_i).$$

**Proposition 1.5.3** *Let $B = \bigoplus_{j \in J} B_j$ be a left $R$-module. Then, there exists an isomorphism of $S$-modules*

$$\phi_!(B) = S \otimes_R B \simeq \bigoplus_{j \in J} (S \otimes_R B_j) = \bigoplus_{j \in J} \phi_!(B_j).$$

**Proof.** It follows immediately from Proposition 1.3.5. $\square$

Finally, for a ring homomorphism $\phi$ as above, we can consider $S$ as an $(R, S)$-bimodule (via $\phi$) and define a third functor

$$\phi_* = \mathrm{Hom}_R(S, -)\colon R\text{-}\mathsf{mod} \longrightarrow S\text{-}\mathsf{mod}$$

as in Remark 1.2.4 *d*). Now, given a left $S$-module $A$ and a left $R$-module $B$, we can compare the abelian groups $\mathrm{Hom}_R(\phi^* A, B)$ and $\mathrm{Hom}_S(A, \mathrm{Hom}_R(S, B))$. Let $\alpha \in \mathrm{Hom}_R(\phi^* A, B)$ and let $a \in A$. Define $\psi_a\colon S \longrightarrow B$ by $\psi_a(s) = \alpha(sa)$. Then, it is easy to check that the $\psi_a$'s are $R$-homomorphisms. Moreover, they satisfy the relations $\psi_{a+a'} = \psi_a + \psi_{a'}$ and $\psi_{sa} = s\psi_a$; hence, the map $\alpha'\colon A \longrightarrow \mathrm{Hom}_R(S, B)$ defined by $\alpha'(a) = \psi_a$ is an $S$-homomorphism from $A$ to $\mathrm{Hom}_R(S, B)$. We now have a mapping

$$\tau_{A,B}\colon \mathrm{Hom}_R(\phi^*(A), B) \longrightarrow \mathrm{Hom}_S(A, \mathrm{Hom}_R(S, B)),$$

defined by $\tau_{A,B}(\alpha) = \alpha'$, which can easily be shown to be an isomorphism of abelian groups. This gives a proof of the following.

**Theorem 1.5.4** *There is an isomorphism of abelian groups, natural in $A$ and $B$:*

$$\mathrm{Hom}_R(\phi^*(A), B) \simeq \mathrm{Hom}_S(A, \phi_*(B)).$$

*In particular, the functor $\phi_*$ is right adjoint to $\phi^*$.*

## 1.6 Exact Sequences

We now introduce the notions of *kernel* and *cokernel* of a module homomorphism. Formally, they do not differ from the analogous notions for abelian groups (after all, modules *are* abelian groups, with some extra structure). In fact, the isomorphism theorems do still hold in this setting, although in proving them one has to verify the extra condition imposed by the action of the ring. So, let $\phi\colon A \longrightarrow B$ be an $R$-homomorphism.

**Definition 1.6.1** The *kernel* of $\phi$ is the submodule $\ker(\phi)$ of $A$ defined as

$$\ker(\phi) = \{a \in A : \phi(a) = 0\};$$

the *image* of $\phi$ is the submodule $\mathrm{im}(\phi)$ of $B$ defined as

$$\mathrm{im}(\phi) = \{b \in B : \text{there exists } a \in A \text{ with } \phi(a) = b\}.$$

In particular, a submodule is a subgroup; hence, we can consider the quotient $B/\mathrm{im}(\phi)$ as abelian groups. It is easy to check that the action of $R$ on $B$ induces one on the quotient; namely, $r \cdot \overline{a} = \overline{r \cdot a}$ (where $\overline{a}$ is the equivalence class of $a$). We define the *cokernel* of $\phi$ to be the quotient module

$$\mathrm{coker}(\phi) = B/\mathrm{im}(\phi).$$

**Definition 1.6.2** A (possibly infinite) sequence

$$\cdots \longrightarrow A_{i+1} \xrightarrow{\phi_{i+1}} A_i \xrightarrow{\phi_i} A_{i-1} \longrightarrow \cdots \tag{1.4}$$

of $R$-modules and $R$-homomorphisms is *exact at $A_i$* if $\ker(\phi_i) = \mathrm{im}(\phi_{i+1})$. It is *exact* if it is exact at $A_i$ for all $i$.

A *short exact sequence* is an exact sequence of the special form

$$0 \longrightarrow A' \xrightarrow{\phi} A \xrightarrow{\psi} A'' \longrightarrow 0. \tag{1.5}$$

**Example 1.6.3** The presence of 0's in an exact sequence gives some information about the adjacent maps. Namely,

*a)* A sequence $0 \longrightarrow A \xrightarrow{\phi} B$ is exact if and only if $\phi$ is injective;

*b)* A sequence $A \xrightarrow{\psi} B \longrightarrow 0$ is exact if and only if $\psi$ is surjective.

Short exact sequences can be "composed", and long ones decomposed, as in:

**Proposition 1.6.4 (Splicing of exact sequences)**

*a) If the short sequences*

$$0 \longrightarrow A \xrightarrow{\phi} B \xrightarrow{\psi} C \longrightarrow 0,$$

$$0 \longrightarrow C \xrightarrow{\alpha} D \xrightarrow{\beta} E \longrightarrow 0$$

*are exact, then there is a (long) exact sequence*

$$0 \longrightarrow A \xrightarrow{\phi} B \xrightarrow{\alpha\psi} D \xrightarrow{\beta} E \longrightarrow 0.$$

*b) Conversely, a long sequence as in (1.4) is exact at $A_i$ if and only if the following is a short exact sequence:*

$$0 \longrightarrow \mathrm{im}(\phi_{i+1}) \longrightarrow A_i \longrightarrow \mathrm{im}(\phi_i) \longrightarrow 0.$$

The proof is left as an exercise at the end of this section.

**Definition 1.6.5** Let $A$ and $B$ be $R$-modules, then $A$ is a *retract* of $B$ if there exist $R$-homomorphisms $s\colon A\longrightarrow B$ and $r\colon B\longrightarrow A$ such that $r\circ s=\mathsf{id}_A$. When this happens, we say that $s$ is a *section* of $r$, and $r$ is a *retraction* of $s$.

**Proposition 1.6.6** *For a short exact sequence*

$$0\longrightarrow A\xrightarrow{\ f\ }C\xrightarrow{\ g\ }B\longrightarrow 0$$

*of left $R$-modules, the following are equivalent:*

*a) $f$ has a retraction $r\colon C\longrightarrow A$;*

*b) $g$ has a section $s\colon B\longrightarrow C$;*

*c) there is an isomorphism $C\simeq A\oplus B$ under which the short exact sequence rewrites as*

$$0\longrightarrow A\xrightarrow{\ \sigma_A\ }A\oplus B\xrightarrow{\ \pi_B\ }B\longrightarrow 0.$$

The proof is left as Exercise *b*) below.

**Definition 1.6.7** We shall call *split* a short exact sequence satisfying the equivalent properties of Proposition 1.6.6.

**Proposition 1.6.8 (5-lemma)** In any commutative diagram

$$\begin{tikzcd}
A' \arrow[r] \arrow[d, "a"', "\cong"] & B' \arrow[r] \arrow[d, "b"', "\cong"] & C' \arrow[r] \arrow[d, "c"'] & D' \arrow[r] \arrow[d, "d"', "\cong"] & E' \arrow[d, "e"', "\cong"] \\
A \arrow[r] & B \arrow[r] & C \arrow[r] & D \arrow[r] & E
\end{tikzcd}$$

with exact rows, if $a$, $b$, $d$ and $e$ are isomorphisms, then so is $c$. More precisely, if $b$ and $d$ are injective and $a$ is surjective, then $c$ is injective, and dually, if $b$ and $d$ are surjective and $e$ is injective, then $c$ is surjective.

**Proof.** See Exercise *c*) below. □

Let $\mathsf{C}$ and $\mathsf{D}$ be categories of modules (or, more generally, categories in which the notion "exact sequence" is defined: see Chapter 4), and $F\colon\mathsf{C}\longrightarrow\mathsf{D}$ a functor between them. In this case, it makes sense to ask whether $F$ preserves exactness of sequences.

**Definition 1.6.9** $F$ is said to be *left exact* if, whenever a short sequence as in (1.5) is exact in $\mathsf{C}$, the following is exact in $\mathsf{D}$:

$$0\longrightarrow F(A')\xrightarrow{F\phi}F(A)\xrightarrow{F\psi}F(A'').$$

We say that $F$ is *right exact* if, when (1.5) is exact, we have an exact sequence:

$$F(A')\xrightarrow{F\phi}F(A)\xrightarrow{F\psi}F(A'')\longrightarrow 0.$$

Finally, $F$ is said *exact* if it is both left exact and right exact, i.e. if it preserves the exactness of short sequences (or equivalently, by Proposition 1.6.4, of long ones).

**Proposition 1.6.10** *If a functor $F: \mathsf{C} \longrightarrow \mathsf{D}$ between categories of modules is left (respectively, right) exact, then it preserves monomorphisms (resp. epimorphisms).*

**Remark 1.6.11** Bearing in mind that a contravariant functor $F: \mathsf{C} \longrightarrow \mathsf{D}$ is just a covariant functor from $\mathsf{C}^{\mathrm{op}}$ to $\mathsf{D}$ (see the Appendix for definitions), and that clearly a short sequence is exact in $\mathsf{C}$ if and only if it is in $\mathsf{C}^{\mathrm{op}}$, one can easily state the definition of left and right exactness for contravariant functors. It will then follow that a left exact contravariant functor converts monomorphisms into epimorphisms, and a right exact contravariant functor does the converse.

Next, we are going to analyse the exactness properties of the Hom and tensor product functors.

**Proposition 1.6.12** *Let $B$ be a left $R$-module. Then, the contravariant functor*

$$\mathrm{Hom}_R(-, B): R\text{-}\mathsf{mod} \longrightarrow \mathsf{Ab}$$

*is left exact.*

**Proof.** Consider a short exact sequence of left $R$-modules as in (1.5). Then, we want to show the following sequence to be exact:

$$0 \longrightarrow \mathrm{Hom}_R(A'', B) \xrightarrow{\psi^*} \mathrm{Hom}_R(A, B) \xrightarrow{\phi^*} \mathrm{Hom}_R(A', B). \tag{1.6}$$

To see that $\psi^*$ is injective, let $f, g: A \longrightarrow B$ be two $R$-homomorphisms such that $\psi^*(f) = \psi^*(g)$, i.e. $f\psi = g\psi$. Since (1.5) is exact, we know that $\psi$ is surjective, therefore $f = g$, and the sequence (1.6) is exact at $\mathrm{Hom}_R(A'', B)$.

For exactness at $\mathrm{Hom}_R(A, B)$, we need to show $\ker(\phi^*) = \mathrm{im}(\psi^*)$. By Remark 1.2.3, $\phi^*\psi^* = (\psi\phi)^*$, but we know that $\psi\phi$ is the null map, by the exactness of (1.5), hence $\phi^*\psi^* = 0$, and this shows that $\mathrm{im}(\psi^*) \subset \ker(\phi^*)$.

For the reverse inclusion, take a map $f: A \longrightarrow B$ such that $\phi^*(f) = f\phi = 0$. We want to show that $f = g\psi$ for some $g: A' \longrightarrow B$. Since $f\phi = 0$, $f$ factors through the cokernel of $\phi$, i.e. $f = f'\pi$ for some $f'$, where $\pi: A \longrightarrow A/\mathrm{im}(\phi)$ is the quotient projection. On the other hand, surjectivity of $\psi$ determines an isomorphism $h: A'' \xrightarrow{\simeq} A/\ker(\psi) = A/\mathrm{im}(\phi)$, for which it holds $h\psi = \pi$. Then, by taking $g = f'h$ we have $f = f'\pi = f'h\psi = g\psi$, and the result is proved. $\square$

By analogous techniques, one proves the following.

**Proposition 1.6.13** *For a left $R$-module $B$, the functor*

$$\mathrm{Hom}_R(B, -): R\text{-}\mathsf{mod} \longrightarrow \mathsf{Ab}$$

*is left exact.*

**Proposition 1.6.14** *Let $B$ be a left $R$-module. Then, the functor*

$$- \otimes_R B: \mathsf{mod}\text{-}R \longrightarrow \mathsf{Ab}$$

*is right exact.*

### Exercises

*a)* Prove Proposition 1.6.4.

*b)* Prove Proposition 1.6.6.

*c)* Prove Proposition 1.6.8.

*d)* Recall from A.0.3 the definition of mono and epimorphisms; then, using the characterisation of Example 1.6.3, give a proof of Proposition 1.6.10.

*e)* Make precise and verify the claims of Remark 1.6.11.

*f)* Prove Proposition 1.6.13.

*g)* Prove Proposition 1.6.14.

*h)* Show by an example that the functor $- \otimes_R B$ of Proposition 1.6.14 is not left exact. (Hint: you can choose $R$ to be the ring $\mathbb{Z}$)

*i)* Give examples to show that neither the covariant nor the contravariant Hom functors are right exact. (Hint: once again, you can choose $R$ to be $\mathbb{Z}$)

## 1.7 Projective Modules

The last exercise to the previous Section makes clear that the Hom functors are not always exact. In this section, we focus our attention on those modules for which the covariant Hom is.

**Definition 1.7.1** A left $R$-module $P$ is called *projective* if the functor

$$\mathrm{Hom}_R(P, -) \colon R\text{-}\mathsf{mod} \longrightarrow \mathsf{Ab}$$

is exact. An analogous definition holds for right $R$-modules.

More concretely, by Proposition 1.6.13 and Example 1.6.3 $P$ is projective if and only if, for any surjection $\tau \colon A \longrightarrow B$ and any map $\phi \colon P \longrightarrow B$, there exists a lifting $\psi \colon P \longrightarrow A$ making the following commute:

$$\begin{tikzcd} & A \arrow[d, two heads, "\tau"] \\ P \arrow[ur, dotted, "\psi"] \arrow[r, "\phi"'] & B. \end{tikzcd}$$

**Example 1.7.2** Check that every free $R$-module is projective.

It is also straightforward to verify the following:

**Lemma 1.7.3** *Every retract of a projective module is projective.*

In $R$-$\mathsf{mod}$, we can give a complete characterisation of projective modules.

**Proposition 1.7.4** *An $R$-module is projective if and only if it is a retract of a free module.*

**Proof.** One direction easily follows from Example 1.7.2 and Lemma 1.7.3 above. As for the other implication, let $P$ be a projective and suppose we can cover it by a free $R$-module $F$. Then, we can draw the diagram

$$\begin{tikzcd} & F \arrow[d, "r"] \\ P \arrow[r, "\mathsf{id}"'] & P, \end{tikzcd}$$

where $r$ is an epimorphism, and because $P$ is projective, we get a factorisation $\mathsf{id} = rs$ for some $R$-homomorphism $s$, showing $P$ as a retract of $F$. The existence of such a covering is given by the following lemma. $\square$

**Lemma 1.7.5** *Every $R$-module $A$ is* covered *by a free one, i.e. there is a free module $M$ and a surjective $R$-homomorphism*

$$M \longrightarrow A \longrightarrow 0.$$

In particular, this shows that every $R$-module is covered by a projective one. This is an important property of the category of $R$-modules. We shall express it by saying that $R$-$\mathsf{mod}$ *has enough projectives.*

**Definition 1.7.6** A right $R$-module $A$ is called *flat* if the functor $A \otimes_R -$ is exact. Equivalently, by Proposition 1.6.14 and Example 1.6.3, $A$ is flat if $A \otimes_R -$ preserves injective $R$-homomorphisms.

**Proposition 1.7.7** *Every projective right $R$-module is flat.*

### Exercises

*a)* Prove Lemma 1.7.5.

*b)* Show that a direct sum of flat modules is flat; use this to show that every free module is flat.

*c)* Use exercise *b*) to prove Proposition 1.7.7.

*d)* Let $A$ be a $\mathbb{Z}$-module, i.e. an abelian group. Prove that if $A$ is flat then it is torsion free (i.e. for any non-zero elements $a \in A$ and $n \in \mathbb{Z}$, $n \cdot a \neq 0$). See Exercise b) at the end of Section 1.11 for the converse.

*e)* Recall that a subgroup of a free abelian group is free. Prove that an abelian group is projective if and only if it is free.

*f)* Use the classification of finitely generated abelian groups to show that, for such groups, being torsion free, being free and being flat are all equivalent.

## 1.8 Injective Modules

In this Section, we are going to look at those modules for which the contravariant Hom functor is exact.

**Definition 1.8.1** A left $R$-module $I$ is called *injective* if the functor

$$\mathrm{Hom}_R(-, I)\colon R\text{-}\mathsf{mod} \longrightarrow \mathsf{Ab}$$

is exact. The definition is easily adapted to the case of right $R$-modules.

More concretely, $I$ is injective if, for any two $R$-homomorphisms $i\colon A \longrightarrow B$ and $\phi\colon A \longrightarrow I$ with $i$ injective, there exists an extension $\psi\colon B \longrightarrow I$ making the following commute:

$$\begin{tikzcd} A \arrow[r, "\phi"] \arrow[d, tail, "i"'] & I. \\ B \arrow[ur, dotted, "\psi"'] & \end{tikzcd}$$

Notice that the notions of injective and projective module are *dual*; that is, we can write one from the other by "reversing the arrows". More formally, an object is injective in a category if and only if it is projective in its opposite (see Definition A.0.8). This enables us to translate a few results for projective modules to dual statements for injective ones. For instance, since the dual of a retract is again a retract, we have the following.

**Proposition 1.8.2** *Every retract of an injective module is injective.*

**Proposition 1.8.3** *Let $(A_j)_{j\in J}$ be a family of $R$-modules and let $A$ be their direct product. Then, $A$ is injective if and only if each $A_i$ is.*

When $R$ is the commutative ring $\mathbb{Z}$, we can give a precise characterisation of injective modules.

**Definition 1.8.4** An abelian group $D$ is called *divisible* if, for any $d \in D$ and any non-zero integer $n$, there is an element $x \in D$ such that $nx = d$.

**Example 1.8.5** The additive group $\mathbb{Q}$ of rational numbers is divisible. Also, any quotient of a divisible group is itself divisible.

Any element $d$ in an abelian group $D$ determines a (unique) group homomorphism $f\colon \mathbb{Z} \longrightarrow D$ such that $f(1) = d$. Likewise, any $n \in \mathbb{Z}$ determines a morphism $n \cdot -\colon \mathbb{Z} \longrightarrow \mathbb{Z}$, and this will be injective if $n \neq 0$. The group $D$ is then divisible if there exists a $g\colon \mathbb{Z} \longrightarrow D$ making the following triangle commute (so that, by setting $x = g(1)$, we have $d = f(1) = g(n \cdot 1) = nx$):

$$\begin{tikzcd} \mathbb{Z} \arrow[r, "f"] \arrow[d, tail, "n\cdot -"'] & D. \\ \mathbb{Z} \arrow[ur, dotted, "g"'] & \end{tikzcd}$$

This proves the following.

**Proposition 1.8.6** *Every injective $\mathbb{Z}$-module is divisible.*

The converse is also true.

**Proposition 1.8.7** *Every divisible abelian group is injective as a $\mathbb{Z}$-module.*

**Proof.** Let $D$ be a divisible group an consider a diagram of abelian groups

$$\begin{tikzcd} A' \arrow[r, "\phi"] \arrow[d, tail, "i"'] & D \\ A & \end{tikzcd}$$

where $i$ is injective. We wish to extend $\phi$ along $i$. Consider the set $X$ of pairs $(B, \psi)$ such that $B$ is a subgroup of $A$ containing $A'$ and $\psi: B \longrightarrow D$ extends $\phi$. Define a partial order on $X$ by saying that $(B_1, \psi_1) \leq (B_2, \psi_2)$ if $B_1 \subset B_2$ and $\psi_2$ extends $\psi_1$. Then, $(X, \leq)$ is clearly an inductive set, hence by Zorn's Lemma it has a maximal element $(B_0, \psi_0)$.

Suppose $B_0 \neq A$, and let $a \in A - B_0$. Then, the subgroup

$$B' = \{b + na : \, b \in B_0, n \in \mathbb{Z}\}$$

properly contains $B_0$. If $na \notin B_0$ for all non-zero integers $n$, then $\psi_0$ can be extended to a homomorphism $\psi': B' \longrightarrow D$ by defining

$$\psi'(b + na) = \psi_0(b).$$

Otherwise, let $m$ be the least positive integer such that $ma \in B_0$, and let $d = \psi_0(ma)$. Then, define the extension $\psi': B' \longrightarrow D$ of $\psi_0$ as

$$\psi'(b + na) = \psi_0(b) + nx,$$

where $x \in D$ is such that $mx = d$.

In either case, the pair $(B', \psi')$ violates the maximality of $(B_0, \psi_0)$, therefore $B_0 = A$. This shows that $D$ is injective. $\square$

**Example 1.8.8** By Example 1.8.5, it follows immediately that the *rational circle* $\mathbb{Q}/\mathbb{Z}$ is divisible, hence injective.

We are now interested in showing that, every $R$-module can be embedded into an injective one. In order to prove this result, we shall first restrict our attention to the case $R = \mathbb{Z}$. Then, we shall be able to transpose the property to other categories of modules by means of an appropriate adjunction.

For the moment, let us define, for any abelian group $A$, a group $A\check{}$ as

$$A\check{} = \mathrm{Hom}_{\mathbb{Z}}(A, \mathbb{Q}/\mathbb{Z}).$$

Notice that there is a canonical homomorphism of abelian groups

$$\alpha: A \longrightarrow (A\check{})\check{} \tag{1.7}$$

defined by $\alpha(a)(\phi) = \phi(a)$ for any $a$ in $A$ and $\phi \in \mathrm{Hom}_{\mathbb{Z}}(A, \mathbb{Q}/\mathbb{Z})$.

**Lemma 1.8.9** *The group homomorphism $\alpha$ of (1.7) is injective.*

**Proof.** It is clearly enough to prove that for any $a \neq 0$ in $A$ there is a morphism $\phi: A \longrightarrow \mathbb{Q}/\mathbb{Z}$ with $\phi(a) \neq 0$. Using injectivity of $\mathbb{Q}/\mathbb{Z}$, it is enough to define $\phi$ on the subgroup $B$ of $A$ generated by $a$. If $na \neq 0$ for all $n \in \mathbb{Z}$, then $B$ is a free $\mathbb{Z}$-module with base $a$, and we may choose $\phi$ so that $\phi(a)$ is any non-zero element of $\mathbb{Q}/\mathbb{Z}$. Otherwise, let $m$ be the least positive integer such that $ma = 0$ and define $\phi(na)$ as the class of $n/m$ in $\mathbb{Q}/\mathbb{Z}$. In both cases, $\phi$ satisfies the required property. $\square$

**Lemma 1.8.10** *If $A$ is a flat $\mathbb{Z}$-module, then $A\check{}$ is divisible.*

**Proof.** Equivalently, we show that $A^{\vee}$ is an injective abelian group. To this purpose, suppose

$$\begin{tikzcd} B \arrow[r, "f"] \arrow[d, "i"'] & A^{\vee} \\ B', & \end{tikzcd}$$

are group homomorphisms, with $i$ injective. Then, the diagram transposes through the isomorphism (1.2) to give

$$\begin{tikzcd} B \otimes_{\mathbb{Z}} A \arrow[r, "f'"] \arrow[d, "i\otimes \mathsf{id}_A"'] & \mathbb{Q}/\mathbb{Z} \\ B' \otimes_{\mathbb{Z}} A, & \end{tikzcd}$$

where $i \otimes \mathsf{id}_A$ is injective because $A$ is flat.

Hence, by Example 1.8.8, there is an extension $g' \colon B' \otimes_{\mathbb{Z}} A \longrightarrow \mathbb{Q}/\mathbb{Z}$, which transposes back to give a group homomorphism

$$g \colon B \longrightarrow A^{\vee}$$

such that $gi = f$. □

**Theorem 1.8.11** *Every abelian group can be embedded into an injective one.*

**Proof.** By Lemma 1.7.5, we can cover the $\mathbb{Z}$-module $A^{\vee}$ with a free one, i.e. there is an exact sequence

$$F \xrightarrow{q} A^{\vee} \longrightarrow 0 \tag{1.8}$$

where $F$ is free. By applying the contravariant functor $\mathrm{Hom}_{\mathbb{Z}}(-, \mathbb{Q}/\mathbb{Z})$ to (1.8), we obtain an exact sequence

$$0 \longrightarrow (A^{\vee})^{\vee} \xrightarrow{q^*} F^{\vee}.$$

Hence, $q^* \colon (A^{\vee})^{\vee} \longrightarrow F^{\vee}$ is injective and, by Lemma 1.8.9, so is the composite

$$A \xrightarrow{\alpha} (A^{\vee})^{\vee} \xrightarrow{q^*} F^{\vee}.$$

Moreover, $F^{\vee}$ is injective by Lemma 1.8.10, whence the result. □

Analogously to the case of projectives, this property of categories of modules is very important, and we shall express it by saying that $\mathbb{Z}$-$\mathsf{mod}$ *has enough injectives.*

Now, we can use Theorem 1.8.11 to prove that the category $R$-$\mathsf{mod}$ has enough injectives for *any* ring $R$. This will use a standard proof technique, which relies on the following categorical lemma (see the Appendix for the relevant definitions).

**Lemma 1.8.12** *If $F \dashv G\colon \mathsf{C} \longrightarrow \mathsf{D}$ is a pair of adjoint functors and $F$ preserves monos, then $G$ preserves injectives.*

**Theorem 1.8.13** *Every left $R$-module can be embedded into an injective one.*

**Proof.** Let $\phi\colon \mathbb{Z} \longrightarrow R$ be the unique morphism of rings fixing $u(1) = 1$. Then, by Theorem 1.5.4, we have an adjunction

$$\begin{tikzcd} \mathbb{Z}\text{-}\mathsf{mod} \arrow[r, bend right, "\phi_*"'] \arrow[r, phantom, "\perp"] & R\text{-}\mathsf{mod} \arrow[l, bend right, "\phi^*"'] \end{tikzcd}$$

and by Theorem 1.8.11 we can, for any $R$-module $A$, consider the embedding

$$\phi^* A \overset{\gamma}{\rightarrowtail} D$$

of the abelian group $\phi^* A$ into an injective $D$.

Since right adjoint functors preserve monomorphisms, $\phi_* \gamma$ is again mono. Moreover, the unit of the adjunction $\eta\colon A \longrightarrow \phi_* \phi^* A = \mathrm{Hom}_{\mathbb{Z}}(R, A)$ computes as $\eta(a)(r) = ra$, therefore it is obviously injective. The transpose

$$\widehat{\gamma} = A \xrightarrow{\eta} \phi_* \phi^* A \xrightarrow{\phi_* \gamma} \phi_* D$$

of $\gamma$ along the adjunction is then itself mono, and $\phi_* D$ is injective by Lemma 1.8.12, since $\phi^*$ preserves monos; therefore, we have the desired embedding. □

### Exercises

*a)* Prove Proposition 1.8.3.

*b)* Prove Lemma 1.8.12.

## 1.9 Complexes

In Section 1.6 we have investigated the concept of an exact sequence. In particular, these chains of morphisms satisfy the property that the composite of any two consecutive maps is null. Complexes, and their homology, enable one to study how far a sequence of modules is from being exact.

**Definition 1.9.1** A *chain complex* $C_\star$ of $R$-modules is a $\mathbb{Z}$-indexed sequence of $R$-modules and $R$-homomorphism

$$\cdots \longleftarrow C_{n-1} \xleftarrow{d_n} C_n \xleftarrow{d_{n+1}} C_{n+1} \longleftarrow \cdots \tag{1.9}$$

such that $d_n \circ d_{n+1} = 0$ for all $n \in \mathbb{Z}$. The maps $d_n$ are called *boundary maps*, and we shall omit the index whenever possible. An element of $C_n$ is called an *$n$-chain*.

A *morphism of chain complexes* $f_\star\colon C_\star \longrightarrow C'_\star$ is a family $(f_n\colon C_n \longrightarrow C'_n)_{n\in\mathbb{Z}}$ of $R$-homomorphisms, such that the following diagram commutes for all $n \in \mathbb{Z}$:

$$\begin{tikzcd} C_n \arrow[r, "d_n"] \arrow[d, "f_n"'] & C_{n-1} \arrow[d, "f_{n-1}"] \\ C'_n \arrow[r, "d'_n"'] & C'_{n-1}. \end{tikzcd}$$

We shall often drop the subscripts, and express this condition by saying that $fd = df$. Two morphisms of chain complexes $f_\star\colon C_\star \longrightarrow C'_\star$ and $g_\star\colon C'_\star \longrightarrow C''_\star$ can be composed degree by degree, to give the morphism $g_\star \circ f_\star = (g_n \circ f_n)_{n\in\mathbb{Z}}$. Moreover, for any chain complex $C_\star$ there is an obvious identity morphism $\mathsf{id}_{C_\star} = (\mathsf{id}_{C_n})_{n\in\mathbb{Z}}$. Thus, we have a category of chain complexes, which we denote by $\mathsf{Ch}(R)$.

**Definition 1.9.2** The category $\mathsf{Ch}_+(R)$ of *positive chain complexes* is defined to be the (full) subcategory of $\mathsf{Ch}(R)$ whose objects are those chain complexes $C_\star$ with $C_n = 0$ for $n < 0$.

Similarly, we define the category $\mathsf{Ch}_b(R)$ of *bounded chain complexes*, whose objects are those chain complexes $C_\star$ such that $C_n = 0$ for all but finitely many values of $n$.

As for many categorical notions, that of a chain complex has a dual.

**Definition 1.9.3** A *cochain complex* $C^\star$ of $R$-modules is a sequence of modules and maps

$$\cdots \longrightarrow C^{n-1} \xrightarrow{d^{n-1}} C^n \xrightarrow{d^n} C^{n+1} \longrightarrow \cdots \tag{1.10}$$

with the property $d \circ d = 0$. An element of $C^n$ is called an *$n$-cochain*.

Morphisms of cochain complexes are defined in the obvious way, by the condition $fd = df$, and they compose, to form a category of cochain complexes, denoted by $\mathsf{CCh}(R)$. This admits a subcategory of *positive cochain complexes* $\mathsf{CCh}_+(R)$ and a subcategory of *bounded cochain complexes* $\mathsf{CCh}_b(R)$, defined in the obvious way.

Notice that, for categories of (co)chain complexes, it makes sense to speak of short exact sequences. In particular, we shall say that

$$0 \longrightarrow A_\star \xrightarrow{f} B_\star \xrightarrow{g} C_\star \longrightarrow 0$$

is an *exact sequence of (co)chain complexes* if the short sequence of $R$-modules

$$0 \longrightarrow A_n \xrightarrow{f_n} B_n \xrightarrow{g_n} C_n \longrightarrow 0$$

is exact for each $n \in \mathbb{Z}$.

**Example 1.9.4** The *standard $p$-simplex* $\Delta_p \subset \mathbb{R}^{p+1}$ is defined as the set

$$\Delta_p = \left\{ (x_0, \ldots, x_p) : \sum_{i=0}^{p} x_i = 1,\ x_i \geq 0 \text{ for all } i = 0, \ldots, p \right\}.$$

The *face operators* $\partial_i\colon \Delta_{p-1} \longrightarrow \Delta_p$ $(i = 0, \ldots, p)$ are defined by

$$\partial_i(x_0, \ldots, x_{p-1})) = (x_0, \ldots, x_{i-1}, 0, x_i, \ldots, x_{p-1}).$$

A *singular $p$-simplex* in a topological space $X$ is a continuous function $s\colon \Delta_p \longrightarrow X$. For any $p \geq 0$, the set of all singular $p$-simplices in $X$ is denoted by $\Delta_p(X)$. The space $C_p(X)$ of *singular $p$-chains on $X$* is the free $\mathbb{Z}$-module generated by $\Delta_p(X)$.

Precomposition with $\partial_i$ takes any $p$-simplex $s\colon \Delta_p \longrightarrow X$ to a $(p-1)$-simplex $s \circ \partial_i$. Since $\Delta_p(X)$ is a basis of $C_p(X)$, these maps extend uniquely to linear maps

$$d_i\colon C_p(X) \longrightarrow C_{p-1}(X).$$

The *boundary operator* $d\colon C_p(X) \longrightarrow C_{p-1}(X)$ $(p > 0)$ is by definition the linear map

$$d = \sum_{i=0}^{p} (-1)^i d_i$$

taking a singular $p$-chain $\alpha$ to $d\alpha = \sum_{i=0}^{p} (-1)^i \alpha \circ \partial_i$.

The sequence of chain groups

$$\cdots \xleftarrow{\;d\;} C_{p-1}(X) \xleftarrow{\;d\;} C_p(X) \xleftarrow{\;d\;} \cdots$$

is a positive chain complex. We call $C_\star(X)$ the *singular complex of $X$*.

Moreover, any continuous map $f\colon X \longrightarrow Y$ determines by postcomposition a morphism of chain complexes $f_\star\colon C_\star(X) \longrightarrow C_\star(Y)$. We shall look at this in more detail in Chapter 3.

**Example 1.9.5** If $M$ is an $n$-dimensional smooth manifold, define:

$$\mathcal{O}(M) = \{f\colon M \longrightarrow \mathbb{R} : f \text{ is a } C^\infty\text{-function}\}.$$

We denote by $\Omega^p(M)$ $(p \geq 0)$ the $\mathcal{O}(M)$-module of *differential $p$-forms* on $M$, which is generated by elements $dx_{i_1} \wedge \cdots \wedge dx_{i_p}$, subject to the condition

$$dx_{i_1} \wedge \cdots \wedge dx_{i_p} = (-1)^l dx_{\tau i_1} \wedge \cdots \wedge dx_{\tau i_p}$$

for any permutation $\tau$ of $i_1, \ldots, i_p$ of degree $l$. Therefore, any element $\omega$ in $\Omega^p(M)$ is uniquely written as

$$\omega = \sum_{i_1 < \cdots < i_p} f_{i_1 \cdots i_p} dx_{i_1} \wedge \cdots \wedge dx_{i_p} \tag{1.11}$$

with $f_{i_1 \ldots i_p}$ a $C^\infty$-function.

The *exterior derivative* $d\colon \Omega^p(M) \longrightarrow \Omega^{p+1}(M)$ is inductively defined as follows. If $f \in \Omega^0(M) = \mathcal{O}(M)$, then

$$df = \sum_{j=1}^{n} \frac{\partial f}{\partial x_j} dx_j.$$

If $p > 0$ and $\omega \in \Omega^p(M)$ is a $p$-form as in (1.11), then we define

$$d\omega = \sum_{i_1 < \cdots < i_p} df_{i_1 \cdots i_p} \wedge dx_{i_1} \wedge \cdots \wedge dx_{i_p}.$$

The exterior derivative gives a bounded cochain complex $\Omega^\star(M)$, called the *de Rham complex* of $M$:

$$0 \longrightarrow \Omega^0(M) \xrightarrow{d} \Omega^1(M) \xrightarrow{d} \cdots \xrightarrow{d} \Omega^n(M) \longrightarrow 0.$$

Note that the condition $d_n \circ d_{n+1} = 0$ for a chain complex is equivalent to $\operatorname{im}(d_{n+1}) \subset \ker(d_n)$. This is half of the condition needed for exactness. In particular, it enables us to form the quotient groups $\ker(d_n)/\operatorname{im}(d_{n+1})$, which will somehow measure "how far a complex is from being exact".

**Definition 1.9.6** The *n-th homology group* of the chain complex $C_\star$ is defined as:

$$H_n(C_\star) = \ker(d_n)/\operatorname{im}(d_{n+1}).$$

The elements of $H_n(C_\star)$ are called *homology classes*. They are represented by $n$-chains in $\ker(d_n)$ (these are called *n-cycles*), up to a summand in $\operatorname{im}(d_{n+1})$ (which are called *n-boundaries*). The homology class represented by a cycle $c$ is denoted by $[c] = c + \operatorname{im}(d_{n+1}) \in H_n(C_\star)$. The sum of $[x]$ and $[y]$ is given by the class $[x + y]$, and the null class 0 is represented by $x$ if and only if $x = dy$ for some $y \in C_{n+1}$.

Analogously, for a cochain complex $C^\star$, we define its *cohomology group in degree n* as the quotient

$$H^n(C^\star) = \ker(d^n)/\operatorname{im}(d^{n-1}).$$

A cochain $x$ in $C^n$ is a *cocycle* if $d^n x = 0$ and a *coboundary* if $x = dy$ for some $y \in C^{n-1}$. The equivalence class of a cocycle $x$ is called the *cohomology class* of $x$, and it is denoted by $[x]$.

**Example 1.9.7**

*a)* The homology groups $H_n(C_\star(X))$ of the singular complex of Example 1.9.4 above are called the *singular homology groups* of the space $X$, and are usually denoted by $H_i(X)$;
*b)* The $p$-th cohomology group $H^p(\Omega^\star(M))$ of the complex of Example 1.9.5 is denoted by $H^p_{\mathrm{dR}}(M)$ and it is called the *p-th de Rham cohomology group* of the manifold $M$.

A morphism of (co)chain complexes $f_\star\colon C_\star \longrightarrow C'_\star$ clearly maps (co)cycles to (co)cycles and (co)boundaries to (co)boundaries. Therefore, it induces in each dimension $n$ a homomorphism

$$H_n(f)\colon H_n(C_\star) \longrightarrow H_n(C'_\star)$$

defined as $H_n(f)([c]) = [f_n(c)]$. We shall often simply write $f$, or $f_*$, for $H_n(f)$. As usual, we have functoriality: if $f_\star\colon C_\star \longrightarrow C'_\star$ and $g_\star\colon C'_\star \longrightarrow C''_\star$ are morphisms of (co)chain complexes, then $(g \circ f)_* = g_* \circ f_*$ and $\mathsf{id}_* = \mathsf{id}_{H_n(C_\star)}$ for each $n$. Thus each $H_n$ is a covariant functor from $\mathsf{Ch}(R)$ (respectively, $\mathsf{CCh}(R)$) to $R$-$\mathsf{mod}$.

**Proposition 1.9.8** *Let*

$$0 \longrightarrow A_\star \xrightarrow{f_\star} B_\star \xrightarrow{g_\star} C_\star \longrightarrow 0$$

*be a short exact sequence of chain complexes. Then, there are canonically induced* connecting homomorphisms

$$\delta_n \colon H_n(C_\star) \longrightarrow H_{n-1}(A_\star)$$

*making the following long sequence exact:*

$$\cdots \xrightarrow{f_*} H_n(B_\star) \xrightarrow{g_*} H_n(C_\star) \xrightarrow{\delta_n} H_{n-1}(A_\star) \xrightarrow{f_*} \cdots .$$

*Moreover, the connecting morphisms are "natural", in the sense that whenever*

$$\begin{tikzcd}
0 \arrow[r] & A_\star \arrow[r, "f_\star"] \arrow[d, "\phi_\star"'] & B_\star \arrow[r, "g_\star"] \arrow[d, "\psi_\star"'] & C_\star \arrow[r] \arrow[d, "\gamma_\star"'] & 0 \\
0 \arrow[r] & A'_\star \arrow[r, "f'_\star"'] & B'_\star \arrow[r, "g'_\star"'] & C'_\star \arrow[r] & 0
\end{tikzcd}$$

*is a commutative diagram with both rows exact and connecting morphisms $\delta$ and $\delta'$, then the following commutes for each $n$:*

$$\begin{tikzcd}
H_n(C_\star) \arrow[r, "\delta_n"] \arrow[d, "\gamma_*"'] & H_{n-1}(A_\star) \arrow[d, "\phi_*"] \\
H_n(C'_\star) \arrow[r, "\delta'_n"'] & H_{n-1}(A'_\star).
\end{tikzcd}$$

**Proof.** We show how to find $\delta_n([c]) \in H_{n-1}(A_\star)$, where $[c] \in H_n(C_\star)$, and leave further details as an exercise. Consider the commutative diagram

$$\begin{tikzcd}
 & \vdots \arrow[d] & \vdots \arrow[d] & \vdots \arrow[d] & \\
0 \arrow[r] & A_n \arrow[r, "f_n"] \arrow[d] & B_n \arrow[r, "g_n"] \arrow[d] & C_n \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & A_{n-1} \arrow[r, "f_{n-1}"] \arrow[d] & B_{n-1} \arrow[r, "g_{n-1}"] \arrow[d] & C_{n-1} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & A_{n-2} \arrow[r, "f_{n-2}"] \arrow[d] & B_{n-2} \arrow[r, "g_{n-2}"] \arrow[d] & C_{n-2} \arrow[r] \arrow[d] & 0 \\
 & \vdots & \vdots & \vdots & .
\end{tikzcd}$$

We choose $c' \in C_n$ representing $[c]$. Then, $d(c') = 0$ and, by surjectivity of $g$, we can pick $b \in B_n$ such that $g_n(b) = c'$. Then, $g_{n-1}(d(b)) = d(g_n(b)) = d(c') = 0$ and exactness of the middle row implies that there is a unique $a \in A_{n-1}$ such that $f_{n-1}(a) = d(b)$. Therefore, $f_{n-2}(d(a)) = d(f_{n-1}(a)) = d(d(b)) = 0$, and by injectivity of $f_{n-2}$ we get $d(a) = 0$, so we define $\delta_n([c]) = [a] \in H_{n-1}(A_\star)$. More diagram chasing proves that $[a]$ is independent of the choices of $c' \in [c]$ and $b \in B_n$ such that $g_n(b) = c'$. Naturality of $\delta$ is proved by analogous techniques. $\square$

### Exercise

*a)* Fill in the details of the proof of Proposition 1.9.8. Then, formulate the analogous result for cochain complexes.

## 1.10 Homotopy of chain complexes

We shall now focus on chain complexes, although this is inessential, and we could easily restate every result for cochain complexes.

**Definition 1.10.1** Let $f, g\colon C_\star \longrightarrow C'_\star$ be two morphisms of chain complexes. Then, a *chain homotopy* between $f$ and $g$ is a sequence of homomorphisms $h_n\colon C_n \longrightarrow C'_{n+1}$ ($n \in \mathbb{Z}$) such that, for all $n$,

$$f_n - g_n = d'_{n+1} \circ h_n + h_{n-1} \circ d_n. \tag{1.12}$$

We shall often omit the subscripts, and write (1.12) as $f - g = hd + dh$. Pictorially, the situation is represented in the diagram below:

$$\begin{tikzcd}
\cdots & C_{n-1} \arrow[l] \arrow[d, shift right, "f_{n-1}"'] \arrow[d, shift left, "g_{n-1}"] \arrow[dr, dotted, "h_{n-1}"'] & C_n \arrow[l, "d_n"'] \arrow[d, shift right, "f_n"'] \arrow[d, shift left, "g_n"] \arrow[dr, dotted, "h_n"'] & C_{n+1} \arrow[l, "d_{n+1}"'] \arrow[d, shift right, "f_{n+1}"'] \arrow[d, shift left, "g_{n+1}"] & \cdots \arrow[l] \\
\cdots & C'_{n-1} \arrow[l] & C_n \arrow[l, "d'_n"] & C_{n+1} \arrow[l, "d'_{n+1}"] & \cdots \arrow[l]
\end{tikzcd}$$

**Example 1.10.2** Homotopies between maps of topological spaces induce chain homotopies. We shall investigate this further in Chapter 3.

**Remark 1.10.3** We define an equivalence relation between maps of chain complexes by saying that $f$ is equivalent to $g$ if there is a homotopy from $f$ to $g$. (This is indeed an equivalence relation, by Exercise *a*) below.) We denote this relation by writing $f \sim g$, and say in this case that *$f$ and $g$ are homotopic.* Notice that it is respected by composition:

**Proposition 1.10.4** *Consider the following diagram of chain complexes:*

$$C_\star \underset{g}{\overset{f}{\rightrightarrows}} C'_\star \underset{i}{\overset{h}{\rightrightarrows}} C''_\star.$$

*If $f \sim g$ and $h \sim i$, then $h \circ f \sim i \circ g$.*

The proof of this result is left as an exercise.

**Remark 1.10.5** This shows that the relation of being homotopic is preserved by composition. In particular, this allows one to define the quotient categories

$$\mathsf{Ho}(R),\ \mathsf{Ho}_+(R) \text{ and } \mathsf{Ho}_b(R)$$

of $\mathsf{Ch}(R)$, $\mathsf{Ch}_+(R)$ and $\mathsf{Ch}_b(R)$ respectively. These will have the same objects as their corresponding categories of complexes, but as maps homotopy equivalence classes of maps of chain complexes.

Homology can not distinguish homotopic maps:

**Proposition 1.10.6** *If $f \sim g\colon C_\star \longrightarrow C'_\star$ are homotopic maps of chain complexes, then they induce the same maps between the homology groups:*

$$f_* = g_*\colon H_n(C_\star) \longrightarrow H_n(C'_\star).$$

**Proof.** Suppose $h$ is a homotopy between $f$ and $g$. Then, we want to show that the difference $f_* - g_*$ is the null map between the homology groups. To this purpose, let $[c] \in H_n(C_\star)$ for some $c \in \ker d_n$. Then,

$$(f_n - g_n)(c) = (d'_{n+1}h_n + h_{n-1}d_n)(c) = d'_{n+1}(h_n(c));$$

that is, $(f_n - g_n)(c)$ is an $n$-boundary in $C'_n$. Hence,

$$(f_* - g_*)([c]) = [(f_n - g_n)(c)] = 0$$

for all $[c] \in H_n(C_\star)$. $\square$

**Definition 1.10.7** A morphism $f_\star\colon C_\star \longrightarrow C'_\star$ of chain complexes is a *homotopy equivalence* if there exists another morphism $g_\star\colon C'_\star \longrightarrow C_\star$ such that $f \circ g \sim \mathsf{id}_{C'}$ and $g \circ f \sim \mathsf{id}_C$.

From Proposition 1.10.6 we obtain immediately:

**Corollary 1.10.8** *A homotopy equivalence $f_\star\colon C_\star \longrightarrow C'_\star$ induces isomorphisms $f_*\colon H_n(C_\star) \longrightarrow H_n(C'_\star)$ between the homology groups in each degree $n$.*

**Definition 1.10.9** A morphism $f_\star\colon C_\star \longrightarrow C'_\star$ of chain complexes is said to be a *quasi-isomorphism* if the corresponding homology maps $f_*\colon H_n(C_\star) \longrightarrow H_n(C'_\star)$ are isomorphisms for all $n \in \mathbb{Z}$.

**Remark 1.10.10** It is obvious by the definitions and Proposition 1.10.6 that every homotopy equivalence is also a quasi-isomorphism. The converse is not true (see Exercise *d*) below).

**Definition 1.10.11** A *resolution* $A \xleftarrow{\epsilon} P_\star$ of a module $A$ is an exact sequence of modules

$$0 \longleftarrow A \xleftarrow{\ \epsilon\ } P_0 \longleftarrow P_1 \longleftarrow \cdots$$

The resolution is called *projective* if so are all the modules $P_n$, for $n \in \mathbb{Z}$. Dually, an *injective resolution* $A \xrightarrow{\eta} I^\star$ of $A$ is an exact sequence:

$$0 \longrightarrow A \xrightarrow{\ \eta\ } I^0 \longrightarrow I^1 \longrightarrow \cdots$$

where each $I^n$ is injective.

**Lemma 1.10.12** *Given a map $f: A \longrightarrow A'$ of modules, a projective resolution $A \xleftarrow{\epsilon} P_\star$ of $A$ and any resolution $A' \xleftarrow{\epsilon'} B_\star$ of $A'$, there is a morphism $f_\star: P_\star \longrightarrow B_\star$ of exact sequences lifting $f$, in the sense that $\epsilon' f_0 = f\epsilon$:*

$$\begin{tikzcd}
0 & A \arrow[l] \arrow[d, "f"'] & P_0 \arrow[l, "\epsilon"'] \arrow[d, dotted, "f_0"'] & P_1 \arrow[l, "d_1"'] \arrow[d, dotted, "f_1"'] & P_2 \arrow[l, "d_2"'] \arrow[d, dotted, "f_2"'] & \cdots \arrow[l, "d_3"'] \\
0 & A' \arrow[l] & B_0 \arrow[l, "\epsilon'"] & B_1 \arrow[l, "d'_1"] & B_2 \arrow[l, "d'_2"] & \cdots . \arrow[l, "d'_3"]
\end{tikzcd} \tag{1.13}$$

*Furthermore, the morphism $f_\star$ is unique up to homotopy.*

**Proof.** Since $P_0$ is projective and $\epsilon'$ is surjective, we can find $f_0$ to make

$$\begin{tikzcd}
A \arrow[d, "f"'] & P_0 \arrow[l, "\epsilon"'] \arrow[d, dotted, "f_0"] \\
A' & B_0 \arrow[l, "\epsilon'"]
\end{tikzcd}$$

commute. It is easy to check that $f_0$ restricts to a map $F_0: \ker(\epsilon) \longrightarrow \ker(\epsilon')$, and by surjectivity of $B_1 \longrightarrow \ker(\epsilon')$ we can find $f_1$ making

$$\begin{tikzcd}
\ker(\epsilon) \arrow[d, "F_0"'] & P_1 \arrow[l] \arrow[d, dotted, "f_1"] \\
\ker(\epsilon') & B_1 \arrow[l]
\end{tikzcd}$$

commute. By iterating this construction, we define $f_n$ for all $n \in \mathbb{Z}$.

If $g_\star: P_\star \longrightarrow B_\star$ is another morphism of exact sequences making (1.13) commute, then $f_\star - g_\star$ covers the morphism $0 = f - f: A \longrightarrow A'$. Therefore, uniqueness follows if we show that $f_\star$ is homotopic to zero whenever $f = 0$. To give a homotopy, we need maps $h_n: P_n \longrightarrow B_{n+1}$ $(n \geq 0)$ so that

$$f_n = d'_{n+1} h_n + h_{n-1} d_n.$$

We are going to define them inductively. For $n = 0$, this reads $f_0 = d'_1 h_0$, and since $P_0$ is projective, the existence of $h_0$ follows from the exactness of the bottom row and the commutativity of the following:

$$\begin{tikzcd}
A \arrow[d, "0"'] & P_0 \arrow[l] \arrow[d, "f_0"'] \arrow[dr, dotted, "h_0"] & \\
A' & B_0 \arrow[l] & B_1 . \arrow[l, "d'_1"]
\end{tikzcd}$$

For the inductive step, suppose the maps $h_i$ are defined for $i < n$ and consider the diagram

$$\begin{tikzcd}
P_{n-1} \arrow[d, "f_{n-1}"'] \arrow[dr, "h_{n-1}"] & P_n \arrow[l, "d_n"'] \arrow[d, "f_n"'] \arrow[dr, dotted, "h_n"] & \\
B_{n-1} & B_n \arrow[l, "d'_n"] & B_{n+1} . \arrow[l, "d'_{n+1}"]
\end{tikzcd}$$

The bottom row is exact and $P_n$ is projective, so we can find a factorisation $f_n - h_{n-1}d_n = d'_{n+1}h_n$ provided we verify that $d'_n(f_n - h_{n-1}d_n) = 0$. But, by the inductive hypothesis, we know that $d'_n h_{n-1} = f_{n-1} - h_{n-2}d_{n-1}$, therefore

$$d'_n(f_n - h_{n-1}d_n) = d'_n f_n - f_{n-1}d_n + h_{n-2}d_{n-1}d_n = d'_n f_n - f_{n-1}d_n = 0.$$

□

**Proposition 1.10.13** *Every module $A$ has a projective resolution, which is unique up to homotopy.*

**Proof.** We know by the comment after Lemma 1.7.5 that $A$ can be covered by a projective, i.e. there is a projective module $P_0$ and an exact sequence

$$0 \longleftarrow A \xleftarrow{\epsilon=\epsilon_0} P_0 \xleftarrow{\iota_1} \ker(\epsilon_0) \longleftarrow 0$$

Similarly, the module $\ker(\epsilon_0)$ gives rise to an exact sequence

$$0 \longleftarrow \ker(\epsilon_0) \xleftarrow{\epsilon_1} P_1 \xleftarrow{\iota_2} \ker(\epsilon_1) \longleftarrow 0$$

and so on for $\ker(\epsilon_n)$, for each $n \in \mathbb{Z}$. By Proposition 1.6.4, we can splice all these short exact sequences into one long exact sequence:

$$\begin{tikzcd}
0 & A \arrow[l] & P_0 \arrow[l, "\epsilon"'] & & P_1 \arrow[ll, "d_1"'] \arrow[dl, bend left, "\epsilon_1"] & & P_2 \arrow[ll, "d_2"'] \arrow[dl, bend left, "\epsilon_2"] & \cdots \arrow[l] \\
& & & \ker(\epsilon) \arrow[ul, bend left, "\iota_1"] & & \ker(\epsilon_1) \arrow[ul, bend left, "\iota_2"] & &
\end{tikzcd}$$

This gives the desired resolution, and uniqueness follows by Lemma 1.10.12. □

The dual of the last two results, for injective resolutions, is also valid. We will come back to this duality in a more general context in Chapter 4. The proofs of the following statements are analogous to the previous ones, and are left as an exercise.

**Lemma 1.10.14** *Given a map $f\colon A' \longrightarrow A$ of modules, an injective resolution $A \xrightarrow{\eta} I^\star$ of $A$ and a resolution $A' \xrightarrow{\eta'} B^\star$ of $A'$, there is a morphism $f^\star\colon B^\star \longrightarrow I^\star$ of exact sequences lifting $f$, in the sense that $f_0\eta' = \eta f$:*

$$\begin{tikzcd}
0 \arrow[r] & A' \arrow[r, "\eta'"] \arrow[d, "f"'] & B^0 \arrow[r, "d'^0"] \arrow[d, dotted, "f^0"'] & B^1 \arrow[r, "d'^1"] \arrow[d, dotted, "f^1"'] & \cdots \\
0 \arrow[r] & A \arrow[r, "\eta"'] & I^0 \arrow[r, "d^0"'] & I^1 \arrow[r, "d^1"'] & \cdots .
\end{tikzcd}$$

*Furthermore, the morphism $f^\star$ is unique up to homotopy.*

**Proposition 1.10.15** *Every module $A$ has an injective resolution, which is unique up to homotopy.*

### Exercises

*a)* Show the relation $\sim$ of Remark 1.10.3 to be in fact an equivalence relation.
*b)* Prove Proposition 1.10.4.
*c)* Prove Corollary 1.10.8.
*d)* Show by an example that a quasi-isomorphism between chain complexes need not be a homotopy equivalence.

## 1.11 Double Complexes

Just as we have exact sequences of chain complexes, we can formalise the concept of a chain complex of chain complexes. This is usually achieved by using two indexes. For this reason, we call these structures *double complexes*.

**Definition 1.11.1** A *double complex* of $R$-modules $C^{\star\star}$ is a family $\{C^{p,q}\}$ of modules together with maps:

$$d_h\colon C^{p,q}\longrightarrow C^{p,q+1} \text{ and } d_v\colon C^{p,q}\longrightarrow C^{p+1,q}$$

such that $d_h d_h = d_v d_v = d_v d_h + d_h d_v = 0$. Pictorially, $C^{\star\star}$ is a diagram

$$\begin{tikzcd}
 & \vdots \arrow[d] & \vdots \arrow[d] & \vdots \arrow[d] & \\
\cdots \arrow[r] & C^{p-1,q-1} \arrow[r, "d_h"] \arrow[d, "d_v"'] & C^{p-1,q} \arrow[r, "d_h"] \arrow[d, "d_v"'] & C^{p-1,q+1} \arrow[r] \arrow[d, "d_v"'] & \cdots \\
\cdots \arrow[r] & C^{p,q-1} \arrow[r, "d_h"] \arrow[d, "d_v"'] & C^{p,q} \arrow[r, "d_h"] \arrow[d, "d_v"'] & C^{p,q+1} \arrow[r] \arrow[d, "d_v"'] & \cdots \\
\cdots \arrow[r] & C^{p+1,q-1} \arrow[r, "d_h"] \arrow[d] & C^{p+1,q} \arrow[r, "d_h"] \arrow[d] & C^{p+1,q+1} \arrow[r] \arrow[d] & \cdots \\
 & \vdots & \vdots & \vdots &
\end{tikzcd}$$

in which each row $C^{\star,q}$ and each column $C^{p,\star}$ is a cochain complex and, furthermore, each square anti-commutes. Morphisms of double complexes are defined in an obvious way.

A double complex is called *bounded*, if there are only finitely many non-zero modules on each diagonal $p + q = n$. A special case of this situation is when both the rows and the columns are positive complexes, that is to say, $C^{p,q} = 0$ unless $p, q \geq 0$. We call $C^{\star\star}$ *positive* in this case.

**Definition 1.11.2** Given a bounded double complex $C^{\star\star}$, define the *total complex* $\mathrm{Tot}(C)^\star$ by:

$$\mathrm{Tot}(C)^n = \bigoplus_{p+q=n} C^{p,q}.$$

The formula $d = d_h + d_v$ defines a *total differential*

$$D\colon \mathrm{Tot}(C)^n \longrightarrow \mathrm{Tot}(C)^{n+1} \tag{1.14}$$

making $\mathrm{Tot}(C)^\star$ into a cochain complex. The action of $D$ on an element $x \in C^{p,q}$ maps it to $(d_h x, d_v x) \in C^{p,q+1} \oplus C^{p+1,q}$. Naively, we could think of the total complex as "summing up $C^{\star\star}$ along the diagonals $p + q = n$".

**Remark 1.11.3** In the definition of double complexes we could have just as well required commutativity of each square, instead of anti-commutativity. In other words, the equation $d_h d_v + d_v d_h = 0$ could be replaced by $d_v d_h - d_h d_v = 0$. We can always move from one convention to the other by replacing signs in the appropriate way. For instance, we can replace the maps $d_h\colon A^{p,q} \longrightarrow A^{p+1,q}$ by the maps

$$\delta_h = (-1)^p d_h\colon C^{p,q} \longrightarrow C^{p,q+1},$$

or analogously for the vertical maps $d_v$. The definition of the differential maps in the associated total complex will then have to be adjusted according to the convention.

**Remark 1.11.4** Notice that a positive double complex $C^{\star\star}$ can always be *augmented* by adding in front of each row and each column the kernel of the appropriate map. If we denote them by

$$H_h^0(C^{\star,0}) = \ker(d_h^{\star,0}) \quad \text{and} \quad H_v^0(C^{0,\star}) = \ker(d_h^{0,\star}),$$

then the complex takes the shape

$$\begin{tikzcd}
 & H_v^0(C^{0,0}) \arrow[r, "d_h"] \arrow[d] & H_v^0(C^{0,1}) \arrow[r] \arrow[d] & \cdots \\
H_h^0(C^{0,0}) \arrow[r] \arrow[d, "d_v"'] & C^{0,0} \arrow[r, "d_h"] \arrow[d, "d_v"'] & C^{0,1} \arrow[r] \arrow[d, "d_v"'] & \cdots \\
H_h^0(C^{1,0}) \arrow[r] \arrow[d] & C^{1,0} \arrow[r, "d_h"] \arrow[d] & C^{1,1} \arrow[r] \arrow[d] & \cdots \\
\vdots & \vdots & \vdots & .
\end{tikzcd} \tag{1.15}$$

The top row and the leftmost column of this augmented double complex are themselves cochain complexes, which we denote by $H_v^0(C^{\star\star})$ and $H_h^0(C^{\star\star})$, respectively. The cohomology of these complexes are denoted by $H_v^n(H_h^0(C^{\star\star}))$ and $H_h^n(H_v^0(C^{\star\star}))$.

The cohomology of these two cochain complexes is helpful in calculating the cohomology of the total complex associated to $C^{\star\star}$.

**Lemma 1.11.5 (Double Complex Lemma)** *If $C^{\star\star}$ is a positive double complex with exact columns and exact rows, then there are canonical isomorphisms, for all $n \geq 0$:*

$$H^n(\mathrm{Tot}(C)^\star) \simeq H_v^n(H_h^0(C^{\star\star})) \simeq H_h^n(H_v^0(C^{\star\star})).$$

**Proof.** We picture the double complex $C^{p,q}$ as in (1.15), with the $p$'s running vertically and the $q$'s horizontally. By symmetry, it is sufficient to show that

$$H^n(\mathrm{Tot}(C)^\star) \simeq H^n_v(H^0_h(C^{\star\star})).$$

There is a natural group homomorphism

$$\phi\colon H^n_v(H^0_h(C^{\star\star})) \longrightarrow H^n(\mathrm{Tot}(C)^\star)$$

taking the class $[a^{n,0}]$ to $[(a^{n,0}, 0, \ldots, 0)]$. To prove that $\phi$ is an isomorphism, we are going to produce its inverse $\psi$. To this purpose, it is enough to show that any equivalence class $x = [(a^0, \ldots, a^n)] \in H^n(\mathrm{Tot}(C)^\star)$ can be uniquely written as $[(b^0, 0, \ldots, 0)]$ for an opportune $b^0$, and then define $\psi(x) = [b^0]$. More generally, we prove that if $x = [(a^0, \cdots, a^k, 0, \cdots, 0)]$ for some $1 \leq k \leq n$, then $x = [(b^0, \cdots, b^{k-1}, 0, \cdots, 0)]$, for opportune $b^i$'s. The thesis will then follow by iterating the argument $n$-many times.

So, suppose $x = [(a^0, \cdots, a^k, 0, \cdots, 0)]$; then, post-composing $D$ with the projection $\pi\colon \mathrm{Tot}(C)^{n+1} \longrightarrow C^{n-k,k+1}$, we get

$$d_h(a^k) = d_h(a^k) + d_v(0) = \pi D(x) = 0,$$

and by exactness of the $(n-k)$-th row, there exists $e \in C^{n-k,k-1}$ such that $d_h(e) = a^k$. Now we have

$$\begin{aligned}
[(a^0, \ldots, a^k, 0, \ldots, 0)] - [(a^0, \ldots, a^{k-1} - d_v(e), 0, \ldots, 0)] \\
&= [(0, \ldots, 0, d_v(e), a^k, 0, \ldots, 0)] \\
&= [(0, \ldots, 0, d_v(e), d_h(e), 0, \ldots, 0)] \\
&= D[(0, \ldots, 0, e, 0, \ldots, 0)];
\end{aligned}$$

hence, $x$ is represented by the class $[(a^0, \ldots, a^{k-1} - d_v(e), 0, \ldots, 0)]$.

We leave it to the reader to check that these maps $\phi$ and $\psi$ are well-defined and inverse to each other.

Note that we only use the fact that the rows are exact. Similarly, in order to prove $H^n(\mathrm{Tot}(C)^\star) \simeq H^n_h(H^0_v(C^{\star\star}))$, it is suffices to have the exactness of the columns. $\square$

To close this section, we give an application of double complexes to the study of $R$-modules.

Let $A$, $B$ be $R$-modules. Let $A \xleftarrow{\epsilon} P_\star$ be a projective resolution of $A$ and $B \xrightarrow{\eta} I^\star$ an injective resolution of $B$. We can then form the double complex $\mathrm{Hom}_R(P_\star, I^\star)$. The morphisms $\epsilon$ and $\eta$ induce maps from $\mathrm{Hom}_R(P_\star, I^\star)$ to

$\mathrm{Hom}_R(A, I^\star)$ and $\mathrm{Hom}_R(P_\star, B)$:

$$\begin{tikzcd}
 & \mathrm{Hom}_R(A,I^0) \arrow[r] \arrow[d, "\epsilon^*"'] & \mathrm{Hom}_R(A,I^1) \arrow[r] \arrow[d, "\epsilon^*"'] & \cdots \\
\mathrm{Hom}_R(P_0,B) \arrow[r, "\eta_*"] \arrow[d] & \mathrm{Hom}_R(P_0,I^0) \arrow[r] \arrow[d] & \mathrm{Hom}_R(P_0,I^1) \arrow[r] \arrow[d] & \cdots \\
\mathrm{Hom}_R(P_1,B) \arrow[r, "\eta_*"] \arrow[d] & \mathrm{Hom}_R(P_1,I^0) \arrow[r] \arrow[d] & \mathrm{Hom}_R(P_1,I^1) \arrow[r] \arrow[d] & \cdots \\
\vdots & \vdots & \vdots & .
\end{tikzcd}$$

By projectivity of the $P_n$'s and injectivity of the $I^n$'s, we have that all the rows and columns of the augmented double complex above are exact. Therefore, by the double complex lemma, we have that

$$H^n(\mathrm{Hom}_R(A, I^\star)) \simeq H^n(\mathrm{Hom}_R(P_\star, B)).$$

**Definition 1.11.6** The cohomology groups we have just constructed are called the *Ext groups*, and are denoted by $\mathrm{Ext}^n(A, B)$. More precisely, for each $A$, $B$ in $R$-$\mathsf{mod}$ we define:

$$\mathrm{Ext}^n(A, B) = H^n(\mathrm{Hom}_R(A, I^\star)) \simeq H^n(\mathrm{Hom}_R(P_\star, B)).$$

Analogously, given projective resolutions $A \xleftarrow{\epsilon} P_\star$ and $B \xleftarrow{\delta} Q_\star$ of two $R$-modules $A$ and $B$, we define the *Tor groups* $\mathrm{Tor}_n(A, B)$ as the homology groups

$$\mathrm{Tor}_n(A, B) = H_n(A \otimes_R Q_\star) \simeq H_n(P_\star \otimes_R B).$$

**Proposition 1.11.7** *Let $B$ be an $R$-module.*

*(i) For every $R$-module $A$ we have* $\mathrm{Tor}_0(A, B) \simeq A \otimes_R B$.

*(ii) Given a short exact sequence $0 \to A' \to A \to A'' \to 0$ of $R$-modules, there exists a natural long exact '*$\mathrm{Tor}$*-sequence'*

$$\cdots \to \mathrm{Tor}_1(A'', B) \to A' \otimes_R B \to A \otimes_R B \to A'' \otimes_R B \to 0.$$

**Proof.** Let $B \xleftarrow{\delta} Q_\star$ be a projective resolution of $B$. Then, for every $R$-module $A$, the sequence

$$A \otimes_R Q_1 \xrightarrow{\mathsf{id}\otimes d_1} A \otimes_R Q_0 \xrightarrow{\mathsf{id}\otimes \delta} A \otimes_R B \to 0$$

is exact since $A \otimes -$ is right-exact. Hence $\mathrm{coker}\, \mathsf{id} \otimes d_1 \simeq A \otimes_R B$. This proves item (i) because $\mathrm{Tor}_0(A, B) = \mathrm{coker}\, \mathsf{id} \otimes d_1$.

To prove item (ii) suppose we are given a short exact sequence $0 \to A' \to A \to A'' \to 0$ of $R$-modules. Let $B \xleftarrow{\delta} Q_\star$ be a projective resolution of $B$ and consider the following short exact sequence of chain complexes.

$$\begin{tikzcd}
 & \vdots \arrow[d] & \vdots \arrow[d] & \vdots \arrow[d] & \\
0 \arrow[r] & A' \otimes Q_1 \arrow[r] \arrow[d] & A \otimes Q_1 \arrow[r] \arrow[d] & A'' \otimes Q_1 \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & A' \otimes Q_0 \arrow[r] \arrow[d] & A \otimes Q_0 \arrow[r] \arrow[d] & A'' \otimes Q_0 \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 &
\end{tikzcd}$$

Applying Proposition 1.9.8 on this short exact sequence then gives the desired Tor-sequence. □

### Exercise

*a)* Show, using the double complex lemma, that the Tor groups are well defined.

*b)* Working over the ring $R = \mathbb{Z}$, prove that

$$\mathrm{Tor}_1(A, \mathbb{Z}/n) = \{a \in A | n \cdot a = 0\}.$$

Use the classification of finitely generated abelian groups to conclude that $A$ is torsion free if and only if $\mathrm{Tor}_1(A, B) = 0$ for every finitely generated abelian group $B$.

## 1.12 The Künneth formula

Let $C$ and $C'$ be chain complexes of right and left $R$-modules respectively. The total complex of the double complex $C_{p,q} = C_p \otimes_R C'_q$ with boundary maps

$$d \otimes \mathsf{id}\colon C_{p,q} \longrightarrow C_{p-1,q} \qquad \text{and} \qquad (-1)^p \mathsf{id} \otimes d\colon C_{p,q} \longrightarrow C_{p,q-1}$$

is called the *tensor product* of $C$ and $C'$ and is denoted by $C \otimes_R C'$. Remember its boundary formula is given by

$$D(c \otimes c') = dc \otimes c' + (-1)^p c \otimes dc' \qquad \text{for } c \in C_p \text{ and } c' \in C'_q. \tag{1.16}$$

**Remark 1.12.1** (i) Let $A$ be a right $R$-module, $B$ an $(R, S)$-bimodule and $C$ a left $S$-module. There is an isomorphism

$$(A \otimes_R B) \otimes_S C \to A \otimes_R (B \otimes_S C)$$

determined by

$$(a \otimes b) \otimes c \mapsto a \otimes (b \otimes c).$$

(ii) The above generalizes as follows. Let $C'$ and $C''$ be chain complexes of right $R$-modules and left $S$-modules respectively, and let $C$ be a chain complex of $(R, S)$-bimodules. Then there is an isomorphism of chain complexes

$$(C' \otimes_R C) \otimes_S C'' \simeq C' \otimes_R (C \otimes_S C'').$$

(iii) Let $A$ and $B$ be $R$-bimodules. There is an isomorphism

$$A \otimes B \to B \otimes A$$

determined by $a \otimes b \mapsto b \otimes a$. Given two chain complexes of $R$-bimodules $C$ and $C'$, this generalizes as follows: there is an isomorphism $C \otimes C' \simeq C' \otimes C$, determined by

$$c \otimes c' \mapsto (-1)^{pq} c' \otimes c, \qquad \text{where } c \in C_p \text{ and } c' \in C'_q.$$

In the remainder of this section we will, when no confusion can arise, suppress reference to the ring $R$ in the notations for tensor products of $R$-modules and tensor products of chain complexes.

Let $C$ and $C'$ be chain complexes of right and left $R$-modules respectively. From the boundary formula (1.16), it follows that the tensor product $c \otimes c'$ of two cycles is a cycle in $C \otimes C'$ and that the tensor product of a cycle and a boundary is a boundary. We thus have a well-defined homomorphism

$$\bigoplus_{p+q=n} H_p(C) \otimes H_q(C') \to H_n(C \otimes C') : \ [c] \otimes [c'] \mapsto [c \otimes c'],$$

which is called the *homology product*.

**Theorem 1.12.2 (The Künneth formula)** *Let $R$ be a principal ideal domain and let $C$ and $C'$ be chain complexes of right and left $R$-modules respectively. If the $R$-modules $C_i$ are all free then, for each $n$, there is a natural short exact 'Künneth sequence'*

$$0 \to \bigoplus_{p+q=n} H_p(C) \otimes H_q(C') \to H_n(C \otimes C') \to \bigoplus_{p+q=n-1} \mathrm{Tor}_1(H_p(C), H_q(C')) \to 0.$$

**Definition 1.12.3** Generalizing the direct sum of modules we define the *direct sum* $\bigoplus_{i\in I} C_i$ of a family of chain complexes of left (resp. right) $R$-modules $(C_i)_{i\in I}$ by

$$(\bigoplus_{i\in I} C_i)_n = \bigoplus_{i\in I} (C_i)_n \qquad \text{and} \qquad d(c_i) = (dc_i).$$

Clearly $\bigoplus_{i\in I} C_i$ is a chain complex of left (resp. right) $R$-modules and it can easily be shown there exists a canonical isomorphism

$$H_n(\bigoplus_{i\in I} C_i) \simeq \bigoplus_{i\in I} H_n(C_i)$$

for each $n$. Moreover, if all of the $C_i$ are chain complexes of right $R$-modules and $C'$ is a chain complex of left $R$-modules then, using Proposition 1.3.5, one can prove there is a canonical isomorphism of chain complexes

$$(\bigoplus_{i\in I} C_i) \otimes C' \simeq \bigoplus_{i\in I} (C_i \otimes C').$$

*Proof of Theorem 1.12.2.* We shall use the fact that every submodule of a free $R$-module is free if $R$ is a principal ideal domain, see [3, Theorem III.7.1] for example.

First assume $C$ has trivial boundary morphisms. Then $H_p(C) = C_p$ is free for all $p$, hence the $\mathrm{Tor}_1$-term vanishes from the Künneth sequence and we have to show that the homology product is an isomorphism. Since the boundary maps of $C$ are trivial the boundary formula of $C \otimes C'$ becomes

$$D(c \otimes c') = (-1)^p c \otimes dc' \qquad \text{for } c \in C_p \text{ and } c' \in C'_q,$$

and one can easily show there is a canonical isomorphism of chain complexes

$$C \otimes C' \simeq \bigoplus_{p \in \mathbb{Z}} C_p \otimes C'[p],$$

where $C_p$ denotes the constant chain complex $C_p$ with trivial boundary maps, and $C'[p]$ is given by $C'[p]_n = C'_{n-p}$, with boundary maps $(-1)^p d$. Since $C_p$ is free it can be written as a disjoint sum of a family of $R$-modules $R_i$ indexed by a set $I$, all of which are isomorphic to $R$. Hence

$$H_n(C_p \otimes C'[p]) \simeq \bigoplus_i H_{n-p}(C') \simeq C_p \otimes H_{n-p}(C') = H_p(C) \otimes H_{n-p}(C'),$$

where we have used that $R_i \otimes C'_q \simeq C'_q$ and $H_n(C'[p]) = H_{n-p}(C')$. Summing over $p$ thus gives an isomorphism

$$H_n(C \otimes C') \simeq \bigoplus_{p+q=n} (H_p(C) \otimes H_q(C')),$$

which one checks to be given by the homology product.

In the general case, denote the kernels and the images of the boundary homomorphisms of $C$ by $Z_p \subset C_p$ and $B_p \subset C_p$ respectively. They form chain complexes $Z$ and $B$ with trivial boundary maps and we have a short exact sequence $0 \to Z_p \to C_p \xrightarrow{d} B_{p-1} \to 0$ in each degree $p$. Because $B_{p-1}$ is free, we have $\mathrm{Tor}_1(B_{p-1}, C'_q) = 0$ in the associated long exact Tor-sequence (cf. Proposition 1.11.7) and hence

$$0 \to Z_p \otimes C'_q \to C_p \otimes C'_q \to B_{p-1} \otimes C'_q \to 0$$

is exact for each pair $p$ and $q$. Summing over $p + q = n$, these assemble to give short exact sequences

$$0 \to (Z \otimes C')_n \to (C \otimes C')_n \to (B \otimes C')_{n-1} \to 0.$$

Since $Z$ and $B$ are chain complexes with trivial boundary maps we have by the special case above

$$Z \otimes C' \simeq \bigoplus_{p \in \mathbb{Z}} Z_p \otimes C'[p] \qquad \text{and} \qquad B \otimes C' \simeq \bigoplus_{p \in \mathbb{Z}} B_p \otimes C'[p].$$

For each degree $n$, these isomorphisms convert the exact sequence above into an exact sequence that commutes with the respective boundary maps, so they form an exact sequence of chain complexes

$$0 \to \bigoplus_{p \in \mathbb{Z}} Z_p \otimes C'[p] \to C \otimes C' \to \bigoplus_{p \in \mathbb{Z}} B_{p-1} \otimes C'[p] \to 0.$$

Hence we have a long exact sequence in homology (cf. Proposition 1.9.8)

$$\cdots \to \bigoplus_{p+q=n} B_p \otimes H_q(C') \xrightarrow{\delta_n} \bigoplus_{p+q=n} Z_p \otimes H_q(C') \to H_n(C \otimes C')$$
$$\to \bigoplus_{p+q=n-1} B_p \otimes H_q(C') \xrightarrow{\delta_{n-1}} \bigoplus_{p+q=n-1} Z_p \otimes H_q(C') \to \cdots .$$

Notice that the connecting morphism $\delta_n$ is just the homomorphism given by the homomorphisms $B_p \otimes H_q(C') \to Z_p \otimes H_q(C')$ coming from the inclusion $B_p \subset Z_p$. Indeed, let $b \otimes [c']$ belong to $B_p \otimes H_q(C')$. Pulling the cycle $b \otimes c'$ in $B_p \otimes C'_q$ back under $d \otimes \mathsf{id}$ gives $c \otimes c'$ in $C_{p+1} \otimes C'_q$, where $d(c) = b$, and $D(c \otimes c') = b \otimes c'$ in $C_p \otimes C'_q$ since $c'$ is a cycle. Pulling the latter back under $Z_p \otimes C'_q \to C_p \otimes C'_q$, which is induced by the inclusion $Z_p \subset C_p$, gives $b \otimes c'$, now seen as a tensor product in $Z_p \otimes C'_q$. The image of $b \otimes [c']$ in $B_p \otimes H_q(C')$ under $\delta_n$ is thus $b \otimes [c']$ in $Z_p \otimes H_q(C')$.

Now the exactness in $H_n(C \otimes C')$ of the long exact sequence above is equivalent to the exactness of the short exact sequence

$$0 \to \operatorname{coker} \delta_n \to H_n(C \otimes C') \to \ker \delta_{n-1} \to 0,$$

which we will show to be the sequence of the theorem.

For each $p$ we have an exact sequence $0 \to B_p \to Z_p \to H_p(C) \to 0$, and tensoring with $H_q(C')$ gives the exact sequence

$$B_p \otimes H_q(C') \to Z_p \otimes H_q(C') \to H_p(C) \otimes H_q(C') \to 0.$$

Summing over $p + q = n$ these assemble to give the exact sequence

$$\bigoplus_{p+q=n} B_p \otimes H_q(C') \xrightarrow{\delta_n} \bigoplus_{p+q=n} Z_p \otimes H_q(C') \to \bigoplus_{p+q=n} H_p(C) \otimes H_q(C') \to 0,$$

hence $\operatorname{coker} \delta_n = \bigoplus_{p+q=n} H_p(C) \otimes H_q(C')$.

To determine $\ker \delta_{n-1}$, again consider the exact sequence $0 \to B_p \to Z_p \to H_p(C) \to 0$. Since $Z_p$ is free we have $\operatorname{Tor}_1(Z_p, H_q(C)) = 0$ in the associated long exact Tor-sequence and hence the sequence below is exact.

$$0 \to \operatorname{Tor}_1(H_p(C), H_q(C)) \to B_p \otimes H_q(C) \to Z_p \otimes H_q(C) \to H_p(C) \otimes H_q(C) \to 0$$

Summing over $p + q = n - 1$ we thus find

$$\ker \delta_{n-1} = \bigoplus_{p+q=n-1} \operatorname{Tor}_1(H_p(C), H_q(C)),$$

which completes the proof of the theorem. The reader can check that every step in the argument above is natural, so that the Künneth sequence itself is natural.

□

Since every module (vector space) over a field is free, we obtain

**Corollary 1.12.4** *Let $K$ be a field and $C$ and $C'$ be chain complexes left and right $K$-modules respectively. Then the homology product*

$$\bigoplus_{p+q=n} H_p(C) \otimes H_q(C') \to H_n(C \otimes C')$$

*is an isomorphism.*

**Definition 1.12.5** Let $C$ be a chain complex of left $R$-modules. If $M$ is any right $R$-module, the homology of $C$ *with coefficients in* $M$ is given by

$$H_*(C; M) = H_*(C \otimes M).$$

**Theorem 1.12.6 (Universal coefficient theorem for homology)** *Let $C$ be a chain complex of free left modules over $R$ and let $M$ be a right $R$-module. Then there is a natural short exact sequence*

$$0 \to H_n(C) \otimes M \to H_n(C; M) \to \mathrm{Tor}_1(H_{n-1}(C), M) \to 0.$$

*In particular, if $R$ is a field then $H_n(C; M) \simeq H_n(C) \otimes M$.*

**Proof.** Apply Theorem 1.12.2 on $C$ and $C'$, where $C'_0 = M$ and $C'_n = 0$ for $n \neq 0$. □

**Proposition 1.12.7** *The Künneth sequence of Theorem 1.12.2 splits, yielding a direct sum composition*

$$H_n(C \otimes C') \simeq \left( \bigoplus_{p+q=n} H_p(C) \otimes H_q(C') \right) \oplus \left( \bigoplus_{p+q=n-1} \mathrm{Tor}_1(H_p(C), H_q(C')) \right).$$

**Proof.** We will show that the Künneth sequence splits if both $C$ and $C'$ are free. This suffices for our applications. It is not hard to generalize the argument below to show splitting when $C'$ is not free, see [2, Section V.2].

The exact sequence $0 \to Z_p \to C_p \to B_{p-1} \to 0$ splits since $B_{p-1}$ is a free. Hence $C_p \simeq Z_p \oplus B_{p-1}$ and we can thus extend the quotient map $Z_p \to H_p(C)$ to a homomorphism $C_p \to H_p(C)$. Viewing $H_*(C)$ as a chain complex $H(C)$ with trivial boundary maps, these homomorphisms form a morphism of chain complexes $C \to H(C)$. Similarly we have a morphism of chain complexes $C' \to H(C')$ and tensoring these morphisms gives a morphism

$$C \otimes C' \to H(C) \otimes H(C').$$

Now the chain complex $H(C) \otimes H(C')$ equals its own homology because its boundary maps are trivial and the induced morphism on homology for the morphism of chain complexes above is the splitting map we sought. □

**Remark 1.12.8** The splitting of the Künneth sequence is not natural, as is demonstrated by the following example.

If the splitting were natural then we would have, for any two morphisms of chain complexes $\phi: C \longrightarrow D$ and $\psi: C' \longrightarrow D'$, that $(\phi \otimes \psi)_* = 0$ if $\phi_* \otimes \psi_* = 0$ and $\mathrm{Tor}_1(\phi_*, \psi_*) = 0$. To give a counter-example to this implication, consider the following situation. Take $R = \mathbb{Z}$, $C_0 = \mathbb{Z} = C_1$, $C_n = 0$ for $n \neq 0, 1$ and $d_1(1) = 2$; $D_1 = \mathbb{Z}$, $D_n = 0$ for $n \neq 1$ and $\phi_1 = \mathsf{id}$; $C'_0 = \mathbb{Z}/2\mathbb{Z}$, $C'_n = 0$ for $n \neq 0$, $D' = C'$ and $\psi = \mathsf{id}$. Clearly $\phi_* = 0$, hence, assuming that the splitting is natural, we would have $(\phi \otimes \psi)_* = 0$. However, $(C \otimes C')_2 = 0 = (D \otimes D')_2$ and hence $H_1(C \otimes C') = C_1 \otimes C'_0 = \mathbb{Z} \otimes \mathbb{Z}/2\mathbb{Z}$ and $H_1(C \otimes C') = D_1 \otimes D'_0 = \mathbb{Z} \otimes \mathbb{Z}/2\mathbb{Z}$, so that $(\phi \otimes \psi)_*: H_1(C \otimes C') \longrightarrow H_1(D \otimes D')$ is an isomorphism.

## 1.13 Additional exercises

1. Let $R$ be a ring with unit, $A$ a right $R$-module and $B$ a left $R$-module. Prove that $R \otimes_R B \cong B$ and $A \otimes_R R \cong A$.

2. Compute $\mathbb{Z}/n \otimes_{\mathbb{Z}} \mathbb{Q}$ and $\mathbb{Q} \otimes_{\mathbb{Z}} \mathbb{Q}$.

3. Show that $\mathbb{Z}/n \otimes_{\mathbb{Z}} \mathbb{Z}/m \cong \mathbb{Z}/d$, where $d = \gcd(n, m)$.

4. Give an example of $R$-modules $A$ and $\{B_i\}_{i \in I}$ such that

$$A \otimes_R \left( \prod_{i \in I} B_i \right) \ncong \prod_{i \in I} (A \otimes_R B_i) \, .$$

# 2. Cohomology of groups

In this Chapter, we use the machinery of cohomology introduced in Chapter 1 in order to study properties of groups. To fix notation, our default group $G$ (in multiplicative notation) will have elements $g$, $h$, and so on; its unit will be denoted by 1.

## 2.1 Cohomology of a group

In order to introduce the concept of cohomology for an arbitrary group, we first need to generate a cochain complex. This will arise from the following definitions.

**Definition 2.1.1** Given a group $G$, we denote by $\mathbb{Z}[G]$ the free abelian group on the underlying set of $G$. Multiplication of $G$ clearly extends by distributivity to a multiplication on $\mathbb{Z}[G]$, making it into a ring. We call this the *group ring* on $G$.

There is an obvious ring epimorphism $\epsilon\colon \mathbb{Z}[G] \longrightarrow \mathbb{Z}$, called the *augmentation* of $G$, determined by mapping every $g \in G$ to 1. As we saw in Section 1.5, this induces a functor $\epsilon^*\colon \mathbb{Z}\text{-}\mathsf{mod} = \mathsf{Ab} \longrightarrow \mathbb{Z}[G]\text{-}\mathsf{mod}$.

**Definition 2.1.2** A *$G$-module* $A$ is an abelian group $A$ (in additive notation), equipped with a left action by $G$ for which the distributive law holds, i.e. a map

$$G \times A \longrightarrow A \quad (g, a) \mapsto g \cdot a$$

such that, for all $a, b \in A$ and $g, h \in G$,

*a)* $g \cdot (a + b) = g \cdot a + g \cdot b$;
*b)* $1 \cdot a = a$;
*c)* $g \cdot (h \cdot a) = (gh) \cdot a$.

A *morphism of $G$-modules* $f\colon A \longrightarrow B$ is a morphism of abelian groups which respects the action of $G$; namely, for all $g \in G$ and $a, a_1, a_2 \in A$:

*a)* $f(a_1 + a_2) = f(a_1) + f(a_2)$;
*b)* $f(g \cdot a) = g \cdot f(a)$.

$G$-modules and $G$-module morphisms form a category, denoted by $G$-$\mathsf{mod}$.

**Remark 2.1.3** Alternatively, we could define an action of a group $G$ on an abelian group $A$ as a group homomorphism

$$G \longrightarrow \mathrm{Aut}(A),$$

where $\mathrm{Aut}(A)$ is the group of automorphisms of $A$, with composition as multiplication and the identity on $A$ as unit.

**Example 2.1.4** Every abelian group $A$ can be viewed as a *trivial $G$-module*, with the action $g \cdot a = a$, for all $g \in G$ and $a \in A$. This determines a trivial $G$-module functor from $\mathsf{Ab}$ to $G$-$\mathsf{mod}$, which is the functor $\epsilon^*$ of Definition 2.1.1.

**Remark 2.1.5** The name $G$-module for such structures is not inappropriate. In fact, to give a $G$-module structure on an abelian group $A$ is the same as giving a $\mathbb{Z}[G]$-module structure on it. More precisely, there is an isomorphism of categories $G$-$\mathsf{mod} \simeq \mathbb{Z}[G]$-$\mathsf{mod}$.

Appealing to this isomorphism, we shall henceforth make no distinction between a $G$-module and a $\mathbb{Z}[G]$-module structure on an abelian group $A$.

**Definition 2.1.6** For a $G$-module $A$, the subgroup

$$A^G = \{a \in A \mid ga = a \text{ for all } g \in G\} \tag{2.1}$$

is called the *subgroup of invariants* of $A$. The quotient group

$$A_G = A/\langle ga - a \mid g \in G, a \in A\rangle$$

is the *group of coinvariants* of $A$. The assignments $A \mapsto A^G$ and $A \mapsto A_G$ define functors $(-)^G$ and $(-)_G$ from $G$-$\mathsf{mod}$ to $\mathsf{Ab}$.

You should check that these functors have the property stated in the following proposition (cf. Exercise *b*) below):

**Proposition 2.1.7** *For a group $G$ with augmentation $\epsilon \colon \mathbb{Z}[G] \longrightarrow \mathbb{Z}$, there is a chain of adjunctions*

$$(-)_G \dashv \epsilon^* \dashv (-)^G.$$

By uniqueness of adjoints (see Exercise *c*) in the Appendix) and the results of Section 1.5, we obtain:

**Corollary 2.1.8** *For any group $G$, there are natural isomorphisms*

$$A_G \simeq \mathbb{Z} \otimes_{\mathbb{Z}[G]} A \quad \textit{and} \quad A^G \simeq \mathrm{Hom}_{\mathbb{Z}[G]}(\mathbb{Z}, A).$$

Now, we have all the tools needed to define the homology and cohomology of groups. If $A$ is a $G$-module, we can consider it as a $\mathbb{Z}[G]$-module, by Remark 2.1.5. Then, by Proposition 1.10.15, we can take an injective resolution of $A$

$$0 \longrightarrow A \longrightarrow I^0 \longrightarrow I^1 \longrightarrow I^2 \longrightarrow \cdots.$$

Applying now the functor $(-)^G \colon G\text{-}\mathsf{mod} \longrightarrow \mathsf{Ab}$ to this resolution, we obtain the cochain complex

$$(I^0)^G \longrightarrow (I^1)^G \longrightarrow (I^2)^G \longrightarrow \cdots . \tag{2.2}$$

**Definition 2.1.9** We define the *cohomology groups of $G$ with coefficients in $A$* as the cohomology of the cochain complex (2.2), denoted by $H^*(G; A)$.

**Remark 2.1.10** Note that the notion is well-defined, because, by Proposition 1.10.15, the injective resolution of $A$ is unique up to homotopy, and we know that homotopic maps induce the same morphisms in homology.

**Remark 2.1.11** We know from Corollary 2.1.8 that $(-)^G$ is naturally isomorphic to $\mathrm{Hom}_{\mathbb{Z}[G]}(\mathbb{Z}, -)$, so it follows that

$$H^*(G; A) \simeq H^*(\mathrm{Hom}_{\mathbb{Z}[G]}(\mathbb{Z}, I^\star)) = \mathrm{Ext}^*_{\mathbb{Z}[G]}(\mathbb{Z}, A).$$

From the double complex lemma 1.11.5, we know that we can also compute $\mathrm{Ext}^*_{\mathbb{Z}[G]}(\mathbb{Z}, A)$ by considering a projective resolution of $\mathbb{Z}$ in the category of (left) $\mathbb{Z}[G]$-modules

$$0 \longleftarrow \mathbb{Z} \longleftarrow P_0 \longleftarrow P_1 \longleftarrow \cdots$$

and taking the cohomology of the cochain complex $\mathrm{Hom}_{\mathbb{Z}[G]}(P_\star, A)$:

$$\mathrm{Hom}_{\mathbb{Z}[G]}(P_0, A) \longrightarrow \mathrm{Hom}_{\mathbb{Z}[G]}(P_1, A) \longrightarrow \mathrm{Hom}_{\mathbb{Z}[G]}(P_2, A) \longrightarrow \cdots .$$

It follows, that $H^*(G; A) \simeq \mathrm{Ext}^*_{\mathbb{Z}[G]}(\mathbb{Z}, A) \simeq H^*(\mathrm{Hom}_{\mathbb{Z}[G]}(P_\star, A))$, and these isomorphisms are natural in $A$.

So far, we considered only cohomology of groups. Using the covariant functor $(-)_G\colon G\text{-}\mathsf{mod} \longrightarrow \mathsf{Ab}$, we can define the homology groups of $G$ with coefficients in $A$ in a dual way.

First, we consider a projective resolution of $A$ as a $\mathbb{Z}[G]$-module

$$0 \longleftarrow A \longleftarrow Q_0 \longleftarrow Q_1 \longleftarrow \cdots ;$$

then, we apply the functor $(-)_G$ to this complex to get

$$(Q_0)_G \longleftarrow (Q_1)_G \longleftarrow \cdots \tag{2.3}$$

**Definition 2.1.12** The *homology groups of $G$ with coefficients in $A$*, denoted $H_*(G; A)$, are defined to be the homology of the chain complex (2.3); hence: $H_*(G; A) = H_*((Q_\star)_G)$.

**Remark 2.1.13** As in the case of cohomology, this definition is correct, i.e. it does not depend on the choice of the projective resolution $Q_\star$. In fact, for any other resolution ${Q'}_\star$, the groups $H_n((Q_\star)_G)$ and $H_n(({Q'}_\star)_G)$ are naturally isomorphic.

Moreover, for any projective resolution $0 \longleftarrow \mathbb{Z} \longleftarrow P_0 \longleftarrow P_1 \longleftarrow \cdots$ of $\mathbb{Z}$ as a $\mathbb{Z}[G]$-module, we can compute these homology groups from the complex

$$P_0 \otimes_{\mathbb{Z}[G]} A \longleftarrow P_1 \otimes_{\mathbb{Z}[G]} A \longleftarrow \cdots$$

Again by the double complex lemma, it follows that $H_*(G; A) \simeq \mathrm{Tor}_*(\mathbb{Z}, A)$.

In these lectures, we shall focus on the cohomology of groups and leave homology aside.

### Exercises

*a)* Let $U\colon \mathsf{Rng} \longrightarrow \mathsf{Grp}$ be the functor taking a ring $R$ to the group $U(R)$ of invertible elements in $R$ (what is its action on maps?). Show that there is an adjunction $\mathbb{Z}[-] \dashv U$.

*b)* Complete the definition of the functors $(-)^G$ and $(-)_G$ and show that there is a chain of adjunctions $(-)_G \dashv \epsilon^* \dashv (-)^G$, where $\epsilon^*$ is the trivial $G$-module functor of Example 2.1.4. Using uniqueness of adjoints (see exercise *c*) in the Appendix), explain how Corollary 2.1.8 follows.

## 2.2 Functoriality of cohomology

In this Section, we show that the cohomology groups $H^*(G; A)$ are functorial in both $A$ and $G$.

### 2.2.1 Functoriality in $A$

Suppose the group $G$ is fixed. Then, for every $n \in \mathbb{N}$, $H^n(G; -)\colon G\text{-}\mathsf{mod} \longrightarrow \mathsf{Ab}$ is a functor which takes a $G$-module $A$ to the $n$-th cohomology group of $G$ with coefficients in $A$, and a $G$-module morphism $\varphi\colon A \longrightarrow B$ to the map

$$\varphi_* = H^n(G; \varphi)\colon H^n(G; A) \longrightarrow H^n(G; B) \tag{2.4}$$

induced by $\mathrm{Hom}_{\mathbb{Z}[G]}(P_n, A) \longrightarrow \mathrm{Hom}_{\mathbb{Z}[G]}(P_n, B)$.

In detail, let $0 \longleftarrow \mathbb{Z} \longleftarrow P_\star$ be a projective resolution of the trivial left $\mathbb{Z}[G]$-module $\mathbb{Z}$. By applying $\mathrm{Hom}_{\mathbb{Z}[G]}(-, A)$ and $\mathrm{Hom}_{\mathbb{Z}[G]}(-, B)$ to this sequence, and merging the two with the morphism $\varphi_*$ of Remark 1.2.3, we get a morphism of cochain complexes

$$\begin{tikzcd}
\mathrm{Hom}_{\mathbb{Z}[G]}(P_0, A) \arrow[r] \arrow[d, "\varphi_*"'] & \mathrm{Hom}_{\mathbb{Z}[G]}(P_1, A) \arrow[r] \arrow[d, "\varphi_*"] & \cdots \\
\mathrm{Hom}_{\mathbb{Z}[G]}(P_0, B) \arrow[r] & \mathrm{Hom}_{\mathbb{Z}[G]}(P_1, B) \arrow[r] & \cdots .
\end{tikzcd}$$

The desired map $\varphi_*$ is now defined by functoriality of $H^n$ for these chain complexes and this chain map.

Note that $\varphi_*$ does not depend on the projective resolution of $\mathbb{Z}$, for if there is an other projective resolution $0 \longleftarrow \mathbb{Z} \longleftarrow Q^\star$, then we have natural isomorphisms

$$\begin{tikzcd}
H^n(\mathrm{Hom}_{\mathbb{Z}[G]}(P_n, A)) \arrow[r, "\varphi_{*P}"] \arrow[d, "\simeq"'] & H^n(\mathrm{Hom}_{\mathbb{Z}[G]}(P_n, B)) \arrow[d, "\simeq"] \\
H^n(\mathrm{Hom}_{\mathbb{Z}[G]}(Q_n, A)) \arrow[r, "\varphi_{*Q}"'] & H^n(\mathrm{Hom}_{\mathbb{Z}[G]}(Q_n, B)).
\end{tikzcd}$$

**Remark 2.2.1** We could have computed the map $\varphi_*$ using injective resolutions of $A$ and $B$ and lifting $\varphi$ to a chain map. However, using a projective resolution of $\mathbb{Z}$ reveals an important property of the cohomology of groups. Namely, if there is a short exact sequence of $G$-modules

$$0 \longrightarrow A \xrightarrow{\varphi} B \xrightarrow{\psi} C \longrightarrow 0,$$

then we can construct the short exact sequence of cochain complexes

$$0 \longrightarrow \mathrm{Hom}_{\mathbb{Z}[G]}(P_\star, A) \xrightarrow{\varphi_*} \mathrm{Hom}_{\mathbb{Z}[G]}(P_\star, B) \xrightarrow{\psi_*} \mathrm{Hom}_{\mathbb{Z}[G]}(P_\star, C) \longrightarrow 0$$

because each $P_i$ is projective; so, by Proposition 1.9.8, there is a long exact sequence in cohomology

$$\cdots \xrightarrow{\delta} H^n(G;A) \xrightarrow{\varphi_*} H^n(G;B) \xrightarrow{\psi_*} H^n(G;C) \xrightarrow{\delta} H^{n+1}(G;A) \longrightarrow \cdots .$$

### 2.2.2 Functoriality in $G$

Now, we fix a $G$-module $A$, and consider functoriality of cohomology groups in the first variable. Suppose we have a group homomorphism $\rho\colon G' \longrightarrow G$. Then, $A$ can be viewed as a $G'$-module *via* $\rho$. More precisely, the action of $G'$ on $A$ is defined by $g'a = \rho(g')a$ for all $g' \in G'$ and $a \in A$. This determines a functor $\rho^*\colon G\text{-}\mathsf{mod} \longrightarrow G'\text{-}\mathsf{mod}$, just as in Section 1.5.

We want to construct maps $\rho^{\#}\colon H^n(G;A) \longrightarrow H^n(G';\rho^*(A))$, for $n \in \mathbb{N}$. To this end, consider an injective resolution of the $G$-module $A$

$$0 \longrightarrow A \longrightarrow I^0 \longrightarrow I^1 \longrightarrow \cdots$$

Because $\rho^*$ is an exact functor, this resolution yields a (not necessarily injective) resolution of the $G'$-module $\rho^*(A)$:

$$0 \longrightarrow \rho^*(A) \longrightarrow \rho^*(I^0) \longrightarrow \rho^*(I^1) \longrightarrow \cdots .$$

Now, we can choose an injective resolution of the $G'$-module $\rho^*(A)$ and lift $\mathsf{id}_{\rho^*(A)}$ to a chain map $\widetilde{\rho}$:

$$\begin{tikzcd}
0 \arrow[r] & \rho^*(A) \arrow[r] \arrow[d, "\mathsf{id}_{\rho^*(A)}"'] & \rho^*(I^0) \arrow[r] \arrow[d, "\widetilde{\rho}"'] & \rho^*(I^1) \arrow[r] \arrow[d, "\widetilde{\rho}"'] & \cdots \\
0 \arrow[r] & \rho^*(A) \arrow[r] & J^0 \arrow[r] & J^1 \arrow[r] & \cdots .
\end{tikzcd} \tag{2.5}$$

By applying the functor $(-)^{G'}$ and taking cohomology, this induces, in turn, a natural map $\rho^\dagger\colon H^n(\rho^*(I^\star)^{G'}) \longrightarrow H^n(G';\rho^*(A))$ for every $n \in \mathbb{N}$.

On the other hand, given any $G$-module $B$, there is a natural inclusion $i\colon B^G \longrightarrow \rho^*(B)^{G'}$; so, we have a map of cochain complexes

$$\begin{tikzcd}
0 \arrow[r] & A^G \arrow[r] \arrow[d, "i"'] & (I^0)^G \arrow[r] \arrow[d, "i"'] & (I^1)^G \arrow[r] \arrow[d, "i"'] & \cdots \\
0 \arrow[r] & \rho^*(A)^{G'} \arrow[r] & \rho^*(I^0)^{G'} \arrow[r] & \rho^*(I^1)^{G'} \arrow[r] & \cdots
\end{tikzcd}$$

which induces a map

$$i^*: H^n(G; A) \longrightarrow H^n(\rho^*(I^\star)^{G'}).$$

Composing this map with $\rho^\dagger$, we have the desired canonical map

$$\rho^\#:=\rho^\dagger i^*: H^n(G; A) \longrightarrow H^n(G'; \rho^*(A)). \tag{2.6}$$

So, $H^n$ is a contravariant functor in $G$ for all $n \in \mathbb{N}$. Similarly to the case of functoriality in $A$, the constructed $\rho^\#$ does not depend on the chosen injective resolutions $I^\star$ and $J^\star$.

**Remark 2.2.2** In the special case where $G' = H$ is a subgroup of $G$ and $\rho$ is the inclusion, we call $\rho^\#$ the *restriction* from $G$ to $H$ and we write $\mathrm{res}_G^H$ instead of $\rho^\#$.

### Exercises

*a)* Verify the details of Remark 2.2.1.

*b)* Verify that $H^n(-; A)$, with the action on maps $(-)^\#$ defined in (2.6) above, is indeed a contravariant functor.

*c)* Give a dual proof for projective resolutions of functoriality in $G$.

## 2.3 Cohomology of cyclic groups

We shall deal with finite cyclic groups and leave the infinite case as an exercise. Let $C_n = \{1, \tau, \tau^2, \ldots, \tau^{n-1}\}$ be the cyclic group of order $n$ generated by $\tau$. Then, clearly, $\mathbb{Z}[C_n] \cong \mathbb{Z}[\tau]/(\tau^n - 1)$. Consider in the group ring $\mathbb{Z}[C_n]$ the elements $N = 1 + \tau + \cdots + \tau^{n-1}$, called the *norm element*, and $D = \tau - 1$. Multiplication with $N$ and $D$ induces maps, which we denote again by $N$ and $D$, from $\mathbb{Z}[C_n]$ to $\mathbb{Z}[C_n]$. These maps determine a free resolution of the trivial $C_n$-module $\mathbb{Z}$:

$$0 \longleftarrow \mathbb{Z} \xleftarrow{\ \epsilon\ } \mathbb{Z}[C_n] \xleftarrow{\ D\ } \mathbb{Z}[C_n] \xleftarrow{\ N\ } \mathbb{Z}[C_n] \xleftarrow{\ D\ } \mathbb{Z}[C_n] \xleftarrow{\ N\ } \cdots .$$

In fact, this is a chain complex, because $DN = ND$ is the map "multiplication with $\tau^n - 1 = 0$", and $\epsilon D = 0$. To show its exactness, write the elements of $\mathbb{Z}[C_n]$ in the form $a = \sum_{i=0}^{n-1} a_i \tau^i$. Then, we have:

*a)* if $\epsilon(a) = 0$ then $a = D(b)$, where $b_i = a_{i+1} + \cdots + a_{n-1}$ for $0 \le i \le n-2$ and $b_{n-1} = 0$; so, $\ker(\epsilon) = \mathrm{im}(D)$;

*b)* if $D(a) = 0$ then $a = N(b)$, where $b_0 = a_0$ and $b_i = 0$, $1 \le i \le n-1$; so, $\ker(D) = \mathrm{im}(N)$;

*c)* if $N(a) = 0$ then $\epsilon(a) = 0$, hence, $\ker(N) = \ker(\epsilon) = \mathrm{im}(D)$.

We can now compute the cohomology groups $H^k(C_n; A)$ from the cochain complex

$$\mathrm{Hom}_{\mathbb{Z}[C_n]}(\mathbb{Z}[C_n]; A) \xrightarrow{D^*} \mathrm{Hom}_{\mathbb{Z}[C_n]}(\mathbb{Z}[C_n]; A) \xrightarrow{N^*} \mathrm{Hom}_{\mathbb{Z}[C_n]}(\mathbb{Z}[C_n]; A) \xrightarrow{D^*} \cdots ,$$

where, of course, $D^* = - \circ D$ and $N^* = - \circ N$. Under the obvious isomorphism $\mathrm{Hom}_{\mathbb{Z}[C_n]}(\mathbb{Z}[C_n]; A) \cong A$, this becomes the complex of abelian groups

$$A \xrightarrow{D} A \xrightarrow{N} A \xrightarrow{D} \cdots ,$$

Therefore, we have the following computation for the cohomology groups:

$$H^k(C_n; A) = \begin{cases} A^{C_n} & \text{if } k = 0 \\ \{a \in A \mid Na = 0\}/(\tau - 1)A & \text{if } k \text{ is odd} \\ A^{C_n} / \operatorname{im}(N) & \text{if } k \text{ is even and } k \geq 2, \end{cases}$$

where $A^{C_n}$ is the subgroup of invariants defined in (2.1).

### Exercises

*a)* Let $C$ be the infinite cyclic group generated by $\tau$. Show that we can identify $\mathbb{Z}[C]$ with the Laurent polynomial ring

$$\mathbb{Z}[\tau, \tau^{-1}] = \{\textstyle\sum_{i=-n}^{n} a_i \tau^i \mid a_i \in \mathbb{Z}, n \in \mathbb{N}\}.$$

*b)* Show that we have a free resolution

$$0 \longrightarrow \mathbb{Z}[C] \xrightarrow{D} \mathbb{Z}[C] \xrightarrow{\epsilon} \mathbb{Z} \longrightarrow 0$$

of the trivial $\mathbb{Z}[C]$-module $\mathbb{Z}$; use this in order to compute the cohomology groups $H^n(C; A)$, where $A$ is an arbitrary $C$-module.

## 2.4 The bar resolution

Now, let $G$ be a fixed group. We are going to introduce a specific (projective) resolution of $G$, namely the *bar resolution*, which will be useful in calculating its cohomology. In order to understand the bar resolution, we start with the following considerations.

We recall first of all that an *action* of $G$ on a set $S$ is a map

$$G \times S \to S, \quad (g, s) := gs$$

such that, for all $g, h \in G$, $s \in S$,

$$\begin{aligned} g(hs) &= (g \cdot h)s, \\ 1s &= s. \end{aligned}$$

Equivalently, an action is described by a group homomorphism

$$G \to \mathrm{Aut}(S),$$

where $\mathrm{Aut}(S)$ is the group of bijective functions from $S$ to $S$ with composition as multiplication and the identity as unit.

We denote by $G$-$\mathsf{Set}$ the category whose objects are sets $S$ with an action by $G$, and arrows are maps preserving the action, in the sense that $f(gs) = g(f(s))$ for any $g \in G$ and $s \in S$.

An action of $G$ on a set $S$ is called *free*, or we say that $S$ is a *free $G$-set*, whenever the equality $gs = s$ for *some* $s \in S$ implies that $g = 1$. If for any $s, s' \in S$ there is a $g \in G$ such that $gs = s'$, then the action is called *transitive*, or we say that $G$ acts *transitively* on $S$.

The *orbit* of an element $s \in S$ is the subset $\{gs \mid g \in G\}$ of $S$. Clearly, the action of $G$ on $S$ restricts to a transitive action on each orbit.

For $G$-sets, there are analogous constructions to those of the subgroup of invariants and the quotient group of coinvariants. These are the subset of *fix-points*

$$S^G = \{s \in S \mid gs = s, \forall g \in G\}$$

and the *set of orbits* $S/G$, which is the quotient of $S$ by the equivalence relation $\sim$, defined by $s \sim s'$ if and only if there is a $g \in G$ such that $gs = s'$.

Now, observe that for any $G$-set $S$ the free abelian group $\mathbb{Z}[S] = \bigoplus_{s \in S} \mathbb{Z}$ can be given the structure of a $G$-module: the elements of $\mathbb{Z}[S]$ have the form

$$\sum_{i=1}^{m} n_i s_i,$$

where $m \in \mathbb{N}$, $n_i \in \mathbb{Z}$, $s_i \in S$ for all $1 \leq i \leq m$, and $G$ acts on $\mathbb{Z}[S]$ as

$$g\left(\sum_{i=1}^{m} n_i s_i\right) = \sum_{i=1}^{m} n_i (g s_i), \tag{2.7}$$

for all $g \in G$.

Therefore, the adjunction

$$\begin{tikzcd} \mathsf{Set} \arrow[r, bend left, "{\mathbb{Z}[-]}"] \arrow[r, phantom, "\perp"] & \mathsf{Ab} \arrow[l, bend left, "U"] \end{tikzcd}$$

between the forgetful functor $U$ and the free abelian group functor $\mathbb{Z}[-]$, extends to an adjunction

$$\begin{tikzcd} G\text{-}\mathsf{Set} \arrow[r, bend left, "{\mathbb{Z}[-]}"] \arrow[r, phantom, "\perp"] & G\text{-}\mathsf{mod} \arrow[l, bend left, "U"] \end{tikzcd} \tag{2.8}$$

where $\mathbb{Z}[S]$ is the free $G$-module generated by the $G$-set $S$ as in (2.7) above, and $U(A)$ is the underlying $G$-set of a $G$-module $A$.

**Proposition 2.4.1** *If $S$ is a free $G$-set, then the $G$-module $\mathbb{Z}[S]$ is projective.*

**Proof.** Let $A$ and $B$ be $G$-modules and $\varphi\colon B \longrightarrow A$ a surjective $G$-module homomorphism. We want to show that, for any homomorphism $\alpha\colon \mathbb{Z}[S] \longrightarrow A$ there is a map of $G$-modules $\beta$ making the following triangle commute:

$$\begin{tikzcd} & B \arrow[d, two heads, "\varphi"] \\ \mathbb{Z}[S] \arrow[ur, dotted, "\beta"] \arrow[r, "\alpha"'] & A. \end{tikzcd}$$

Equivalently, through adjunction (2.8), we show that for any $f\colon S \longrightarrow U(A)$ in $G$-$\mathsf{Set}$ there is a map of $G$-sets $g$ making the following commute:

$$\begin{tikzcd} & U(B) \arrow[d, two heads, "U\varphi"] \\ S \arrow[ur, dotted, "g"] \arrow[r, "f"'] & U(A). \end{tikzcd}$$

Now, we write $S$ as a disjoint union of orbits $S = \coprod_{i \in I} S_i$. The group $G$ then acts transitively and freely on each orbit $S_i$. If we choose an element $s_i \in S_i$ for every $i \in I$, we have that any map of $G$-sets out of $S$ is completely determined by its value at the elements $s_i$. Now, using the fact that the underlying map of $\varphi$ is surjective, we can pick, for every $i \in I$, an element $b_i$ in $B$ such that $U\varphi(b_i) = f(s_i)$. The map $g$ will then be determined by the association $g(s_i) = b_i$. □

Now, we look at the cartesian product $G^{n+1}$ as a $G$-set, with the free diagonal action defined by

$$g(g_0, g_1, \ldots, g_n) = (gg_0, gg_1, \ldots, gg_n).$$

The sets $G^{n+1}$, for $n \geq 0$, form a simplicial $G$-set. In fact, there are maps of $G$-sets

$$d_i\colon G^{n+1} \longrightarrow G^n \qquad (0 \leq i \leq n)$$

given by

$$d_i(g_0, \ldots, g_n) = (g_0, \ldots, \hat{g}_i, \ldots, g_n),$$

where the notation $\hat{g}_i$ means that we delete the element $g_i$ from the list; moreover, these satisfy the simplicial identities

$$d_i d_j = d_{j-1} d_i \qquad (i < j). \tag{2.9}$$

Denote by $B_n(G)$ the $G$-module $\mathbb{Z}[G^{n+1}]$, for $n \geq 0$. Then, the maps $d_i$ induce homomorphisms of $G$-modules, which, by an abuse of notation, we call again $d_i$:

$$d_i\colon B_n(G) \longrightarrow B_{n-1}(G), \qquad (0 \leq i \leq n)$$

and we define, for every $n \geq 1$, the map

$$d = \sum_{i=0}^{n} (-1)^i d_i\colon B_n(G) \longrightarrow B_{n-1}(G).$$

**Theorem 2.4.2** *The sequence*

$$0 \longleftarrow \mathbb{Z} \xleftarrow{\ \varepsilon\ } B_0(G) \xleftarrow{\ d\ } B_1(G) \xleftarrow{\ d\ } B_2(G) \xleftarrow{\ d\ } \cdots \tag{2.10}$$

*is a projective resolution of* $\mathbb{Z}$ *by* $G$*-modules, called the* bar *or* standard resolution *of* $G$.

**Proof.** The proof consists of three simple steps:

*a)* The sequence (2.10) is a chain complex;

*b)* The $G$-modules $B_n(G)$ are projective;

*c)* The chain complex is exact.

Using relations (2.9), we immediately get $\varepsilon d = dd = 0$, hence *a*) is checked. As for *b*), this follows by Proposition 2.4.1, since $G$ acts freely on $G^{n+1}$ for all $n \in \mathbb{N}$. So, we can focus on *c*).

To show exactness of (2.10), we construct a *contracting homotopy* $h$ of chain complexes:

$$\begin{tikzcd}
0 & \mathbb{Z} \arrow[l] \arrow[d, "\mathsf{id}"'] \arrow[dr, dashed, "h_{-1}"] & B_0(G) \arrow[l, "\varepsilon"'] \arrow[d, "\mathsf{id}"] \arrow[dr, dashed, "h_0"] & B_1(G) \arrow[l, "d"'] \arrow[d, "\mathsf{id}"] \arrow[dr, dashed, "h_1"] & B_2(G) \arrow[l, "d"'] \arrow[d, "\mathsf{id}"] & \cdots \arrow[l, "d"'] \\
0 & \mathbb{Z} \arrow[l] & B_0(G) \arrow[l, "\varepsilon"] & B_1(G) \arrow[l, "d"] & B_2(G) \arrow[l, "d"] & \cdots ; \arrow[l, "d"]
\end{tikzcd}$$

that is, a homotopy between the identity and the 0 chain maps. By Proposition 1.10.6, this implies that $\mathsf{id}\colon B_n(G) \longrightarrow B_n(G)$ induces both the identity and the zero map between the homology groups, thus proving that they are trivial and the chain complex is exact.

Writing $B_{-1}(G)$ for $\mathbb{Z}$, the maps

$$h_n\colon B_n(G) \longrightarrow B_{n+1}(G) \qquad (n \geq -1)$$

are defined as follows.

For $n = -1$, $h_{-1}(m) = m \cdot 1$ for every $m \in \mathbb{Z}$. When $n \geq 0$, we define $h_n$ on generators by

$$h_n(g_0, \ldots, g_n) = (1, g_0, \ldots, g_n).$$

All we have to do, now, is to check the homotopy condition (1.12). Clearly, $\varepsilon h_{-1}$ is the identity map and $\mathsf{id} = h_{-1}\varepsilon + dh_0$. Moreover, for $n > 0$ and $0 \leq i \leq n$, we have

$$d_{i+1}h_n - h_{n-1}d_i = 0;$$

hence, $h_{n-1}d + dh_n = d_0h_n + \sum_{i=0}^{n}(-1)^i(d_{i+1}h_n - h_{n-1}d_i) = d_0h_n = \mathsf{id}$. $\square$

The standard projective resolution of $\mathbb{Z}$ as a $\mathbb{Z}[G]$-module allows us to calculate explicitly the cohomology $H^n(G; A)$ of $G$ with coefficients in any $G$-module $A$. We are now going to give an alternative description of these cohomology groups.

Let $A$ be a $G$-module. We define the positive cochain complex $C^\star(G, A)$ of groups as follows. For $n \in \mathbb{N}$, we have a group (with pointwise addition)

$$C^n(G, A) = \mathrm{Hom}_{\mathsf{Set}}(G^n, A);$$

the boundary map

$$\delta\colon C^{n-1}(G, A) \longrightarrow C^n(G, A) \tag{2.11}$$

is defined as $\delta = \sum_{i=0}^{n}(-1)^i\delta_i$, where the maps $\delta_i\colon C^{n-1}(G, A) \longrightarrow C^n(G, A)$ are defined by

$$\delta_i(f)(g_1, \ldots, g_n) = \begin{cases} g_1 f(g_2, \ldots, g_n) & i = 0 \\ f(g_1, \ldots, g_{i-1}, g_i g_{i+1}, g_{i+2}, \ldots, g_n) & i = 1, \ldots, n-1 \\ f(g_1, \ldots, g_{n-1}) & i = n \end{cases}$$

Notice that, for $n = 0$, we have $C^0(G, A) = A$ and the boundary map $\delta\colon C^0(G, A) \longrightarrow C^1(G, A)$ is given by $\delta(a)(g_1) = g_1 a - a$ for every $a \in A$ and $g_1 \in G$. The fact that this is really a cochain complex, i.e. that $\delta\delta = 0$, follows from the proof of the following theorem.

**Theorem 2.4.3** *The cohomology group in degree $n$ of the cochain complex $C^\star(G, A)$ is isomorphic to $H^n(G; A)$.*

**Proof.** We have the standard projective resolution $B_\star(G)$ of $\mathbb{Z}$, so $H^n(G; A)$ can be computed from the complex $\mathrm{Hom}_{\mathbb{Z}[G]}(B_\star(G), A)$ with boundary maps

$$d^* = - \circ d\colon \mathrm{Hom}_{\mathbb{Z}[G]}(B_{n-1}(G), A) \longrightarrow \mathrm{Hom}_{\mathbb{Z}[G]}(B_n(G), A).$$

In fact, the complex $\mathrm{Hom}_{\mathbb{Z}[G]}(B_\star(G), A)$ is isomorphic to $C^\star(G, A)$. To see this, define

$$\alpha\colon \mathrm{Hom}_{\mathbb{Z}[G]}(B_n(G), A) \longrightarrow C^n(G, A)$$

by $\alpha(c)(g_1, \ldots, g_n) = c(1, g_1, g_1g_2, \ldots, g_1 \ldots g_n)$, and

$$\beta\colon C^n(G, A) \longrightarrow \mathrm{Hom}_{\mathbb{Z}[G]}(B_n(G), A)$$

by $\beta(f)(h_0, \ldots, h_n) = h_0 f(h_0^{-1}h_1, \ldots, h_{n-1}^{-1}h_n)$.

It is immediate to see that $\alpha$ and $\beta$ are mutually inverse, and the result is proved, once we show that they are homomorphisms of cochain complexes; but this follows by the fact that the diagram

$$\begin{tikzcd}
\mathrm{Hom}_{\mathbb{Z}[G]}(B_{n-1}(G), A) \arrow[r, "d_i^*"] & \mathrm{Hom}_{\mathbb{Z}[G]}(B_n(G), A) \arrow[d, "\alpha"] \\
C^{n-1}(G, A) \arrow[u, "\beta"] \arrow[r, "\delta_i"'] & C^n(G, A)
\end{tikzcd}$$

commutes for all $n \in \mathbb{N}$ and all $0 \le i \le n$. To see this, let $f \in C^{n-1}(G, A)$ and $(g_1, \ldots, g_n) \in G^n$. Then, we have

$$\begin{aligned}
&(\alpha d_i^* \beta)(f)(g_1, \ldots, g_n) = \\
&= (d_i^*\beta)(f)(1, g_1, g_1g_2, \ldots, g_1 \ldots g_n) \\
&= \beta(f) d_i(1, g_1, g_1g_2, \ldots, g_1 \ldots g_n) \\
&= \begin{cases} \beta(f)(g_1, g_1g_2, \ldots, g_1 \cdots g_n) & i = 0 \\ \beta(f)(1, g_1, \ldots, \widehat{g_1 \cdots g_i}, \ldots, g_1 \cdots g_n) & 1 \le i \le n-1 \\ \beta(f)(1, g_1, \ldots, g_1 \cdots g_{n-1}) & i = n \end{cases} \\
&= \begin{cases} g_1 f(g_2, g_3, \ldots, g_n) & i = 0 \\ f(g_1, \ldots, g_{i-1}, g_i g_{i+1}, g_{i+2}, \ldots, g_n) & 1 \le i \le n-1 \\ f(g_1, g_2 \ldots, g_{n-1}) & i = n \end{cases} \\
&= \delta_i(f)(g_1, \ldots, g_n).
\end{aligned}$$

□

Using the previous result, we can compute the following homology groups.

**Proposition 2.4.4** *For a group $G$ and a $G$-module $A$,* *a)* $H^0(G,A) = A^G$;

*b)* *if* $A$ *is a trivial* $G$*-module (i.e.* $ga = a$ *for all* $a, g$*), then*

$$H^1(G,A) = \mathrm{Hom}_{\mathsf{Grp}}(G,A),$$

*the set (abelian group) of group homomorphisms from* $G$ *to* $A$*.*

## Exercises

*a)* Show naturality of the isomorphisms of Theorem 2.4.3. More specifically, give the evident functoriality of $C^\star(G,A)$ in $G$, and prove it agrees with that of $H^n(G,A)$ as proved in Section 2.2.2.

*b)* Check that there is an adjunction as in (2.8). More generally, look for pairs of adjoint functors relating the following categories, checking for possible commutativity of the diagram

$$\begin{tikzcd} \mathsf{Set} \arrow[r, shift left, "\mathbb{Z}[-]"] \arrow[r, phantom, "\perp"] \arrow[d, shift left] \arrow[d, phantom, "*"] & \mathsf{Ab} \arrow[l, shift left, "U"] \arrow[d, shift left] \arrow[d, phantom, "*"] \\ G\text{-}\mathsf{Set} \arrow[u, shift left] \arrow[r, shift left, "\mathbb{Z}[-]"] \arrow[r, phantom, "\perp"] & G\text{-}\mathsf{mod}. \arrow[l, shift left, "U"] \arrow[u, shift left] \end{tikzcd}$$

To this end, consider the functors

$$\begin{aligned} (-)^G &: G\text{-}\mathsf{Set} \longrightarrow \mathsf{Set}, & (-)^G &: G\text{-}\mathsf{mod} \longrightarrow \mathsf{Ab}, \\ -/G &: G\text{-}\mathsf{Set} \longrightarrow \mathsf{Set}, & (-)_G &: G\text{-}\mathsf{mod} \longrightarrow \mathsf{Ab}. \end{aligned}$$

*c)* Prove Proposition 2.4.4.

# 2.5 Group extensions and $H^2$

In this Section, we study the cohomology group in degree 2. We shall prove that $H^2(G;A)$ classifies group extensions, but first, we need to introduce some new notions.

**Definition 2.5.1** Let $G$ be a group and $A$ an abelian group. An *extension of* $G$ *by* $A$ is a short exact sequence of groups of the form

$$1 \longrightarrow A \xrightarrow{i} E \xrightarrow{p} G \longrightarrow 1. \tag{2.12}$$

Two such extensions (2.12) and

$$1 \longrightarrow A \xrightarrow{i'} E' \xrightarrow{p'} G \longrightarrow 1,$$

are said to be *equivalent* if there is a group homomorphism $\varphi: E \longrightarrow E'$ such that the squares of the following diagram commute:

$$\begin{tikzcd} 1 \arrow[r] & A \arrow[r, "i"] \arrow[d, "\mathrm{id}_A"'] & E \arrow[r, "p"] \arrow[d, "\varphi"] & G \arrow[r] \arrow[d, "\mathrm{id}_G"] & 1 \\ 1 \arrow[r] & A \arrow[r, "i'"'] & E' \arrow[r, "p'"'] & G \arrow[r] & 1. \end{tikzcd} \tag{2.13}$$

The relation between extensions and $G$-modules is given by the following proposition, which we leave to you to check (see Exercise $b$) below):

**Proposition 2.5.2** *An extension as in (2.12) induces a well-defined $G$-module structure on $A$ by*

$$i(ga) = \widetilde{g}i(a)\widetilde{g}^{-1},$$

*where $\widetilde{g}$ is any element in $p^{-1}(g)$.*

If $A$ is already given as a $G$-module, then by an extension of $G$ by $A$ we understand an extension as in (2.12), which induces the given $G$-module structure. Note that there is always an extension of $G$ by any $G$-module $A$; namely, the one given by the *semidirect product* $E = A \rtimes G$ of $G$ with $A$. Recall that $A \rtimes G$ is the group whose underlying set is $A \times G$, and multiplication is given by

$$(a, g) \cdot (b, h) = (a + gb, gh). \tag{2.14}$$

It is then a natural question whether there are more extensions of $G$ by a $G$-module $A$. More precisely, we look for extension classes under the equivalence relation of Definition 2.5.1.

Denote by $\mathrm{Ext}(G, A)$ the set of such extension classes. The following theorem answers the question in a very powerful way.

**Theorem 2.5.3** *There is a one-to-one correspondence between the extensions of the group $G$ by a $G$-module $A$ and $H^2(G; A)$; in other words, we have a bijection of sets*

$$\mathrm{Ext}(G; A) \simeq H^2(G; A).$$

Before giving the proof, we need to make some more considerations about cocycles in $C^2(G, A)$.

**Definition 2.5.4** A cocycle $c \in C^2(G, A)$ is called *normal* if, for all $g \in G$,

$$c(1, g) = 0 = c(g, 1).$$

Notice that, for any cocycle $c \in C^2(G, A)$, we have $\delta c(g, g^{-1}, g) = 0$, and this rewrites to

$$gc(g^{-1}, g) - c(1, g) + c(g, 1) - c(g, g^{-1}) = 0;$$

therefore, when $c$ is normal, we have for all $g \in G$,

$$gc(g^{-1}, g) = c(g, g^{-1}). \tag{2.15}$$

**Lemma 2.5.5** *A cohomology class $u \in H^2(G; A)$ can always be represented by a normal cocycle.*

**Proof.** Suppose $u = [c]$ for some $c \in C^2(G, A)$, and define $f\colon G \longrightarrow A$ by $f(g) = c(1, g)$. Now, consider the map $c' = c - \delta f$ in $C^2(G, A)$. Clearly, $u = [c] = [c']$; then, to prove that $c'$ is normal, we reason as follows. First, observe that

$$\begin{aligned}(\delta f)(h, g) &= hf(g) - f(hg) + f(h)\\ &= hc(1, g) - c(1, hg) + c(1, h)\\ &= hc(1, g) - \delta c(1, h, g)\\ &= hc(1, g),\end{aligned}$$

because $\delta c = 0$. Now, instantiating $h$ to 1, we get $\delta f(1,g) = c(1,g)$. Hence, for all $g \in G$, we have

$$c'(1,g) = 0. \tag{2.16}$$

Analogously, we have that

$$\begin{aligned} 0 \;=\; \delta c'(g,h,1) &= gc'(h,1) - c'(gh,1) + c'(g,h) - c'(g,h) \\ &= gc'(h,1) - c'(gh,1); \end{aligned}$$

whence, putting $h = 1$ and using (2.16), we get $c'(g,1) = 0$. □

**Proof of Theorem 2.5.3.** We shall exhibit maps $\xi\colon \mathrm{Ext}(G;A) \longrightarrow H^2(G;A)$ and $\chi\colon H^2(G;A) \longrightarrow \mathrm{Ext}(G;A)$ and show that they are mutually inverse.

To define $\xi$, suppose given an extension

$$1 \longrightarrow A \xrightarrow{\;i\;} E \xrightarrow{\;p\;} G \longrightarrow 1$$

and choose a section $\sigma\colon G \longrightarrow E$, i.e. a set map with $p\sigma = \mathsf{id}_G$. Note that this is always possible since $p$ is a surjection. Even more, because $p(1) = 1$, we can choose $\sigma$ such that $\sigma(1) = 1$ (and we shall always do that, implicitly).

Now, let $c \in C^2(G,A)$ be the only 2-cochain such that

$$ic(g_1,g_2) = \sigma(g_1)\sigma(g_2)\sigma(g_1g_2)^{-1}. \tag{2.17}$$

(This definition is correct, because $\sigma(g_1)\sigma(g_2)\sigma(g_1g_2)^{-1} \in \ker(p) = \mathrm{im}(i)$.) In a slogan, we could say that $c$ "measures the failure of $\sigma$ to be a group homomorphism". From the definition of $\delta$ in (2.11), we have

$$\delta c(g_0,g_1,g_2) = g_0c(g_1,g_2) - c(g_0g_1,g_2) + c(g_0,g_1g_2) - c(g_0,g_1).$$

If we write $\alpha$, $\beta$, $\gamma$ and $\zeta$ for the image under $i$ of the summands in the right hand-side, then, by Proposition 2.5.2,

$$\alpha = i(g_0 \cdot c(g_1,g_2)) = \widetilde{g_0} \cdot \left(\sigma(g_1)\sigma(g_2)\sigma(g_1g_2)^{-1}\right) \cdot \widetilde{g_0}^{-1},$$

for any $\widetilde{g_0}$ such that $p(\widetilde{g_0}) = g_0$, so we can choose $\widetilde{g_0} = \sigma(g_0)$. Then, by (2.17), we have

$$\begin{aligned} \alpha &= \sigma(g_0)\sigma(g_1)\sigma(g_2)\sigma(g_1g_2)^{-1}\sigma(g_0)^{-1}, \\ \beta &= \sigma(g_0g_1)\sigma(g_2)\sigma(g_0g_1g_2)^{-1}, \\ \gamma &= \sigma(g_0)\sigma(g_1g_2)\sigma(g_0g_1g_2)^{-1}, \\ \zeta &= \sigma(g_0)\sigma(g_1)\sigma(g_0g_1)^{-1}. \end{aligned}$$

Then, $\alpha\gamma\beta^{-1} = \zeta$, and since the image of $A$ is an abelian subgroup of $E$, also $\alpha\beta^{-1}\gamma\zeta^{-1} = 1$; therefore $\delta(c) = 0$. This shows that $c$ is a cocycle, so we can define $\xi(E)$ to be the cohomology class $[c]$. To see that this is a well-defined map, we need to show that $[c]$ does not depend on the choice of the section $\sigma$, and that equivalent extensions generate the same class in cohomology.

Suppose that $\rho\colon G \longrightarrow E$ is another section with $\rho(1) = 1$, giving rise to a cocycle $b \in C^2(G,A)$ such that

$$ib(g_1,g_2) = \rho(g_1)\rho(g_2)\rho(g_1g_2)^{-1},$$

and consider $f\colon G \longrightarrow A$ defined by $if(g) = \sigma(g)\rho(g)^{-1}$. To prove that $c$ and $b$ define the same cohomology class, we show that $i(c - b - \delta f) = 1$, because this implies $c = b + \delta f$. Let $\alpha$, $\beta$ and $\gamma$ be the elements

$$\begin{aligned}
\alpha &= ic(g_1, g_2) = \sigma(g_1)\sigma(g_2)\sigma(g_1g_2)^{-1} \\
\beta &= ib(g_1, g_2) = \rho(g_1)\rho(g_2)\rho(g_1g_2)^{-1} \\
\gamma &= i\delta f(g_1, g_2) = i(g_1 f(g_2) - f(g_1g_2) + f(g_1)) \\
&= \left(\widetilde{g_1}\sigma(g_2)\rho(g_2)^{-1}\widetilde{g_1}^{-1}\right) \cdot \left(\sigma(g_1g_2)\rho(g_1g_2)^{-1}\right)^{-1} \cdot \left(\sigma(g_1)\rho(g_1)^{-1}\right), \\
&= \gamma_1 \cdot \gamma_2^{-1} \cdot \gamma_3,
\end{aligned}$$

where $\widetilde{g_1} \in p^{-1}(g_1)$, so we can choose $\widetilde{g_1} = \rho(g_1)$. It is now easy to compute that $\gamma_3^{-1} \cdot \alpha \cdot \gamma_2 \cdot \beta^{-1}\gamma_1^{-1} = 1$, whence, again by the fact that the image of $A$ is an abelian subgroup of $E$, we get that $\alpha\beta^{-1}\gamma^{-1} = 1$, so $c = b + \delta f$.

Finally, consider an equivalence of extensions $\varphi\colon E \longrightarrow E'$ as in (2.13), and $\sigma$ a section of $p$. Choosing the section $\sigma' = \varphi\sigma$ of $p'$, it is easy to see that the induced cocycles $c$ and $c'$ give the same cohomology class in $H^2(G; A)$.

So, we have a function of sets $\xi\colon \mathrm{Ext}(G; A) \longrightarrow H^2(G; A)$. Notice that the cocycle $c$ defined in (2.17) is in fact normal. Now, we focus our attention on defining an inverse $\chi$ for $\xi$.

Consider a cohomology class $u \in H^2(G; A)$. By Lemma 2.5.5 we can choose a normal cocycle $c \in C^2(G, A)$ such that $[c] = u$. Then, consider on the cartesian product $A \times G$ the multiplication

$$(a, g) \cdot (b, h) = (a + gb + c(g, h), gh).$$

This operation gives $A \times G$ the structure of a group, which we denote by $A \rtimes_c G$. In fact, by the cocycle condition, for $g_1, g_2, g_3 \in G$,

$$0 = \delta c(g_1, g_2, g_3) = g_1c(g_2, g_3) - c(g_1g_2, g_3) + c(g_1, g_2g_3) - c(g_1, g_2)$$

and we have

$$g_1c(g_2, g_3) + c(g_1, g_2g_3) = c(g_1g_2, g_3) + c(g_1, g_2).$$

Hence, we show associativity by

$$\begin{aligned}
&(a_1, g_1) \cdot ((a_2, g_2) \cdot (a_3, g_3)) \\
&\quad= (a_1, g_1) \cdot (a_2 + g_2a_3 + c(g_2, g_3), g_2g_3) \\
&\quad= (a_1 + g_1a_2 + g_1g_2a_3 + g_1c(g_2, g_3) + c(g_1, g_2g_3), g_1g_2g_3) \\
&\quad= (a_1 + g_1a_2 + g_1g_2a_3 + c(g_1, g_2) + c(g_1g_2, g_3), g_1g_2g_3) \\
&\quad= (a_1 + g_1a_2 + c(g_1, g_2), g_1g_2) \cdot (a_3, g_3) \\
&\quad= ((a_1, g_1) \cdot (a_2, g_2)) \cdot (a_3, g_3).
\end{aligned}$$

A simple computation shows the unit of this multiplication to be $(0, 1) \in A \rtimes_c G$ and the inverse of an element $(a, g)$ to be the pair $(-g^{-1}a - c(g^{-1}, g), g^{-1})$.

Moreover, the sequence

$$0 \longrightarrow A \xrightarrow{\ i\ } A \rtimes_c G \xrightarrow{\ p\ } G \longrightarrow 1,$$

where $i(a) = (a, 1)$ and $p(a, g) = g$, is clearly exact, thus defining an extension of $G$ by $A$, which we set as $\chi(u)$.

For the map $\chi$ to be well-defined, we have to show that $\chi(u)$ does not depend on the choice of the normal cocycle $c$. To this end, suppose $c'$ is another normal cocycle such that $[c] = [c'] = u$. Then, $c' = c + \delta f$ and, by normality, $f(1) = 0$. In order to prove that the extensions corresponding to $c$ and $c'$ are equivalent, define a map $\varphi\colon A \rtimes_{c'} G \longrightarrow A \rtimes_c G$ by $\varphi(a, g) := (a + f(g), g)$. Since we have $c' = c + \delta f$, it follows that $c'(g_1, g_2) + f(g_1 g_2) = c(g_1, g_2) + g_1 f(g_2) + f(g_1)$. Therefore, for $(a_1, g_1), (a_2, g_2) \in A \rtimes_{c'} G$,

$$\begin{aligned}
\varphi((a_1, g_1) \cdot (a_2, g_2)) &= \varphi(a_1 + g_1 a_2 + c'(g_1, g_2), g_1 g_2) \\
&= (a_1 + g_1 a_2 + c'(g_1, g_2) + f(g_1 g_2), g_1 g_2) \\
&= (a_1 + g_1 a_2 + c(g_1, g_2) + g_1 f(g_2) + f(g_1), g_1 g_2) \\
&= \varphi(a_1, g_1)\varphi(a_2, g_2),
\end{aligned}$$

proving that $\varphi$ is a group homomorphism. Moreover, we have

$$\varphi i(a) = \varphi(a, 1) = (a + f(1), 1) = (a, 1) = i'(a)$$

and

$$p'\varphi(a, g) = p'(a + f(g), g) = g = p(a, g),$$

showing that $\varphi$ is an equivalence of extensions.

Finally, we are going to show that $\xi$ and $\chi$ are mutually inverse. For a normal cocycle $c$ representing a cohomology class $u \in H^2(G; A)$, choose the "obvious" section $\sigma\colon G \longrightarrow A \rtimes_c G$ (defined by $\sigma(g) = (0, g)$) of the corresponding extension

$$0 \longrightarrow A \xrightarrow{i} A \rtimes_c G \xrightarrow{p} G \longrightarrow 1.$$

This determines another cocycle $c'$, such that $ic'(g_1, g_2) = \sigma(g_1)\sigma(g_2)\sigma(g_1 g_2)^{-1}$. Now, using (2.15), we get that

$$\begin{aligned}
i(c'(g_1, g_2)) &= \sigma(g_1)\sigma(g_2)\sigma(g_1 g_2)^{-1} \\
&= (c(g_1, g_2), g_1 g_2) \cdot (-c((g_1 g_2)^{-1}, g_1 g_2), (g_1 g_2)^{-1}) \\
&= (c(g_1, g_2), 1) \\
&= i(c(g_1, g_2)),
\end{aligned}$$

which proves that $\xi\chi$ is the identity on $H^2(G; A)$.

Conversely, for an extension

$$0 \longrightarrow A \xrightarrow{i} E \xrightarrow{p} G \longrightarrow 1$$

and a section $\sigma\colon G \longrightarrow E$, we have the normal cocycle $c$, satisfying the equality $i(c(g_1, g_2)) = \sigma(g_1)\sigma(g_2)\sigma(g_1 g_2)^{-1}$, and the group extension

$$0 \longrightarrow A \xrightarrow{i'} A \rtimes_c G \xrightarrow{p'} G \longrightarrow 1,$$

where $i'(a) = (a, 1)$ and $p'(a, g) = g$. To prove the equivalence of the two extensions, define the map $\psi\colon A \rtimes_c G \longrightarrow E$ as $\psi(a, g) = i(a)\sigma(g)$. Obviously, we have

$$\begin{aligned}
\psi((a_1,g_1)\cdot(a_2,g_2)) &= \psi(a_1+g_1a_2+c(g_1,g_2),g_1g_2)\\
&= i(a_1)i(g_1a_2)i(c(g_1,g_2))\sigma(g_1g_2)\\
&= i(a_1)\sigma(g_1)i(a_2)\sigma(g_1)^{-1}\sigma(g_1)\sigma(g_2)\sigma(g_1g_2)^{-1}\sigma(g_1g_2)\\
&= \psi(a_1,g_1)\psi(a_2,g_2),
\end{aligned}$$

showing that $\psi$ is a homomorphism of groups. Moreover, we clearly have that $p\psi(a,g) = p(i(a)\sigma(g)) = g = p'(a,g)$ and $\psi(i'(a)) = \psi(a,1) = i(a)\sigma(1) = i(a)$. Therefore, $\psi$ is an equivalence of extensions. This shows that $\chi\xi$ is the identity on $\mathrm{Ext}(G;A)$, and the theorem is proved. $\square$

It follows from Theorem 2.5.3 that $\mathrm{Ext}(G;A)$ has the structure of an abelian group. This structure can be described explicitly as follows:

**Proposition 2.5.6 (The Baer sum)** *Given two extensions*

$$0 \longrightarrow A \xrightarrow{\ i\ } E \xrightarrow{\ p\ } G \longrightarrow 1$$

*and*

$$0 \longrightarrow A \xrightarrow{\ j\ } F \xrightarrow{\ q\ } G \longrightarrow 1$$

*inducing the same $G$-module structure on $A$, define the set*

$$E \times_G F = \{(e,f) \in E \times F \mid p(e) = q(f)\}$$

*and let $E * F$ be the quotient of $E \times_G F$ obtained by the identification*

$$(ei(a), f) \sim (e, fj(a)),$$

*with the notation $e \otimes f$ for the equivalence class of $(e,f)$. We call $E * F$ the* Baer sum *of $E$ and $F$. This defines on* $\mathrm{Ext}(G;A)$ *the structure of an abelian group.*

The proof is outlined in Exercises *e)*–*h)* below.

## Exercises

*a)* Prove that a group homomorphism $\varphi\colon E \longrightarrow E'$ as in (2.13) is automatically an isomorphism, and that "being equivalent" in the sense of Definition 2.5.1 is an equivalence relation.

*b)* Prove Proposition 2.5.2.

*c)* Check that $A \times G$ with the product operation defined in (2.14) is indeed a group. Show that the extension

$$0 \longrightarrow A \xrightarrow{\ i\ } A \rtimes G \xrightarrow{\ p\ } G \longrightarrow 1,$$

where $i(a) = (a,1)$ and $p(a,g) = g$ for all $a \in A$ and $g \in G$, indeed induces the "old" $G$-module structure on $A$.

*d)* State and prove a "normalisation lemma" analogous to Lemma 2.5.5 for the cohomology group $H^n(G;A)$ in each dimension $n \in \mathbb{N}$; here, we call an $n$-cochain $c$ *normal* if $c(g_1,\ldots,g_n) = 0$ as soon as one of the $g_i$ is 1.

*e)* Prove that the operation

$$(e \otimes f) \cdot (e' \otimes f') = (ee' \otimes ff')$$

is a well defined group structure on the Baer sum of two extensions $E$ and $F$. Conclude that $E * F$ fits into an extension

$$0 \longrightarrow A \xrightarrow{s} E * F \xrightarrow{t} G \longrightarrow 1$$

where $s(a) = i(a) \otimes 1 = 1 \otimes j(a)$ and $t(e \otimes f) = p(e) = q(f)$.

*f)* Prove that if $\sigma, \tau$ are sections of two extensions $E, F$ giving cocycles $c, c'$, then there is a section $\sigma \otimes \tau$ of $E * F$ giving the cocycle $c + c'$. Use this to prove Proposition 2.5.6. Show in particular that the semidirect product $A \rtimes G$ is the neutral element of the group $\mathrm{Ext}(G; A)$.

*g)* Show the following universal property characterising $A \rtimes G$: an extension

$$O \longrightarrow A \longrightarrow E \longrightarrow G \longrightarrow 1$$

has a section $\sigma$ which is a group homomorphism if and only if it is equivalent to $A \rtimes G$.

*h)* Check directly that $*$ defines on $\mathrm{Ext}(G; A)$ the structure of an abelian group. Can you describe the inverse $E^{-1}$ of an extension $E$ for this group structure directly in terms of $E$?

## 2.6 Additional exercises

1. Show that if $G$ is a non-trivial finite cyclic group, then $\mathbb{Z}$ does not admit a projective resolution of finite length.

2. Let $\alpha\colon H \longrightarrow G$ be a homomorphism of groups, and write $\alpha\colon \mathbb{Z}[H] \longrightarrow \mathbb{Z}[G]$ for the induced ring homomorphism. Recall that $\alpha^*\colon G\text{-}\mathsf{mod} \longrightarrow H\text{-}\mathsf{mod}$ has two adjoint functors $\alpha_!$ and $\alpha_*$. (For an $H$-module $M$, the modules $\alpha_!(M) = \mathbb{Z}[G] \otimes_{\mathbb{Z}[H]} M$ and $\alpha_*(M) = \mathrm{Hom}_{\mathbb{Z}[H]}(\mathbb{Z}[G], M)$ are also called the induced and coinduced modules in the literature, and denoted $\mathrm{Ind}_H^G M$ and $\mathrm{Coind}_H^G M$.)

   i) Prove that if $\alpha$ is the inclusion of a subgroup, then $\alpha_*$ preserves epis hence $\alpha^*$ preserves projectives.

   ii) Conclude that for any $H$-module $M$, there are canonical isomorphisms

   $$H^*(G, \alpha_*(M)) \xrightarrow{\cong} H^*(H, M)$$

   and

   $$H_*(H, M) \xrightarrow{\cong} H_*(G, \alpha_!(M)).$$

3. Given an extension $0 \to A \longrightarrow E \longrightarrow G \longrightarrow 1$ and a group homomorphism $\alpha\colon G' \longrightarrow G$, construct an extension $0 \to A \longrightarrow E' \longrightarrow G' \longrightarrow 1$ by

pullback, show it is characterized up to equivalence such that the following diagram commutes

$$\begin{tikzcd}
0 \arrow[r] & A \arrow[r] & E \arrow[r] & G \arrow[r] & 1 \\
0 \arrow[r] & A \arrow[r] \arrow[u, equal] & E' \arrow[r] \arrow[u] & G' \arrow[r] \arrow[u, "\alpha"] & 1
\end{tikzcd}$$

and prove that this map corresponds to the induced map in cohomology

$$\alpha^*\colon H^2(G,A) \longrightarrow H^2(G',A).$$

4. **[cf. Brown, pp. 102–103]** Let $E$ and $N$ be groups. We say that $N$ is a *crossed module* over $E$ if there is an action of $E$ on $N$ denoted by $e * n$ and a homomorphism $\partial\colon N \longrightarrow E$ such that $\partial(e * n) = e\partial(n)e^{-1}$ and $\partial(n) * m = nmn^{-1}$.

   i) Prove that $N$ is a crossed module over $\mathrm{Aut}(N)$, the group of automorphisms of $N$, where $\partial(n)$ is given by the inner automorphism associated to $n$.

A map between two crossed modules $(N, E, \partial)$ and $(N', E', \partial')$ is a pair of maps $\varphi\colon N \longrightarrow N'$ and $\psi\colon E \longrightarrow E'$ for which the following diagram commutes

$$\begin{tikzcd}
N \arrow[r, "\partial"] \arrow[d, "\varphi"'] & E \arrow[d, "\psi"] \\
N' \arrow[r, "\partial'"'] & E'
\end{tikzcd}$$

and which together respect the action.

   ii) For a crossed module $(N, E, \partial)$, with $A = \ker(\partial)$ and $G = \mathrm{coker}(\partial)$ (the latter makes sense, since $\mathrm{im}(\partial)$ is normal) check that $A$ is a central subgroup of $N$ (in particular, $A$ is abelian and normal). This gives an exact sequence

$$A \xrightarrow{\ i\ } N \xrightarrow{\ \partial\ } E \xrightarrow{\ \pi\ } G.$$

   iii) Choose set theoretical sections $s\colon G \longrightarrow E$ and $t\colon \mathrm{im}(\partial) \longrightarrow N$ of $\pi$ and $\partial$ respectively, and define $f\colon G \times G \longrightarrow E$ by

$$f(g,h) = s(g) \cdot s(h) \cdot s(gh)^{-1}.$$

   Note that $\mathrm{im}(f) \subseteq \mathrm{im}(\partial)$, and consider $F = t \circ f\colon G \times G \longrightarrow N$. Although $N$ is not abelian, we can mimic the definition of the boundary operator in group cohomology and define $c = \delta(t \circ f)$ by

$$c(g,h,k) = (s(g) * F(h,k))F(g,hk)F(gh,k)^{-1}F(g,h)^{-1}$$

   Check that $\partial c = 1$ so that $c$ defines a map $G \times G \times G \longrightarrow A$. Prove that $c$ is a cocycle, hence defines a cohomology class $[c] \in H^3(G,A)$.

   iv) Prove that $[c]$ does not depend on the choice of the sections $s$ and $t$.

v) For a map between crossed modules $(N, E, \partial)$ and $(N', E', \partial')$ as above, with associated cokernels $G$ and $G'$, and kernels $A$ and $A'$, note that there are maps $\psi: G \longrightarrow G'$ and $\varphi: A \longrightarrow A'$ respecting the action, i.e., $\psi(g) \cdot \varphi(a) = \varphi(g \cdot a)$. Prove that for the induced maps

$$H^3(G, A) \xrightarrow{\varphi_*} H^3(G, A') \xleftarrow{\psi^*} H^3(G', A')$$

we have an equality $\varphi_*[c] = \psi^*[c']$, where $[c]$ is as above, and $[c']$ is the class associated similarly to $\partial': N' \longrightarrow E'$.

**Remark.** It can be proved that any class in $H^3(G, A)$ in fact comes from a crossed module in this way.

# 3. Chain complexes in algebraic topology

## 3.1 Simplicial methods

The use of simplexes to associate a chain complex to a topological space is an important method. In order to describe it properly, we first need to introduce some notation and some basic properties and constructions regarding simplicial sets and simplicial abelian groups.

**Definition 3.1.1** The category $\Delta$ has as objects all the finite non-empty ordered sets

$$[n] = \{0, 1, \ldots, n\} \qquad (n \geq 0)$$

and as arrows the monotone maps; i.e. arrows $\alpha\colon [n] \longrightarrow [m]$ are functions satisfying $\alpha(i) \leq \alpha(j)$ whenever $i \leq j$.

Amongst all maps in $\Delta$, a special role is played by the injective functions

$$\delta^i\colon [n-1] \longrightarrow [n] \quad \text{“omit } i\text{”} \quad (i = 0, \ldots, n)$$

whose image is the set $\{0, \ldots, \widehat{\imath}, \ldots, n\}$, with the number $i$ missing, and the surjective functions

$$\sigma^j\colon [n] \longrightarrow [n-1] \quad \text{“double } j\text{”} \quad (j = 0, \ldots, n-1)$$

which is injective everywhere except at $\sigma^j(j) = j = \sigma^j(j+1)$.

These functions, satisfy the following rules for composition:

$$\begin{cases} \delta^j\delta^i = \delta^i\delta^{j-1} & i < j \\ \sigma^j\sigma^i = \sigma^i\sigma^{j+1} & i \leq j \\ \sigma^j\delta^i = \delta^i\sigma^{j-1} & i < j \\ \sigma^i\delta^i = \mathsf{id} = \sigma^i\delta^{i+1} & \\ \sigma^j\delta^i = \delta^{i-1}\sigma^j & i > j+1. \end{cases} \tag{3.1}$$

Furthermore, every arrow $\alpha\colon [n] \longrightarrow [m]$ in $\Delta$ can be written as a composition of a surjection followed by an injection, and every surjection (resp. injection) is a composite of $\sigma$’s (resp. $\delta$’s). In this way, $\Delta$ is determined by the $\sigma$’s and $\delta$’s, together with the identities (3.1).

**Definition 3.1.2** A *simplicial set* is a functor

$$X\colon \Delta^{\mathrm{op}} \longrightarrow \mathsf{Set}.$$

A *map* $X \to Y$ between simplicial sets is a natural transformation. Together, these data define a category of simplicial sets.

It is common to write $X_n$ for the image $X([n])$, and $\alpha^*\colon X_m \longrightarrow X_n$ for $X(\alpha)$, where $\alpha\colon [n] \longrightarrow [m]$. By functoriality, it is trivial that any simplicial set $X$ is completely determined by its effect on objects (the sets $X_n$) and on the $\delta$'s and $\sigma$'s. One writes

$$\begin{array}{ll} d_i = X(\delta^i) = (\delta^i)^*\colon X_n \longrightarrow X_{n-1} & (i = 0, \ldots, n) \\ s_j = X(\sigma^j) = (\sigma^j)^*\colon X_{n-1} \longrightarrow X_n & (j = 0, \ldots, n-1). \end{array}$$

These operations satisfy identities dual to those in (3.1); the so-called *simplicial identities*:

$$\begin{cases} d_i d_j = d_{j-1} d_i & i < j \\ s_i s_j = s_{j+1} s_i & i \leq j \\ d_i s_j = s_{j-1} d_i & i < j \\ d_i s_i = \mathsf{id} = d_{i+1} s_i & \\ d_i s_j = s_j d_{i-1} & i > j+1. \end{cases} \tag{3.2}$$

**Example 3.1.3**

*a)* Let $S$ be a set. Define

$$S_n = S^{n+1} = S \times \cdots \times S = \mathrm{Hom}_{\mathsf{Set}}(\{0, \ldots, n\}, S),$$

and $\alpha^*\colon S_m \longrightarrow S_n$ by composition with $\alpha\colon \{0, \ldots, n\} \longrightarrow \{0, \ldots, m\}$. Then,

$$\begin{array}{rcl} d_i(s_0, \ldots, s_n) & = & (s_0, \ldots, \widehat{s_i}, \ldots, s_n) \\ s_j(s_0, \ldots, s_{n-1}) & = & (s_0, \ldots, s_j, s_j, \ldots, s_{n-1}). \end{array}$$

This construction is functorial, in the sense that a map of sets $f\colon S \longrightarrow T$ induces a simplicial map $S_* \longrightarrow T_*$.

*b)* Let $\mathsf{C}$ be a (small) category. The *nerve* of $\mathsf{C}$ is the simplicial set $N(\mathsf{C})$, defined by

$$N(\mathsf{C})_n = \{F \mid F\colon [n]^{\mathrm{op}} \longrightarrow \mathsf{C} \text{ is a functor }\},$$

where we view $[n]^{\mathrm{op}}$ as a category pictured as

$$0 \longleftarrow 1 \longleftarrow \cdots \longleftarrow n.$$

The operations $\alpha^*$ are again defined by composition. An element of $N(C)_n$ can be pictured as a string

$$C_0 \xleftarrow{f_1} C_1 \xleftarrow{f_2} \cdots \xleftarrow{f_n} C_n$$

and

$$d_i(C_0 \xleftarrow{f_1} \cdots \xleftarrow{f_n} C_n) = \begin{cases} C_1 \xleftarrow{f_2} \cdots \xleftarrow{f_n} C_n, & i = 0 \\ C_0 \leftarrow \cdots C_{i-1} \xleftarrow{f_i f_{i+1}} C_{i+1} \cdots \leftarrow C_n, & 0 < i < n \\ C_0 \xleftarrow{f_1} \cdots \xleftarrow{f_{n-1}} C_{n-1}, & i = n \end{cases}$$

while

$$s_i(C_0 \leftarrow \cdots \leftarrow C_n) = (C_0 \leftarrow \cdots \leftarrow C_i \xleftarrow{\mathsf{id}} C_i \leftarrow \cdots \leftarrow C_{n-1}).$$

Note that $N(\mathsf{C})_0 = |\mathsf{C}|$ and $N(\mathsf{C})_1$ is the set of arrows of $\mathsf{C}$. The construction is again functorial, in the sense that a functor $\phi\colon \mathsf{C} \longrightarrow \mathsf{D}$ induces a simplicial map $N(\mathsf{C}) \to N(\mathsf{D})$.
*c)* Let

$$\Delta^n = \{(x_0, \ldots x_n) \in \mathbb{R}^{n+1} \mid x_0 + \cdots + x_n = 1,\ 0 \le x_0, \ldots, x_n \le 1\}$$

be (a model for) the *standard Euclidean n-simplex.* It has $n+1$ *vertices*, $v_0, \ldots, v_n$, given by

$$v_i = (0, \ldots, 0, \overset{i}{1}, 0, \ldots, 0).$$

Each $\alpha\colon [n] \longrightarrow [m]$ induces a map

$$\alpha_*\colon \mathrm{Vertices}(\Delta^n) \longrightarrow \mathrm{Vertices}(\Delta^m)$$

by

$$\alpha_*(v_i) = v_{\alpha(i)}.$$

This map $\alpha_*$ extends uniquely to an *affine* map

$$\alpha_*\colon \Delta^n \longrightarrow \Delta^m$$

which makes $n \mapsto \Delta^n$ into a (covariant) functor from $\mathbf{\Delta}$ to topological spaces. For an arbitrary topological space $X$, define

$$\mathrm{Sing}(X)_n = \{f\colon \Delta^n \longrightarrow X \mid f \text{ continuous}\}$$

and $\alpha^*\colon \mathrm{Sing}(X)_m \longrightarrow \mathrm{Sing}(X)_n$ by composition with $\alpha_*$. This defines a simplicial set $\mathrm{Sing}(X)$, called the *singular complex* of $X$. The construction is functorial in $X$: any continuous map $X \to Y$ induces a simplicial map $\mathrm{Sing}(X) \to \mathrm{Sing}(Y)$.
*d)* Inside the category of simplicial sets, there are objects $\Delta[n]$ which play the rôle of *standard simplices.* They are defined by

$$\Delta[n]_k = \mathrm{Hom}_{\mathbf{\Delta}}([k], [n]).$$

Note that this is in fact a special case of Example *b*), because $\Delta[n] = N([n]^{\mathrm{op}})$. The Yoneda Lemma A.0.15 asserts that there is a bijective correspondence, for any simplicial set $X$, between maps $\Delta[n] \to X$ in the category of simplicial sets and elements of $X_n$:

$$\mathrm{Hom}(\Delta[n], X) \cong X_n.$$

For later use, we also remark that for $n = 1$, elements of the simplicial interval $\Delta[1]$ can be written as sequences of zeros and ones: $\alpha \in \Delta[1]_k$ is a sequence $\alpha(0), \ldots, \alpha(k)$ of the form $0, \ldots, 0, 1, \ldots, 1$ (and there are $k+2$ such sequences).

**Remark 3.1.4** Given two simplicial sets $X$ and $Y$, we can form their product $X \times Y$ in the evident way, as

$$(X \times Y)_n = X_n \times Y_n.$$

For a map $\alpha\colon [n] \longrightarrow [m]$, the simplicial operator $\alpha^*\colon (X \times Y)_m \longrightarrow (X \times Y)_n$ takes a pair $(x, y)$ to $(\alpha^* x, \alpha^* y)$. Notice that some of the constructions in Example 3.1.3 preserve products; for example,

$$\begin{aligned} N(\mathsf{C} \times \mathsf{D}) &= N(\mathsf{C}) \times N(\mathsf{D}) \\ \mathrm{Sing}(X \times Y) &= \mathrm{Sing}(X) \times \mathrm{Sing}(Y) \end{aligned}$$

for categories $\mathsf{C}$, $\mathsf{D}$ and spaces $X$, $Y$.

**Definition 3.1.5** A *simplicial abelian group* is a functor

$$A\colon \mathsf{\Delta}^{\mathrm{op}} \longrightarrow \mathsf{Ab}$$

into the category $\mathsf{Ab}$ of abelian groups. Maps between simplicial abelian groups are again defined as natural transformations.

**Remark 3.1.6** Any such simplicial abelian group $A$ gives rise to a (positive) chain complex

$$A_0 \xleftarrow{d} A_1 \xleftarrow{d} A_2 \leftarrow \dots$$

by defining the map $d\colon A_n \longrightarrow A_{n-1}$ as the alternating sum

$$d = \sum_{i=0}^{n} (-1)^i d_i.$$

The condition that $dd = 0$ follows immediately from the simplicial identity $d_i d_j = d_{j-1} d_i$ $(i < j)$. The homology of this complex is denoted by $H_*(A)$.

**Remark 3.1.7** If $X$ is a simplicial set, we can construct a simplicial abelian group $\mathbb{Z}[X]$ by taking the free abelian group on each of the sets $X_n$:

$$\mathbb{Z}[X]_n := \mathbb{Z}[X_n].$$

The homology of $\mathbb{Z}[X]$ is called the *simplicial homology* of $X$, and it is denoted

$$H_n(X).$$

(So, $H_n(X) = H_n(\mathbb{Z}[X])$ by definition.) If $X$ is a topological space, the homology of $\mathbb{Z}[\mathrm{Sing}(X)]$ is called the *singular homology* of the space $X$, and (again) it is simply denoted by $H_n(X)$.

## Exercises

*a)* For a topological space $X$, prove that $H_0(X)$ is the free abelian group $\mathbb{Z}[\pi_0(X)]$ on the set $\pi_0(X)$ of path-components of $X$;

*b)* Define the set of connected components $\pi_0(\mathsf{C})$ for a category $\mathsf{C}$, and prove a similar statement for $H_0(N(\mathsf{C}))$.

*c)* Let $X$ be a connected topological space, and let $x_0 \in X$ be a base-point. Let $\Omega(X, x_0)$ be the set of "loops at $x_0$", i.e. maps $\alpha\colon [0,1] \longrightarrow X$ such that $\alpha(0) = x_0 = \alpha(1)$. Recall that the fundamental group $\pi_1(X, x_0)$ is the quotient of $\Omega(X, x_0)$ obtained by identifying homotopic loops.

1. Show that the inclusion $\Omega(X, x_0) \subseteq \mathrm{Sing}(X)_1$ induces a well-defined group homomorphism $\pi_1(X, x_0) \to H_1(X)$;
2. For any group $G$, let $[G, G]$ be the normal subgroup generated by the commutators $ghg^{-1}h^{-1}$, and let $G^{ab} = G/[G, G]$ be the quotient group. Show that $G \mapsto G^{ab}$ defines a left adjoint to the inclusion of the category of abelian groups into that of all groups;
3. *(Poincaré isomorphism)* Show that the map $\pi_1(X, x_0)^{ab} \to H_1(X)$ is an isomorphism.

### 3.1.1 Simplicial homotopies

Recall that, for two maps $f, g\colon X \longrightarrow Y$ between topological spaces, a homotopy from $f$ to $g$ is a continuous map

$$H\colon X \times [0,1] \longrightarrow Y$$

with $H(x, 0) = f(x)$ and $H(x, 1) = g(x)$ (for all $x \in X$).

Amongst simplicial sets, the *standard simplex* $\Delta[1]$ of Example 3.1.3 *d)* plays the rôle of the "unit interval". The rôle of the point is played by $\Delta[0]$, and the two end points of $\Delta[1]$ are the maps

$$\delta^0, \delta^1\colon \Delta[0] \longrightarrow \Delta[1]$$

corresponding to the sequences 1 and 0 of length 1.

**Definition 3.1.8** A *simplicial homotopy* from a map $f\colon X_* \longrightarrow Y_*$ between simplicial sets to another one $g\colon X_* \longrightarrow Y_*$ is defined as a map

$$H\colon X \times \Delta[1] \longrightarrow Y$$

with the property that the diagram

$$\begin{tikzcd} X \times \Delta[0] \arrow[r, "\mathsf{id}\times\delta^1"] \arrow[d, "\cong"] & X \times \Delta[1] \arrow[d, "H"'] & X \times \Delta[0] \arrow[l, "\mathsf{id}\times\delta^0"'] \arrow[d, "\cong"] \\ X \arrow[r, "f"'] & Y & X \arrow[l, "g"] \end{tikzcd}$$

commutes. In other words, for any $x \in X_n$,

$$H(x, 0, \ldots, 0) = f(x) \quad \text{and} \quad H(x, 1, \ldots, 1) = g(x).$$

**Example 3.1.9**

*a)* If $F$ and $G$ are functors $\mathsf{C} \to \mathsf{D}$, then a natural transformation between $F$ and $G$ can be viewed as a functor $\tau\colon \mathsf{C}\times(0 \leftarrow 1) \longrightarrow \mathsf{D}$, hence gives a simplicial homotopy

$$N(\tau)\colon N(\mathsf{C}\times(0 \leftarrow 1)) = N(\mathsf{C})\times N(0 \leftarrow 1) = N(\mathsf{C})\times\Delta[1] \longrightarrow N(\mathsf{D})$$

between the maps $NF$ and $NG$.

*b)* For topological spaces $X$ and $Y$, let $H: X \times [0,1] \longrightarrow Y$ be a homotopy between two continuous maps $f, g: X \longrightarrow Y$. Then, we obtain a simplicial homotopy between $\mathrm{Sing}(f)$ and $\mathrm{Sing}(g)$ with the natural map of simplicial sets

$$\Delta[1] \to \mathrm{Sing}[0,1]$$

determined by the condition that is sends the identity $\mathsf{id} \in \Delta[1]_1$ to the identity in $\mathrm{Sing}([0,1])_1$.

The following theorem expresses one of the basic properties of singular homology, namely its *homotopy invariance*.

**Theorem 3.1.10** *Let $f, g: X \longrightarrow Y$ be two maps between simplicial sets. If there exists a simplicial homotopy between $f$ and $g$, then they have the same effect in homology; i.e.*

$$H_n(f) = H_n(g): H_n(X) \longrightarrow H_n(Y)$$

*for all $n \geq 0$.*

Notice that Example 3.1.9 *b*) above immediately gives the *homotopy invariance of singular homology* for topological spaces:

**Corollary 3.1.11** *Two homotopic maps $f, g: X \longrightarrow Y$ between topological spaces have the same effect in singular homology: $H_n(X) \to H_n(Y)$.*

We shall say that two topological spaces $X$ and $Y$ are *homotopy equivalent* if there are maps $f: X \longrightarrow Y$ and $g: Y \longrightarrow X$ such that $gf$ and $fg$ are homotopic to the identity (on $X$ and $Y$, respectively). A space $X$ is called *contractible* if it is homotopy equivalent to a one point space. It follows by Corollary 3.1.11 that $H_n(X) \cong H_n(Y)$ if $X$ and $Y$ are homotopy equivalent, and that $H_n(X) = 0$ for $n > 0$ (and $H_0(X) = \mathbb{Z}$) if $X$ is contractible.

**Proof of Theorem 3.1.10.** We will show that $H$ induces a chain homotopy $H'$ between the maps of chain complexes $\mathbb{Z}[X] \rightrightarrows \mathbb{Z}[Y]$ induced as in Remarks 3.1.6 and 3.1.7. The statement is then proved by Proposition 1.10.6. Define

$$H'_n: \mathbb{Z}[X_n] \longrightarrow \mathbb{Z}[Y_{n+1}]$$

on a generator $x \in X_n$ by

$$H'_n(x) = \sum_{i=0}^{n} (-1)^i H(s_i(x), 0, \ldots, \overset{i}{0}, 1, \ldots, 1).$$

Here, $(0, \ldots, \overset{i}{0}, 1, \ldots, 1) \in \Delta[1]_{n+1}$ is a sequence of length $n+2$, with zeros in places $0, \ldots, i$ and ones in places $i+1, \ldots, n+1$. (So, the sequences with zeros only or ones only do not occur in the formula for $H'_n(x)$.) One now calculates that

$$dH'_n(x) + H'_{n-1}(dx) = f(x) - g(x) \tag{3.3}$$

by spelling out the definitions. For $dH'_n(x)$, use the formulas for $d_j$ and $s_i$, and write

$$\begin{aligned}
dH'_n(x) &= \sum_{j=0}^{n+1}(-1)^j\sum_{i=0}^{n}(-1)^i H(d_j s_i(x), d_j(0,\ldots,\overset{i}{0},1,\ldots,1)) \\
&= \sum_{0\le j<i\le n}(-1)^{i+j}H(s_{i-1}d_j(x), d_j(0,\ldots,\overset{i}{0},1,\ldots,1)) \\
&\quad+\sum_{i=0}^{n}(-1)^{2i}H(x, d_i(0,\ldots,\overset{i}{0},1,\ldots,1)) \\
&\quad+\sum_{i=0}^{n}(-1)^{2i+1}H(x, d_{i+1}(0,\ldots,\overset{i}{0},1,\ldots,1)) \\
&\quad+\sum_{\substack{0\le i\\ i+1<j\le n+1}}(-1)^{i+j}H(s_i d_{j-1}(x), d_j(0,\ldots,\overset{i}{0},1,\ldots,1)).
\end{aligned}$$

On the other hand,

$$\begin{aligned}
H'_{n-1}(dx) &= \sum_{i=0}^{n-1}(-1)^i\sum_{j=0}^{n}(-1)^j H(s_i d_j(x), 0,\ldots,\overset{i}{0},1,\ldots,1) \\
&= \sum_{0\le j\le i\le n-1}(-1)^{i+j}H(s_i d_j(x),0,\ldots,\overset{i}{0},1,\ldots,1) \\
&\quad+\sum_{0\le i<j\le n}(-1)^{i+j}H(s_i d_j(x),0,\ldots,\overset{i}{0},1,\ldots,1).
\end{aligned}$$

Substituting $i+1$ for $i$ in the first term and $j+1$ for $j$ in the second, we see that $dH'_n(x) - H'_{n-1}(dx)$ is

$$\sum_{i=0}^{n}(H(x, d_i(0,\ldots,\overset{i}{0},1,\ldots,1)) - H(x, d_{i+1}(0,\ldots,\overset{i}{0},1,\ldots,1))).$$

Since $d_{i+1}(0,\ldots,\overset{i+1}{0},1,\ldots,1) = (0,\ldots,\overset{i}{0},1,\ldots,1) = d_{i+1}(0,\ldots,\overset{i}{0},1,\ldots,1)$, all terms cancel except $H(x,0,\ldots,0) - H(x,1,\ldots,1)$, and we obtain the desired formula, (3.3). This proves Theorem 3.1.10. □

## 3.2 The Mayer-Vietoris sequence

Another basic property of homology is the existence of a long exact sequence for the union of two simplicial sets or topological spaces. As we shall see, this property is quite easy to prove in the simplicial context, but much harder in the context of topological spaces.

In order to define the Mayer-Vietoris sequence, we first need to introduce the following:

**Definition 3.2.1** A *simplicial subset* of a simplicial set $X$ is a simplicial set $Y$ for which $Y_n \subseteq X_n$ for all $n \ge 0$, and for which the simplicial operators of $Y$ are the restrictions of those of $X$.

**Theorem 3.2.2 (Mayer-Vietoris)** *Let $X = Y \cup Z$ be the union of two simplicial subsets $Y$ and $Z$. Then, there is a long exact sequence*

$$\begin{aligned} &0 \longleftarrow H_0(X) \longleftarrow H_0(Y)\oplus H_0(Z) \longleftarrow H_0(Y \cap Z) \xleftarrow{\beta} H_1(X) \longleftarrow \dots \\ &\qquad \dots \longleftarrow H_i(X) \longleftarrow H_i(Y)\oplus H_i(Z) \longleftarrow H_i(Y \cap Z) \xleftarrow{\beta} H_{i+1}(X) \longleftarrow \dots . \end{aligned}$$

**Proof.** Consider for each $n$ the sequence

$$0 \longleftarrow \mathbb{Z}[X_n] \xleftarrow{p} \mathbb{Z}[Y_n]\oplus\mathbb{Z}[Z_n] \xleftarrow{i} \mathbb{Z}[Y_n \cap Z_n] \longleftarrow 0$$

where the maps are defined on generators by $i(y) = (y, -y)$ and $p(y) = y$, $p(z) = z$ (or $p(y, z) = y + z$, depending on notation). We claim that the sequence is exact. Clearly, $p \circ i = 0$, and $p$ is surjective, while $i$ is injective. To show the inclusion $\ker(p) \subseteq \mathrm{im}(i)$, consider an element $a \in ker(p)$, and write $a = (\sum_{i=1}^{k} k_i y_i, \sum_{j=1}^{l} l_j z_j)$, where $k \geq 0$, $k_i \neq 0$, and the $y_i \in Y_n$ are all different, and similarly for $l$, $l_j$, $z_j$. Then, if $p(a) = 0$, i.e. $\sum_i k_i y_i + \sum_j l_j z_j = 0$ in $\mathbb{Z}[X_n]$, each $y_i$ must be cancelled by exactly one $z_j$. So, changing the order of the $z_j$ and $l_j$ as necessary, we find $k = l$, $k_i = -l_i$ and $y_i = z_i$. So, $a$ belongs to the image of $i$.

The short exact sequences above, for all $n$, are compatible with the simplicial operators, hence induce a short exact sequence of chain complexes. The theorem now follows by Proposition 1.9.8. □

We would like to conlcude a similar fact for singular homology of topological spaces, but we cannot do so immediately, because if a space $X$ is the union of two subspaces $X = Y \cup Z$, then

$$\mathrm{Sing}(Y) \cup \mathrm{Sing}(Z) \subseteq \mathrm{Sing}(Y \cup Z) = \mathrm{Sing}(X) \tag{3.4}$$

is in general a strict inclusion of simplicial sets. With a bit of work, however, we will be able to deduce the following theorem.

**Theorem 3.2.3 (Mayer-Vietoris)** *Let $X$ be a topological space, and consider two open subspaces $U, V \subseteq X$ such that $X = U \cup V$. Then, there is a long exact sequence in singular homology*

$$\begin{aligned} &0 \longleftarrow H_0(X) \longleftarrow H_0(U)\oplus H_0(V) \longleftarrow H_0(U \cap V) \xleftarrow{\beta} H_1(X) \longleftarrow \dots \\ &\qquad \dots \longleftarrow H_i(X) \longleftarrow H_i(U)\oplus H_i(V) \longleftarrow H_i(U \cap V) \xleftarrow{\beta} H_{i+1}(X) \longleftarrow \dots . \end{aligned}$$

The maps in the sequence are all induced by the inclusions. More explicitly, denote these inclusions by $U \cap V \xrightarrow{i} U \xrightarrow{j} X$ and $U \cap V \xrightarrow{k} V \xrightarrow{l} X$. Then, for every $n$, the morphism $H_n(U \cap V) \to H_n(U)\oplus H_n(V)$ is induced by the map $\mathbb{Z}\,\mathrm{Sing}_n(U \cap V) \to \mathbb{Z}\,\mathrm{Sing}_n(U)\oplus\mathbb{Z}\,\mathrm{Sing}_n(V)$, sending a generator $x$ to $(i(x), -k(x))$, and likewise, the morphism $H_n(U)\oplus H_n(V) \to H_n(X)$ is induced by $\mathbb{Z}\,\mathrm{Sing}_n(U)\oplus\mathbb{Z}\,\mathrm{Sing}_n(V) \to \mathbb{Z}\,\mathrm{Sing}_n(X)$, which takes $(y, z)$ to $j(y) + l(z)$.

Together with the homotopy invariance property of Theorem 3.1.10, the Mayer-Vietoris sequence is an important tool for calculating homology groups. We give some examples before embarking on the proof of Theorem 3.2.3.

**Corollary 3.2.4** *Let $S^n = \{x \in \mathbb{R}^{n+1} \mid |x| = 1\}$ be the n-sphere. Then, for each $n > 0$ and each $i \geq 0$,*

$$H_i(S^n) = \begin{cases} \mathbb{Z} & i = 0, n \\ 0 & \textit{otherwise.} \end{cases}$$

**Proof.** Note first that for $n = 0$ the space $S^0$ is the union of two points, so $H_0(S^0) = \mathbb{Z}\oplus\mathbb{Z}$ while $H_i(S^0) = 0$ for $i > 0$. Also, $H_0(S^n) = \mathbb{Z}$ for $n > 0$, because $S^n$ is connected (this follows from Exercise Section 3.1-1), as can easily be checked). We now use induction on $n$. Consider, for $n > 0$ the north and south poles $N, S \in S^n$, and write $U = S^n - N$ and $V = S^n - S$. Then, $S^n = U \cup V$, while $U \cong \mathbb{R}^n \cong V$ and $U \cap V \cong \mathbb{R}^n - \{0\}$. This last space is homotopy equivalent to $S^{n-1}$ (check this directly!), so the Mayer-Vietoris sequence for $S^n = U \cup V$ takes the form

$$\begin{aligned} 0 \longleftarrow H_0(S^n) \longleftarrow H_0(\mathbb{R}^n)\oplus H_0(\mathbb{R}^n) \longleftarrow H_0(S^{n-1}) \dots \\ \dots \longleftarrow H_i(\mathbb{R}^n)\oplus H_i(\mathbb{R}^n) \longleftarrow H_i(S^{n-1}) \xleftarrow{\beta} H_{i+1}(S^n) \\ \longleftarrow H_{i+1}(\mathbb{R}^n)\oplus H_{i+1}(\mathbb{R}^n) \longleftarrow \dots. \end{aligned}$$

Since $\mathbb{R}^n$ is contractible, we have $H_i(\mathbb{R}^n) = 0$ for $i > 0$, so $H_{i+1}(S^n) \cong H_i(S^{n-1})$ for $i > 0$. This gives the formula by induction on $n$, once we have proved it for $S^1$. For $n = 1$, the beginning of the sequence looks like

$$0 \longleftarrow \mathbb{Z} \xleftarrow{p} \mathbb{Z}\oplus\mathbb{Z} \xleftarrow{w} \mathbb{Z}\oplus\mathbb{Z} \longleftarrow H_1(S^1) \longleftarrow 0,$$

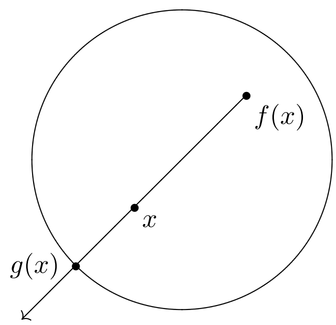
where $p(y, z) = y + z$ and $w(x_1, x_2) = (x_1, -x_1) + (x_2, -x_2)$. So, $w$ is not injective but has kernel $\mathbb{Z}$. $\square$

**Corollary 3.2.5 (Brouwer fixed point theorem)** *Let $B^n$ be the n-disk:*

$$B^n = \{x \in \mathbb{R}^n : |x| \leq 1\}.$$

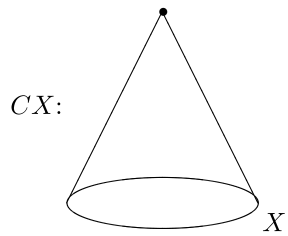
*Every continuous map $f\colon B^n \longrightarrow B^n$ has a fixed point, i.e. a point $x \in B^n$ with $f(x) = x$.*

**Proof.** Suppose $f(x) \neq x$ for every $x$. Then, define a map $g\colon B^n \longrightarrow S^{n-1}$, where $S^{n-1}$ is viewed as the boundary of $B^n$, as follows: for a given point $x$, draw a half-line starting at $f(x)$ and in the direction of $x$, and let $g(x)$ be the point where this half-line hits the boundary of the disk.

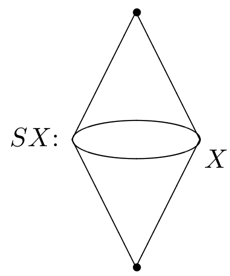


It is easy to check that $g$ is a continuous map. Moreover, $g(x) = x$ if $x$ itself already lies on the boundary. It follows that $g$ is a *retraction* of $B^n$ onto $S^{n-1}$; or, in other words, $g \circ i = \mathsf{id}_{S^{n-1}}$, where $i\colon S^{n-1} \longrightarrow B^n$ is the inclusion. But then, $H_k(S^{n-1})$ is also a retract of $H_k(B^n)$, for each $k > 0$. However, for $k = n-1 > 0$, we have $H_k(S^{n-1}) = \mathbb{Z}$ and $H_k(B^n) = 0$, which is a contradiction. Similarly, if $k = n-1 = 0$, we have $H_k(S^{n-1}) = \mathbb{Z} \oplus \mathbb{Z}$, and $H_k(B^n) = \mathbb{Z}$, again leading to a contradiction. Finally, for $n = 0$ there is nothing to prove. $\square$

Now, consider a topological space $X$. The *cone* on $X$ is the space $CX$ obtained from $X \times [0,1]$ by identifying $X \times \{1\}$ to a point:




The *suspension* $SX$ of $X$ is the space obtained from $X \times [-1,1]$ by identifying $X \times \{-1\}$ to a point and $X \times \{1\}$ to another one. Thus, $SX$ is the union of two cones:




**Corollary 3.2.6 (Freudenthal suspension theorem)** *For every connected space $X$, there is a natural isomorphism*

$$H_n(X) \xrightarrow{\cong} H_{n+1}(SX).$$

This is proved by applying the Mayer-Vietoris sequence to the open cover $SX = U \cup V$, where $U$ is the image of $X \times [-1, \varepsilon)$ and $V$ that of $X \times (-\varepsilon, 1]$ (for some $\varepsilon > 0$). Thus, $U$ is homotopy equivalent to $CX$, as it is $V$, while $U \cap V$ is homotopy equivalent to $X$. Now, one uses that $CX$ is contractible. Further details are left as an exercise (What happens if $X$ is not connected?).

We now start to work towards the proof of Theorem 3.2.3. Recall that the singular homology of a topological space $X$ is defined by means of the complex $\mathbb{Z}[\mathrm{Sing}(X)]$. In topology, one usually writes

$$C_n(X) = \mathbb{Z}[\mathrm{Sing}_n(X)]$$

and calls the elements of $C_n(X)$ *singular $n$-chains* on $X$. The problem noted just above Theorem 3.2.3 can thus be phrased by saying that the sequence

$$0 \longleftarrow C_n(X) \xleftarrow{\;p\;} C_n(U) \oplus C_n(V) \xleftarrow{\;i\;} C_n(U \cap V) \longleftarrow 0 \qquad (3.5)$$

is in general not exact. (compared to (3.4), we have replaced $Y$ by $U$ and $Z$ by $V$ here, and $X = U \cup V$ as in Theorem 3.2.3.) We will force it to be exact by changing $C_n(X)$ a little, without changing its homology.

The generators of $C_n(X)$ are the maps $\alpha: \Delta^n \longrightarrow X$. Call such an $\alpha$ *small* (relative to the open cover $X = U \cup V$) if the image of $\alpha$ is contained either in $U$ or in $V$. Write

$$C'_n(X) \subseteq C_n(X)$$

for the (free) abelian subgroup generated by these small maps $\Delta^n \to X$. Since the boundary $d: C_n(X) \longrightarrow C_{n-1}(X)$ obviously maps small generators to small chains (elements of $C'_n(X)$), the $C'_n(X)$ together form a subcomplex of $C(X)$. (In fact, the small $\alpha$ themselves form a simplicial subset of $\mathrm{Sing}(X)$.)

Now, the following modification of the sequence (3.5),

$$0 \longleftarrow C'_n(X) \xleftarrow{\ p\ } C_n(U) \oplus C_n(V) \xleftarrow{\ i\ } C_n(U \cap V) \longleftarrow 0$$

is evidently exact, and induces a long exact sequence in homology, of the form

$$\ldots \leftarrow H'_n(X) \leftarrow H_n(U) \oplus H_n(V) \leftarrow H_n(U \cap V) \xleftarrow{\beta} H'_{n+1}(X) \leftarrow \ldots$$

This is the desired Mayer-Vietoris sequence, except that $H_*(X)$ has been replaced by the homology $H'_*(X)$ of the small chain complex $C'(X)$. The following proposition thus suffices to prove Theorem 3.2.3.

**Proposition 3.2.7** *The inclusion of chain complexes $C'(X) \to C(X)$ induces an isomorphism in homology.*

We shall prove the result by showing how cocycles in $C(X)$ can be "subdivided" in such a way that they become small, without changing their homology class. We shall do this by means of the so-called *cone construction.*

If $K \subseteq \mathbb{R}^n$ is a convex set and $B \in K$ is a chosen point, we can construct a group homomorphism

$$\mathit{Cone}_B: C_n(K) \longrightarrow C_{n+1}(K) \qquad (\text{all } n \geq 0)$$

by defining, for a generator $\alpha: \Delta^n \longrightarrow K$, the map $\mathit{Cone}_B(\alpha): \Delta^{n+1} \longrightarrow K$ by

$$(t_0, \ldots, t_{n+1}) \mapsto \begin{cases} B & \text{if } t_0 = 1 \\ (1-t_0)\alpha((1-t_0)^{-1}t_1, \ldots, (1-t_0)^{-1}t_{n+1}) + t_0 B & \text{if } t_0 < 1. \end{cases}$$

In other words, $\mathit{Cone}_B(\alpha)$ sends the 0-th vertex of $\Delta^{n+1}$ to $B$, the face opposite to the 0-th vertex is mapped by $\alpha$, and $\mathit{Cone}_B(\alpha)$ is defined by *convex combination* on lines in $\Delta^{n+1}$ connecting the 0-th vertex to the opposite face. We observe the following evident formula (Exercise: check this!):

$$d(\mathit{Cone}_B(\alpha)) = \alpha - \mathit{Cone}_B(d\alpha). \tag{3.6}$$

By induction on $n$, we will now define the *barycentric subdivision*

$$bs = bs_n^X: C_n(X) \longrightarrow C_n(X)$$

for an arbitrary space $X$, and natural in $X$, in the sense that each map $f\colon X \longrightarrow Y$ induces a commutative diagram

$$\begin{tikzcd} C_n(X) \arrow[r, "bs_n^X"] \arrow[d, "f_*"'] & C_n(X) \arrow[d, "f_*"] \\ C_n(Y) \arrow[r, "bs_n^Y"'] & C_n(Y). \end{tikzcd}$$

In particular, if we take for $f$ a generator $\alpha\colon \Delta^n \longrightarrow X$ of $C_n(X)$, we find

$$\begin{tikzcd} C_n(\Delta^n) \arrow[r, "bs_n^{\Delta^n}"] \arrow[d, "\alpha_*"'] & C_n(\Delta^n) \arrow[d, "\alpha_*"] \\ C_n(X) \arrow[r, "bs_n^X"'] & C_n(X). \end{tikzcd}$$

The identity $\mathsf{id}_n\colon \Delta^n \longrightarrow \Delta^n$ can be viewed as an element of $C_n(\Delta^n)$, and $\alpha_*(\mathsf{id}_n) = \alpha$. So,

$$bs_n^X(\alpha) = \alpha_* bs_n^{\Delta^n}(\mathsf{id}_n). \tag{3.7}$$

Thus, naturality implies that the $bs_n^X$ for all spaces are completely determined by $bs_n^{\Delta^n}(\mathsf{id}_n)$.

Using this observation, we will now define $bs_n^X$ by induction on $n$. We start the induction by

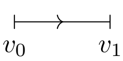
$$bs_0^{\Delta^0}(\mathsf{id}_0) = \mathsf{id}_0\colon \Delta^0 \longrightarrow \Delta^0.$$

This determines $bs_0^X$ for all $X$, as said. Suppose $bs_{n-1}^X$ has been defined for all $X$ and all $\alpha$. Then, set

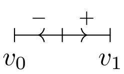
$$bs_n^{\Delta^n}(\mathsf{id}_n) = Cone_{B_n}(bs_{n-1}^{\Delta^n}(d(\mathsf{id}_n)))$$

where $B_n \in \Delta^n$ is the barycenter of the $n$-simplex. By (3.7), this then defines $bs_n^X(\alpha)$ for any $\alpha\colon \Delta^n \longrightarrow X$. The pictures for $n = 1, 2$ are:

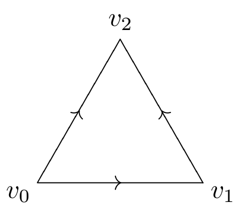
$\mathsf{id}_1$:



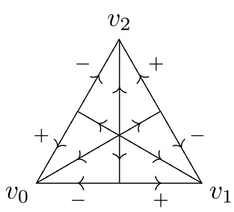

$bs(\mathsf{id}_1)$:




$\mathsf{id}_2$:




$bs(\mathsf{id}_2)$:




**Lemma 3.2.8** *$bs^X\colon C(X) \longrightarrow C(X)$ is a chain map.*

**Proof.** We have to prove that $d\ bs_n = bs_{n-1}d$ for each $n \geq 0$. (The reader should check this first for $n = 1, 2$ on the basis of the pictures above.) We prove by induction on $n$. We omit the superscript $X$ on $bs_n^X$. Take $\alpha\colon \Delta^n \longrightarrow X$. Then,

$$\begin{array}{rcll} d\, bs_n(\alpha) & = & d\, \alpha_* bs_n(\mathsf{id}_n) & \text{(cf (3.7))} \\ & = & \alpha_*(d\, bs_n(\mathsf{id}_n)) & \\ & = & \alpha_* d\, Cone_{B_n}(bs_{n-1}(d\, \mathsf{id}_n)) & \end{array}$$

just by spelling out the definition. Now, write $d\ \mathsf{id}_n = \sum_{j=0}^n (-1)^j \delta^j$, where $\delta^j\colon \Delta^{n-1} \longrightarrow \Delta^n$, so that we can write

$$\begin{array}{rcl} d\, bs_n(\alpha) & = & \alpha_*(\sum_{j=0}^n d\, Cone_{B_n}(bs_{n-1}\delta^j)) \\ & = & \alpha_*(\sum_{j=0}^n (-1)^j bs_{n-1}(\delta^j)) - \alpha_* \sum_{j=0}^n (-1)^j Cone_{B_n}(d\, bs_{n-1}(\delta^j)), \end{array}$$

where the second equality follows from (3.6). The first term here is

$$\sum_{j=0}^n (-1)^j bs_{n-1}(\alpha \circ \delta^j) = bs_{n-1}(d\, \alpha),$$

while, by induction hypothesis, the second is $\alpha_*(Cone\ bs_{n-2}(dd(\mathsf{id}_n))) = 0$. So, $d\, bs_n(\alpha) = bs_{n-1}(d\, \alpha)$, as desired. $\square$

**Lemma 3.2.9** *For any topological space $X$, the chain map $bs^X\colon C(X) \longrightarrow C(X)$ is homotopic to the identity, by a homotopy which is natural in $X$.*

More explicitly, for every $X$ and every $n \geq 0$, there is a homomorphism

$$R_n^X\colon C_n(X) \longrightarrow C_{n+1}(X)$$

for which

$$bs_n^X(\alpha) - \alpha = d\, R_n^X(\alpha) + R_{n-1}^X(d\, \alpha) \tag{3.8}$$

for every $\alpha \in C_n(X)$. Naturality in $X$ is expressed by the commutativity of the diagram

$$\begin{tikzcd} C_n(X) \arrow[r, "R_n^X"] \arrow[d, "f_*"'] & C_{n+1}(X) \arrow[d, "f_*"] \\ C_n(Y) \arrow[r, "R_n^Y"'] & C_{n+1}(Y) \end{tikzcd}$$

for every map $f\colon X \longrightarrow Y$.

**Proof.** Just like for the maps $bs_n^X$, naturality implies that the maps $R_n^X$ for all $X$ are determined completely by

$$R_n^{\Delta^n}(\mathsf{id}_n) \in C_{n+1}(\Delta^n).$$

We define $R_n^X$ by induction on $n$. For $n = 0$ there is only one choice for $R_0^{\Delta^0}(\mathsf{id}_0)\colon \Delta^1 \longrightarrow \Delta^0$, and this determines all $R_0^X$. Suppose $R_n^X$ has been defined for all $X$, and satisfies the homotopy relation (3.8). Consider

$$\alpha = bs_{n+1}^{\Delta^{n+1}}(\mathsf{id}_{n+1}) - \mathsf{id}_{n+1} - R_n^{\Delta^{n+1}}(d\ \mathsf{id}_{n+1}) \in C_{n+1}(\Delta^{n+1}).$$

We want to choose an element $\beta = R_{n+1}^{\Delta^{n+1}}(\mathsf{id}_{n+1})$ with $d\,\beta = \alpha$. Since $\Delta^{n+1}$ is contractible, $H_{n+1}(\Delta^{n+1}) = 0$, so such a $\beta$ will exist if we show that $d\,\alpha = 0$. But, using the homotopy relation (3.8) in degrees $n$ and $n-1$ and Lemma 3.2.8, we find

$$\begin{aligned} d\,\alpha &= bs_n(d(\mathsf{id}_{n+1})) - d(\mathsf{id}_{n+1}) - dR_n(d(\mathsf{id}_{n+1})) \\ &= R_{n-1}(dd\,\mathsf{id}_n) \\ &= 0. \end{aligned}$$

Thus, there is a $\beta$ with $d\,\beta = 0$, and we set $R_{n+1}^{\Delta^{n+1}}(\mathsf{id}_{n+1}) = \beta$, which determines $R_{n+1}^X$ for all $X$, in a way for which the homotopy relation (3.8) holds in degrees $n+1$, $n$. □

**Proof of Proposition 3.2.7.** Recall that this proposition states that the inclusion $C'(X) \to C(X)$ induces an isomorphism $H_n'(X) \to H_n(X)$ for every $n \geq 0$.

First, note that if $\alpha\colon \Delta^n \longrightarrow X$ is a generator of $C_n(X)$, then, after sufficiently many repetitions of the barycentric subdivision operation $bs$, we obtain an element $bs^m(\alpha) = bs \circ \cdots \circ bs(\alpha) \in C_n'(X)$. Indeed, for $m$ large enough, the diameter of the simplices in $bs^m(\mathsf{id}_n)$ becomes arbitrarily small, hence each of these simplices must lie entirely in $\alpha^{-1}(U)$ or entirely in $\alpha^{-1}(V)$ (by compactness of $\Delta^n$: see the following exercise).

Now, we show that the map $H_n'(X) \to H_n(X)$ is surjective. Take $\alpha \in C_n(X)$ with $d\,\alpha = 0$, so that $\alpha$ represents an element $[\alpha] \in H_n(X)$. By Lemma 3.2.9, the same class $[\alpha]$ is also represented by $bs(\alpha)$, and by $bs(bs(\alpha))$, and, in general by $bs^m(\alpha)$ for every $m$. But $bs^m(\alpha)$ represents an elements of $H_n'(X)$, if $m$ is large enough. This shows surjectivity.

Injectivity of $H_n'(X) \to H_n(X)$ is proved similarly: if $\alpha \in C_n'(X)$ represents a class $[\alpha]$ in $H_n'(X)$ which becomes zero in $H_n(X)$, then $\alpha$ is small, $d\,\alpha = 0$, and $\alpha = d\,\beta$ for a not necessarily small $\beta \in C_{n+1}(X)$. For a large enough $m$, however, $bs^m(\beta)$ will be small, and $d(bs^m(\beta)) = bs^m(\alpha)$, so $[bs^m(\alpha)] = 0$ in $H_n'(X)$. But $[bs^m(\alpha)] = [\alpha]$ in $H_n'(X)$, because the homotopy $R$ maps small chains to small chains. This closes the proof. □

## Exercise

*a)* Let $X$ be a compact metric space.

1. Show that every descending chain of non-empty closed subsets of $X$

$$C_0 \supseteq C_1 \supseteq C_2 \supseteq \ldots$$

has a non-empty intersection.

2. Use the previous observation to show that every sequence $(x_n)_{n\in\mathbb{N}}$ in $X$ has a converging subsequence.

3. Finally, conclude that, for every finite open cover $U_i$ $(i = 1, \ldots, n)$ of $X$, there is an $\epsilon > 0$ such that, for any $x \in X$, the ball $B(x, \epsilon)$ is contained in some $U_i$.

## 3.3 The Eilenberg-Zilber isomorphism

Let $X$ and $Y$ be simplicial sets. Recall the product $X \times Y$ is the simplicial set given by $(X \times Y)_n = X_n \times Y_n$ and $\alpha^*(x, y) = (\alpha^*(x), \alpha^*(y))$. We will denote the chain complex computing the homology of this product by $\mathbb{Z}[X \times Y]$, its boundary maps are determined by

$$d(x, y) = \sum_i (-1)^i (d_i(x), d_i(y)).$$

Let $C$ be the double chain complex given by $C_{p,q} = \mathbb{Z}[X_p] \otimes \mathbb{Z}[Y_q]$, with boundary maps

$$d \otimes \mathsf{id}\colon C_{p,q} \longrightarrow C_{p-1,q}, \qquad (-1)^p \mathsf{id} \otimes d\colon C_{p,q} \longrightarrow C_{p,q-1}.$$

The total complex of $C$ will be denoted by $\mathbb{Z}[X] \otimes \mathbb{Z}[Y]$.

Now let $f\colon X \longrightarrow Y$ be a simplicial map, i.e., a natural transformation $X \longrightarrow Y$. Then the induced morphisms $f_n\colon \mathbb{Z}[X_n] \longrightarrow \mathbb{Z}[Y_n]$ commute with the boundary maps since $f$ commutes with the $d_i$'s, so they form a chain map $\mathbb{Z}[f]\colon \mathbb{Z}[X] \longrightarrow \mathbb{Z}[Y]$. Similarly, two simplicial maps $f\colon X_1 \longrightarrow X_2$ and $g\colon Y_1 \longrightarrow Y_2$ induce chain maps

$$\mathbb{Z}[f \times g]\colon \mathbb{Z}[X_1 \times Y_1] \longrightarrow \mathbb{Z}[X_2 \times Y_2]$$

and

$$\mathbb{Z}[f] \otimes \mathbb{Z}[g]\colon \mathbb{Z}[X_1] \otimes \mathbb{Z}[Y_1] \longrightarrow \mathbb{Z}[X_2] \otimes \mathbb{Z}[Y_2].$$

Thus, the assignments

$$(X, Y) \mapsto \mathbb{Z}[X \times Y] \quad \text{and} \quad (X, Y) \mapsto \mathbb{Z}[X] \otimes \mathbb{Z}[Y]$$

determine functors from the category of tuples of simplicial sets to the category of positive chain complexes $\mathsf{Ch}_+(\mathbb{Z})$, which we will denote by $\mathbb{Z}[- \times -]$ and $\mathbb{Z}[-] \otimes \mathbb{Z}[-]$ respectively.

In this section we will prove the following.

**Theorem 3.3.1 (Eilenberg-Zilber)** *For any two simplicial sets $X$ and $Y$ there is a homotopy equivalence*

$$\mathbb{Z}[X] \otimes \mathbb{Z}[Y] \simeq \mathbb{Z}[X \times Y].$$

*Hence the homologies $H_*(\mathbb{Z}[X] \otimes \mathbb{Z}[Y])$ and $H_*(X \times Y)$ are isomorphic.*

We prove this theorem by what is called the *method of acyclic models.*

**Proposition 3.3.2** *The standard simplices $\Delta[n]$ and their products $\Delta[n] \times \Delta[m]$ are contractible: the homology groups $H_i(\Delta[n])$ and $H_i(\Delta[n] \times \Delta[m])$ vanish for all $i > 0$ and $H_0(\Delta[n])$ and $H_0(\Delta[n] \times \Delta[m])$ are both isomorphic to $\mathbb{Z}$.*

**Proof.** See additional exercise 2 at the end of this chapter. $\square$

Let $X$ be a non-empty simplicial set. Let $\varepsilon\colon \mathbb{Z}[X_0] \longrightarrow \mathbb{Z}$ be the morphism that is determined by mapping each generator of $\mathbb{Z}[X_0]$ to 1. Clearly $\varepsilon d = 0$ holds, so

$$0 \longleftarrow \mathbb{Z} \xleftarrow{\varepsilon} \mathbb{Z}[X_0] \xleftarrow{d} \mathbb{Z}[X_1] \longleftarrow \cdots$$

is a chain complex, called the *augmented* chain complex of $X$. Its homology groups are denoted by $\tilde{H}_*(X)$; they form the *reduced homology* of $X$. Note that $\tilde{H}_{-1}(X)$ is the zero group since $X$ is assumed to be non-empty.

**Proposition 3.3.3** *The homology groups $\tilde{H}_i(X)$ and $H_i(X)$ are equal for $i > 0$. Furthermore, $H_0(X) \simeq \tilde{H}_0(X) \oplus \mathbb{Z}$.*

**Proof.** To prove the second statement, consider the following short exact sequence.

$$0 \longrightarrow \ker \varepsilon \longrightarrow \mathbb{Z}[X_0] \xrightarrow{\varepsilon} \mathbb{Z} \longrightarrow 0$$

This sequence splits: any chosen simplex $x \in X_0$ induces a section of $\varepsilon$ determined by $n \mapsto nx$. Hence $\mathbb{Z}[X_0] \simeq \ker \varepsilon \oplus \mathbb{Z}$, recall that this isomorphism is determined by $s \mapsto (s - \varepsilon(s)x, \varepsilon(s))$ and $(s, n) \mapsto s + nx$. Now note that the boundary map $d\colon \mathbb{Z}[X_1] \longrightarrow \mathbb{Z}[X_0]$ rewrites as $s \mapsto (d(s), 0)$ under this isomorphism, and conclude

$$H_0(X) \simeq (\ker \varepsilon \oplus \mathbb{Z}) / \operatorname{im} d \simeq (\ker \varepsilon / \operatorname{im} d) \oplus \mathbb{Z} = \tilde{H}_0(X) \oplus \mathbb{Z}.$$

□

Thus, the reduced homologies of the simplicial sets $\Delta[n]$ and $\Delta[n] \times \Delta[m]$ are both zero in all degrees.

We now turn to the chain complex $\mathbb{Z}[X] \otimes \mathbb{Z}[Y]$. It can be augmented in a similar way by mapping every generator $x \otimes y$ of $\mathbb{Z}[X_0] \otimes \mathbb{Z}[Y_0]$ to 1, giving the following chain complex.

$$0 \longleftarrow \mathbb{Z} \xleftarrow{\varepsilon} \mathbb{Z}[X_0] \otimes \mathbb{Z}[Y_0] \xleftarrow{D} \mathbb{Z}[X_1] \otimes \mathbb{Z}[Y_0] \oplus \mathbb{Z}[X_0] \otimes \mathbb{Z}[Y_1] \longleftarrow \cdots$$

The homology of this complex is again called the reduced homology of $\mathbb{Z}[X] \otimes \mathbb{Z}[Y]$ and its homology groups are denoted by $\tilde{H}_*(\mathbb{Z}[X] \otimes \mathbb{Z}[Y])$. A proof similar to the one above shows that, whenever both $X$ and $Y$ are non-empty,

$$H_i(\mathbb{Z}[X] \otimes \mathbb{Z}[Y]) \simeq \begin{cases} \tilde{H}_i(\mathbb{Z}[X] \otimes \mathbb{Z}[Y]) \oplus \mathbb{Z} & \text{if } i = 0, \\ \tilde{H}_i(\mathbb{Z}[X] \otimes \mathbb{Z}[Y]) & \text{otherwise.} \end{cases}$$

To compute the ordinary homology of the complex $\mathbb{Z}[X] \otimes \mathbb{Z}[Y]$ in case $X = \Delta[n]$ and $Y = \Delta[m]$, consider the following augmentation of the double complex $(\mathbb{Z}[\Delta[n]_p] \otimes \mathbb{Z}[\Delta[m]_q])$.

$$\begin{tikzcd}
 & \mathbb{Z}[\Delta[n]_0] \otimes \mathbb{Z} & \mathbb{Z}[\Delta[n]_1] \otimes \mathbb{Z} \arrow[l, "d\otimes\mathsf{id}"'] & \cdots \arrow[l] \\
\mathbb{Z} \otimes \mathbb{Z}[\Delta[m]_0] & \mathbb{Z}[\Delta[n]_0] \otimes \mathbb{Z}[\Delta[m]_0] \arrow[l, "\varepsilon\otimes\mathsf{id}"'] \arrow[u, "\mathsf{id}\otimes\varepsilon"] & \mathbb{Z}[\Delta[n]_1] \otimes \mathbb{Z}[\Delta[m]_0] \arrow[l, "d\otimes\mathsf{id}"'] \arrow[u, "-\mathsf{id}\otimes\varepsilon"'] & \cdots \arrow[l] \\
\mathbb{Z} \otimes \mathbb{Z}[\Delta[m]_1] \arrow[u, "-\mathsf{id}\otimes d"] & \mathbb{Z}[\Delta[n]_0] \otimes \mathbb{Z}[\Delta[m]_1] \arrow[l, "\varepsilon\otimes\mathsf{id}"'] \arrow[u, "\mathsf{id}\otimes d"] & \mathbb{Z}[\Delta[n]_1] \otimes \mathbb{Z}[\Delta[m]_1] \arrow[l, "d\otimes\mathsf{id}"'] \arrow[u, "-\mathsf{id}\otimes d"'] & \cdots \arrow[l] \\
\vdots \arrow[u] & \vdots \arrow[u] & \vdots \arrow[u] &
\end{tikzcd}$$

Since the rows in the degrees $\geq 0$ are exact, one can translate the proof (or rather its dual) of the Double Complex Lemma (1.11.5) to a proof showing that the homology groups of $\mathbb{Z}[\Delta[n]] \otimes \mathbb{Z}[\Delta[m]]$ are isomorphic to those of

$$0 \longleftarrow \mathbb{Z}[\Delta[n]_0] \otimes \mathbb{Z} \xleftarrow{d\otimes\mathsf{id}} \mathbb{Z}[\Delta[n]_1] \otimes \mathbb{Z} \xleftarrow{d\otimes\mathsf{id}} \mathbb{Z}[\Delta[n]_2] \otimes \mathbb{Z} \longleftarrow \cdots.$$

Clearly this chain complex is isomorphic to $\mathbb{Z}[\Delta[n]]$ and therefore

$$H_i(\mathbb{Z}[\Delta[n]] \otimes \mathbb{Z}[\Delta[m]]) \simeq H_i(\mathbb{Z}[\Delta[n]]) \simeq \begin{cases} \mathbb{Z} & \text{if } i = 0, \\ 0 & \text{otherwise,} \end{cases}$$

which shows that the reduced homology of $\mathbb{Z}[\Delta[n]] \otimes \mathbb{Z}[\Delta[m]]$ vanishes.

The other fact we need is the following. By applying the Yoneda Lemma (A.0.15) on a simplicial set $X$, we find that every $n$-simplex $x \in X_n$ corresponds bijectively to the simplicial map $\tilde{x}\colon \Delta[n] \longrightarrow X$ given by $\tilde{x}(\alpha) = \alpha^* x$. This is useful in the following kind of situation. Suppose $\phi$ is a chain map $\mathbb{Z}[X] \otimes \mathbb{Z}[Y] \longrightarrow \mathbb{Z}[X \times Y]$, natural in both $X$ and $Y$. In particular, for any two simplices $x \in X_p$ and $y \in Y_q$ the diagram below commutes for every $n$.

$$\begin{tikzcd}
\mathbb{Z}[X] \otimes \mathbb{Z}[Y]_n \arrow[r, "\phi_n"] & \mathbb{Z}[X \times Y]_n \\
\mathbb{Z}[\Delta[p]] \otimes \mathbb{Z}[\Delta[q]]_n \arrow[r, "\phi_n"'] \arrow[u, "\mathbb{Z}[\tilde{x}] \otimes \mathbb{Z}[\tilde{y}]_n"] & \mathbb{Z}[\Delta[p] \times \Delta[q]]_n \arrow[u, "\mathbb{Z}[\tilde{x} \times \tilde{y}]_n"']
\end{tikzcd}$$

Taking $n = p + q$, the simplex $\mathsf{id}_{[p]} \otimes \mathsf{id}_{[q]}$ is mapped to $x \otimes y$ by $\mathbb{Z}[\tilde{x}] \otimes \mathbb{Z}[\tilde{y}]_n$, hence

$$\phi_n(x \otimes y) = \mathbb{Z}[\tilde{x} \times \tilde{y}]_n \circ \phi_n(\mathsf{id}_{[p]} \otimes \mathsf{id}_{[q]}).$$

This shows that $\phi_n$, for all $X$ and $Y$, is completely determined by its images of the simplices $\mathsf{id}_{[p]} \otimes \mathsf{id}_{[q]}$ in $\mathbb{Z}[\Delta[p]] \otimes \mathbb{Z}[\Delta[q]]_n$, for all $p, q \geq 0$ with $p + q = n$.

We are now ready to prove the theorem.

*Proof of Theorem 3.3.1.* We will define chain maps

$$\mathbb{Z}[X] \otimes \mathbb{Z}[Y] \underset{\psi}{\overset{\phi}{\rightleftarrows}} \mathbb{Z}[X \times Y]$$

which are each other's inverse up to homotopy: we will also define homotopies

$$h\colon \mathbb{Z}[X] \otimes \mathbb{Z}[Y] \longrightarrow \mathbb{Z}[X] \otimes \mathbb{Z}[Y] \quad \text{and} \quad k\colon \mathbb{Z}[X \times Y] \longrightarrow \mathbb{Z}[X \times Y]$$

such that

$$\psi\phi - \mathsf{id} = Dh + hD \quad \text{and} \quad \phi\psi - \mathsf{id} = dk + kd. \tag{3.9}$$

The idea is to first define $\phi$ and $\psi$ between the corresponding augmented chain complexes $\mathbb{Z} \xleftarrow{\varepsilon} \mathbb{Z}[X] \otimes \mathbb{Z}[Y]$ and $\mathbb{Z} \xleftarrow{\varepsilon} \mathbb{Z}[X \times Y]$, together with homotopies $h$ and $k$ such that the equations above hold. This will be done by induction on the degree, for all $X$ and $Y$ at the same time, and naturally in $X$ and $Y$. Notice that to define $\phi_n$ it suffices, by the remark preceding this proof, to define the images of the simplices $\mathsf{id}_{[p]} \otimes \mathsf{id}_{[q]}$ under $\phi_n$ in case $X = \Delta[p]$ and $Y = \Delta[q]$, for all $p, q \geq 0$ with $p + q = n$. Similarly, $\psi$, $h$ and $k$ are also completely determined by their components in special cases where $X$ and $Y$ are standard simplices. We can thus restrict the definitions to these cases, and make use of the fact that for such $X$ and $Y$ the reduced homologies of the chain complexes $\mathbb{Z}[X \times Y]$ and $\mathbb{Z}[X] \otimes \mathbb{Z}[Y]$ vanish. It will be clear that afterwards, by replacing $\phi_{-1}$, $\psi_{-1}$, $h_{-1}$ and $k_{-1}$ with zero maps, we can obtain the chain maps between $\mathbb{Z}[X] \otimes \mathbb{Z}[Y]$ and $\mathbb{Z}[X \times Y]$ that we seek, together with the homotopies.

First of all, note that there is a canonical identification

$$\mathbb{Z}[X_0] \otimes \mathbb{Z}[Y_0] \simeq \mathbb{Z}[X_0 \times Y_0],$$

so we can define $\phi_0$ and $\psi_0$ to be the corresponding mutually inverse isomorphisms, and define $h_0$ and $k_0$ to be the zero maps. Also define $\phi_{-1}$ and $\psi_{-1}$ to be the identities on $\mathbb{Z}$, and set $h_{-1} = 0 = k_{-1}$. It is clear that defined in this way, $\phi$ and $\psi$ commute with the augmentation maps, and $h$ and $k$ satisfy (3.9) in the degrees $-1$ and 0.

Now suppose $\phi_m$ and $\psi_m$ have been defined for all $m < n$, together with homotopies $h_m$ and $k_m$ so that

$$\psi_m\phi_m - \mathsf{id} = Dh_m + h_m D \quad \text{and} \quad \phi_m\psi_m - \mathsf{id} = dk_m + k_m d,$$

for all $X$ and $Y$, and naturally in $X$ and $Y$. As we have noticed, to define $\phi_n$ it suffices to define the image of the simplices $\mathsf{id}_{[p]} \otimes \mathsf{id}_{[q]}$ under $\phi_n$ for $X = \Delta[p]$ and $Y = \Delta[q]$, for all $p, q \geq 0$ with $p + q = n$. Now $\phi_n$ must also satisfy the definition of a chain map, so $\phi_n(\mathsf{id} \otimes \mathsf{id})$ has to satisfy

$$d\phi_n(\mathsf{id} \otimes \mathsf{id}) = \phi_{n-1}D(\mathsf{id} \otimes \mathsf{id})$$

in $\mathbb{Z}[\Delta[p] \times \Delta[q]]_{n-1}$. But

$$d\phi_{n-1}D(\mathsf{id} \otimes \mathsf{id}) = \phi_{n-1}D^2(\mathsf{id} \otimes \mathsf{id}) = 0,$$

and $\tilde{H}_{n-1}(\mathbb{Z}[\Delta[p]] \otimes \mathbb{Z}[\Delta[q]]) = 0$, so there exists an $s$ in $\mathbb{Z}[\Delta[p]] \otimes \mathbb{Z}[\Delta[q]]_n$ with $ds = \phi_{n-1}D(\mathsf{id} \otimes \mathsf{id})$. We let $\phi_n(\mathsf{id} \otimes \mathsf{id})$ be any such $s$. This defines $\phi_n$ for all $X$ and $Y$.

The map $\psi_n$ is defined similarly: using the same argument as before, we see that $\psi_n$ is completely determined, for all $X$ and $Y$, by its image of $(\mathsf{id}, \mathsf{id})$ in case $X = \Delta[n] = Y$. Now define $\psi_n(\mathsf{id}, \mathsf{id})$ to be any element $s$ such that $Ds = \psi_{n-1}d(\mathsf{id}, \mathsf{id})$, i.e., so that $\psi_n$ commutes with the boundary map. Such an element exists because

$$D\psi_{n-1}d(\mathsf{id}, \mathsf{id}) = \psi_{n-2}d^2(\mathsf{id}, \mathsf{id}) = 0,$$

and $\tilde{H}_{n-1}(\mathbb{Z}[\Delta[n] \times \Delta[n]]) = 0$. This defines $\psi_n$ for all $X$ and $Y$.

The homotopies are constructed in the same way. For example, for the composition $\psi\phi$, we want to define $h_n\colon \mathbb{Z}[X] \otimes \mathbb{Z}[Y]_n \longrightarrow \mathbb{Z}[X] \otimes \mathbb{Z}[Y]_{n+1}$, so that $\psi_n\phi_n - \mathsf{id} = Dh_n + h_{n-1}D$. Again, by naturality, we have the following commutative diagram for any two simplices $x \in X_p$ and $y \in Y_q$.

$$\begin{tikzcd}
\mathbb{Z}[X] \otimes \mathbb{Z}[Y]_n \arrow[r, "h_n"] & \mathbb{Z}[X] \otimes \mathbb{Z}[Y]_{n+1} \\
\mathbb{Z}[\Delta[p]] \otimes \mathbb{Z}[\Delta[q]]_n \arrow[r, "h_n"'] \arrow[u, "\mathbb{Z}[\tilde{x}] \otimes \mathbb{Z}[\tilde{y}]_n"] & \mathbb{Z}[\Delta[p]] \otimes \mathbb{Z}[\Delta[q]]_{n+1} \arrow[u, "\mathbb{Z}[\tilde{x}] \otimes \mathbb{Z}[\tilde{y}]_{n+1}"']
\end{tikzcd}$$

So it is enough to consider the special case $X = \Delta[p]$ and $Y = \Delta[q]$, with $p + q = n$, and define $h_n(\mathsf{id}_{[p]} \otimes \mathsf{id}_{[q]})$ so that

$$Dh_n(\mathsf{id} \otimes \mathsf{id}) = \psi_n\phi_n(\mathsf{id} \otimes \mathsf{id}) - \mathsf{id} \otimes \mathsf{id} - h_{n-1}D(\mathsf{id} \otimes \mathsf{id}).$$

We take $h_n(\mathsf{id} \otimes \mathsf{id})$ to be any element whose boundary is the left-hand side of this equation. Such an element exists, because

$$\begin{aligned}
& D(\psi_n \phi_n(\mathsf{id} \otimes \mathsf{id}) - \mathsf{id} \otimes \mathsf{id} - h_{n-1} D(\mathsf{id} \otimes \mathsf{id})) \\
&= \psi_{n-1}\phi_{n-1} D(\mathsf{id} \otimes \mathsf{id}) - D(\mathsf{id} \otimes \mathsf{id}) - D h_{n-1} D(\mathsf{id} \otimes \mathsf{id}) \\
&= (\psi_{n-1}\phi_{n-1} - \mathsf{id} - D h_{n-1})(D(\mathsf{id} \otimes \mathsf{id})) \\
&= h_{n-2} D^2(\mathsf{id} \otimes \mathsf{id}) = 0.
\end{aligned}$$

The definition of the map $k_n$ is left to the reader. □

Given a simplicial set $X$ and an abelian group $A$, we define

$$H_*(X; A) = H_*(\mathbb{Z}[X] \otimes A).$$

**Theorem 3.3.4 (The Künneth formula for homology)** *For simplicial sets $X$ and $Y$ the Künneth formula yields the following decomposition.*

$$H_n(X \times Y) \simeq \left( \bigoplus_{p+q=n} H_p(X) \otimes_{\mathbb{Z}} H_q(Y) \right) \oplus \left( \bigoplus_{p+q=n-1} \mathrm{Tor}_1(H_p(X), H_q(Y)) \right)$$

*Moreover, for a principal ideal domain $R$, the Künneth formula yields a direct sum decomposition of $H_n(X \times Y; R)$ into*

$$\bigoplus_{p+q=n} H_p(X; R) \otimes_R H_q(Y; R) \quad \textit{and} \quad \bigoplus_{p+q=n-1} \mathrm{Tor}_1(H_p(X; R), H_q(Y; R)).$$

*In particular, if $K$ is a field then*

$$H_n(X \times Y; K) \simeq \bigoplus_{p+q=n} H_p(X; K) \otimes_K H_q(Y; K).$$

**Proof.** The first statement follows directly from the Künneth formula (1.12.2).

Let $X$ and $Y$ be simplicial sets and let $R$ be a principal ideal domain. Applying the Künneth formula on $C = \mathbb{Z}[X] \otimes_{\mathbb{Z}} R$ and $C' = \mathbb{Z}[Y] \otimes_{\mathbb{Z}} R$ we find that $H_n(C \otimes_R C')$ is isomorphic to the direct sum of

$$\bigoplus_{p+q=n} H_p(X; R) \otimes_R H_q(Y; R) \quad \text{and} \quad \bigoplus_{p+q=n-1} \mathrm{Tor}_1(H_p(X; R), H_q(Y; R)).$$

It follows that to prove the second statement of the theorem we have to show that $H_n(C \otimes_R C')$ is isomorphic to $H_n(X \times Y; R)$.

Remember that the chain complexes $\mathbb{Z}[X \times Y]$ and $\mathbb{Z}[X] \otimes \mathbb{Z}[Y]$ are equivalent. That is, there exist chain maps

$$\mathbb{Z}[X] \otimes \mathbb{Z}[Y] \underset{\psi}{\overset{\phi}{\rightleftarrows}} \mathbb{Z}[X \times Y],$$

together with homotopies $h$ and $k$ on $\mathbb{Z}[X] \otimes \mathbb{Z}[Y]$ and $\mathbb{Z}[X \times Y]$ such that

$$\psi\phi - \mathsf{id} = Dh + hD \quad \text{and} \quad \phi\psi - \mathsf{id} = dk + kd.$$

Applying the functor $-\otimes_{\mathbb{Z}} R$ on $\phi$ and $\psi$ we obtain chain maps

$$\mathbb{Z}[X]\otimes_{\mathbb{Z}}\mathbb{Z}[Y]\otimes_{\mathbb{Z}} R \underset{\psi\otimes\mathsf{id}}{\overset{\phi\otimes\mathsf{id}}{\rightleftarrows}} \mathbb{Z}[X\times Y]\otimes_{\mathbb{Z}} R$$

and one can easily check that the morphisms $h\otimes\mathsf{id}$ and $k\otimes\mathsf{id}$ give homotopies

$$(\psi\otimes\mathsf{id})(\phi\otimes\mathsf{id})\sim\mathsf{id}\otimes\mathsf{id},$$
$$(\phi\otimes\mathsf{id})(\psi\otimes\mathsf{id})\sim\mathsf{id}\otimes\mathsf{id}.$$

This shows the chain complexes $\mathbb{Z}[X\times Y]\otimes_{\mathbb{Z}} R$ and $\mathbb{Z}[X]\otimes_{\mathbb{Z}}\mathbb{Z}[Y]\otimes_{\mathbb{Z}} R$ are equivalent and hence

$$H_*(X\times Y;R)\simeq H_*(\mathbb{Z}[X]\otimes_{\mathbb{Z}}\mathbb{Z}[Y]\otimes_{\mathbb{Z}} R).$$

On the other hand,

$$\begin{aligned} C\otimes_R C' &= (\mathbb{Z}[X]\otimes_{\mathbb{Z}} R)\otimes_R(\mathbb{Z}[Y]\otimes_{\mathbb{Z}} R)\\ &\simeq (\mathbb{Z}[X]\otimes_{\mathbb{Z}} R)\otimes_R(R\otimes_{\mathbb{Z}}\mathbb{Z}[Y])\simeq(\mathbb{Z}[X]\otimes_{\mathbb{Z}}(R\otimes_R R))\otimes_{\mathbb{Z}}\mathbb{Z}[Y]\\ &\simeq(\mathbb{Z}[X]\otimes_{\mathbb{Z}} R)\otimes_{\mathbb{Z}}\mathbb{Z}[Y]\simeq\mathbb{Z}[X]\otimes_{\mathbb{Z}}\mathbb{Z}[Y]\otimes_{\mathbb{Z}} R,\end{aligned}$$

where we used Remark 1.12.1. We conclude $H_*(X\times Y;R)$ is isomorphic to $H_*(C\otimes_R C')$, which is what we wanted.

Clearly the last statement follows immediately using Corollary 1.12.4. □

## Exercise

*a)* In this exercise the explicit formulas for the chain maps

$$\mathbb{Z}[X]\otimes\mathbb{Z}[Y]\underset{\psi}{\overset{\phi}{\rightleftarrows}}\mathbb{Z}[X\times Y]$$

will be given.

(i) First note that the chain maps $\phi$ and $\psi$ are to some extent unique. Indeed, prove that if $\phi'\colon\mathbb{Z}[X]\otimes\mathbb{Z}[Y]\longrightarrow\mathbb{Z}[X\times Y]$ is another chain map which agrees with $\phi$ in degree zero and is natural in $X$ and $Y$, then $\phi'$ is chain homotopic to $\phi$. (Hint: show that $\psi$ is a homotopy inverse of $\phi'$ by defining homotopies $\psi\phi'\sim\mathsf{id}$ and $\phi'\psi\sim\mathsf{id}$ in the same way as $h$ and $k$ are defined in the proof of Theorem 3.3.1.) Likewise any natural chain map $\mathbb{Z}[X\times Y]\longrightarrow\mathbb{Z}[X]\otimes\mathbb{Z}[Y]$ that agrees with $\psi$ in degree zero is chain homotopic to $\psi$.

The map $\psi$, the so called *Alexander-Whitney map*, is defined for a simplex $(x,y)\in(X\times Y)_n$ by

$$\psi(x,y)=\sum_{i=0}^{n}\tilde{x}(0\dots i)\otimes\tilde{y}(i\dots n).$$

Here $(0\dots i)$ denotes the inclusion $[i]\longrightarrow[n]$, and $(i\dots n)$ denotes the injection $[n-i]\longrightarrow[n]$ sending $k$ to $i+k$, for $k=0,\dots,n-i$. Notice that

$$\tilde{x}(0\dots i)=d_{i+1}\cdots d_n(x),$$
$$\tilde{y}(i\dots n)=d_0\cdots d_{i-1}(y).$$

(ii) Prove that $\psi$ is a chain map, i.e., $D\psi = \psi d$, prove that $\psi$ is natural in $X$ and $Y$, and note that $\psi_0$ is the canonical isomorphism.

The map $\phi$ is sometimes called the *Mac Lane map*. It is determined by its components $\phi_{p,q}\colon \mathbb{Z}[X_p] \otimes \mathbb{Z}[Y_q] \longrightarrow \mathbb{Z}[X_{p+q} \times Y_{p+q}]$, which are defined as follows. For $p, q \geq 0$ look at all pairs of monotone injections $\alpha\colon [p] \longrightarrow [p+q-1]$ and $\beta\colon [q] \longrightarrow [p+q-1]$ whose images together cover all of $[p+q-1]$. Such a pair $(\alpha, \beta)$ is called a *$p,q$-shuffle*. Define $\phi_{p,q}$ on a generator $x \otimes y$, where $x \in X_p$ and $y \in Y_p$, by

$$\phi(x \otimes y) = \sum_{(\alpha,\beta)} \mathrm{sg}(\alpha, \beta)(s_\alpha(x), s_\beta(y))$$

Here $s_\alpha\colon X_p \longrightarrow X_{p+q}$ and $s_\beta\colon Y_q \longrightarrow Y_{p+q}$ are given by

$$s_\alpha(x) = s_{\alpha(p)} \cdots s_{\alpha(0)}(x),$$
$$s_\beta(y) = s_{\beta(q)} \cdots s_{\beta(0)}(y).$$

Furthermore $\mathrm{sg}(\alpha, \beta)$ denotes the sign of the permutation

$$(\alpha(0) \ldots \alpha(p)\beta(0) \ldots \beta(q)).$$

(iii) Prove that $\phi$ is a chain map which is natural in $X$ and $Y$, and that $\phi_0$ is the canonical isomorphism.

## 3.4 Additional exercises

1. *(Homology of pairs and Excision)*. Let $Y \subseteq X$ be simplicial sets. Define $\mathbb{Z}[X, Y]_n$ as the cokernel of the injective map

$$\mathbb{Z}[Y]_n \longrightarrow \mathbb{Z}[X]_n.$$

Note that $\mathbb{Z}[X, Y]_\bullet$ is a chain complex. Its homology is denoted $H_*(X, Y)$ and is called the *homology of the pair* $(X, Y)$.

i) Prove that there is a long exact sequence

$$\cdots \longrightarrow H_n(Y) \longrightarrow H_n(X) \longrightarrow H_n(X, Y) \longrightarrow H_{n-1}(Y) \longrightarrow \cdots$$

ii) *(Excision)*. Suppose that $Z \subseteq X$ is another simplicial subset and assume that $Y \cup Z = X$. Prove that there is an isomorphism

$$H_*(X \cap Z, Y \cap Z) \longrightarrow H_*(X, Y).$$

Can you deduce this formally from the Mayer-Vietoris sequence of Theorem 3.2.2?

Now, we move to topological spaces and we use the notation $C_*(X) = \mathbb{Z}[\mathrm{Sing}(X)]_*$. Let $X$ be a topological space and $Y \subset X$ a subspace.

iii) Define $C_*(X, Y)$ to be the cokernel of the map $C_*(Y) \longrightarrow C_*(X)$. Deduce from i) that there is a similar long exact sequence for the pair of topological spaces $(X, Y)$.

iv) Let $K$ be a subspace such that $\overline{K} \subset \mathrm{Int}(Y)$. Show that there is an isomorphism

$$H_n(X - K, Y - K) \longrightarrow H_n(X, Y).$$

One way to prove this is to observe that the Mayer-Vietoris sequence for spaces (Theorem 3.2.3), remains true if $U, V \subseteq X$ are (not necessarily open) subsets for which $X = \mathrm{Int}(U) \cup \mathrm{Int}(V)$. The excision isomorphism follows from this by general homological algebra.

(This isomorphism is called "excision". It expresses that we can "excise" (cut out) $K$ without changing the homology.)

2. Suppose that $F\colon \mathsf{C} \rightleftarrows \mathsf{D} \colon G$ are a pair of adjoint functors between small categories.

i) Prove that they induce an isomorphism in the homology of the nerves:

$$H_*(N(\mathsf{C})) \overset{\sim}{\rightleftarrows} H_*(N(\mathsf{D})).$$

(Hint: Use part $a$) of Example 3.1.9.)

ii) Conclude that if $\mathsf{C}$ has an initial or a terminal object, then $N(\mathsf{C})$ is *acyclic*, in the sense that

$$H_i(N(\mathcal{C})) = \begin{cases} \mathbb{Z} & \text{if } i = 0 \\ 0 & \text{if } i > 0 \end{cases}$$

iii) Conclude that for each $n_1, \ldots, n_k \in \mathbb{N}$, the product

$$\Delta[n_1] \times \ldots \times \Delta[n_k]$$

is an acyclic simplicial set. ($\Delta[n]$ is defined in Example 3.1.3 $d$).)

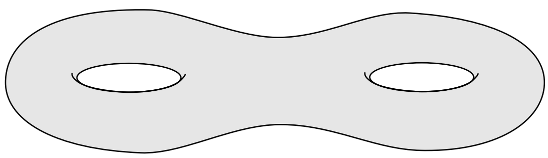
3. Using only the Mayer-Vietoris sequence, excision, homotopy invariance and the homology of the spheres $S^n$, compute the homology of the following topological spaces:

i) The figure eight. (It consists of a pair of $S^1$ connected at a single point.)

ii) The quotient of the sphere $S^2$ obtained by identifying the poles to a point.

iii) The 2-dimensional torus.

iv) The surface of genus 2.



# 4. Abelian categories and derived functors

It should be clear, by now, how useful and important the machinery of homology and cohomology is in mathematics. In developing this theory, we relied on the structure of categories of modules for a ring, or on "module-like" structures. In Chapter 1 (in particular, in Section 1.6), we had already anticipated the notion of a "category of modules", leaving the concept vague, and promising to make it precise at a later stage. The idea behind it was that such categories have enough structure to make it possible to formalise the notions of kernel of a map, of exact sequence and, in general, of homology and cohomology.

It is the purpose of this Chapter to explore the category theory behind homology, bringing out the key categorical concepts involved. We shall introduce the notion of an *abelian category*, and that of a *derived functor*. Using these, we shall formalise the study of homology and cohomology. As an example, we shall study the cohomology of small categories and sheaf cohomology.

## 4.1 Abelian categories

The first step in our process of abstraction is that of identifying the properties of a category that make it possible to formulate the fundamental homological notions. Recall from the Appendix the notions of monic, epic and isomorphic map.

In categories of modules, we had defined the concepts of product and direct sum of objects. These had universal properties, which are stated purely in diagrammatic form (cf. Propositions 1.1.7 and 1.1.8). It is therefore possible to extend them to the context of any category.

**Definition 4.1.1** A *product* of two objects $A$ and $B$ in a category $\mathsf{C}$ is an object $P$ equipped with two maps $p: P \longrightarrow A$ and $q: P \longrightarrow B$, called *projections*, with the property that for any object $C$ and morphisms $f: C \longrightarrow A$ and $g: C \longrightarrow B$, there is a unique morphism $h: C \longrightarrow P$ with $ph = f$ and $qh = g$:

$$\begin{tikzcd} & C \arrow[dl, bend right, "f"'] \arrow[d, dotted, "h"] \arrow[dr, bend left, "g"] & \\ A & P \arrow[l, "p"] \arrow[r, "q"'] & B. \end{tikzcd}$$

A *sum* of two objects $A$ and $B$ in a category $\mathsf{C}$ is an object $S$ equipped with two maps $i: A \longrightarrow S$ and $j: B \longrightarrow S$, called *injections*, with the property that for any object $C$ and morphisms $f: A \longrightarrow C$ and $g: B \longrightarrow C$, there exists a unique morphism $h: S \longrightarrow C$ such that $hi = f$ and $hj = g$:

$$\begin{tikzcd} A \arrow[r, "i"] \arrow[dr, bend right, "f"'] & S \arrow[d, dotted, "h"] & B \arrow[l, "j"'] \arrow[dl, bend left, "g"] \\ & C. & \end{tikzcd}$$

It is immediate, from the definition, that these notions are dual to each other; that is, an object $X$ in a category $\mathsf{C}$ is a product of $A$ and $B$ if and only if it is their coproduct in $\mathsf{C}^{\mathrm{op}}$, and vice versa. The sum and product of two objects $A$ and $B$, when they exist, are unique up to isomorphism and we denote them by $A \oplus B$ and $A \times B$, respectively. In the literature, the notation $A + B$ or $A \coprod B$ is also used for the sum of $A$ and $B$.

**Definition 4.1.2** An object $Z$ in a category $\mathsf{C}$ is called a *zero object* if for any other object $A$ in $\mathsf{C}$ there is a unique morphism $Z \longrightarrow A$ and a unique morphism $A \longrightarrow Z$. It then follows that the zero object in a category, when it exists, is unique up to isomorphism, and we denote it by 0. When $\mathsf{C}$ has a zero object, then between any two objects $A$ and $B$ there is a unique morphism $0_{A,B} \in \mathrm{Hom}_{\mathsf{C}}(A, B)$ which factors through 0; namely, the composite

$$A \longrightarrow 0 \longrightarrow B.$$

We call this the *zero morphism* between $A$ and $B$.

**Remark 4.1.3** For any object $A$ in a category $\mathsf{C}$ with zero object 0, the morphism $0 \longrightarrow A$ is monic and the morphism $A \longrightarrow 0$ is epic.

Another property we encountered for categories of modules was that of having an abelian group structure on the hom-sets. Categories with such structure are called *linear.*

**Definition 4.1.4** A category $\mathsf{L}$ is called *linear* if for any two objects $A$ and $B$, $\mathrm{Hom}_{\mathsf{L}}(A, B)$ is an abelian group, and composition is a bilinear operation; that is, for any $f, f' \in \mathrm{Hom}_{\mathsf{L}}(A, B)$ and $g, g' \in \mathrm{Hom}_{\mathsf{L}}(B, C)$,

$$(g + g')f = gf + g'f \quad \text{and} \quad g(f + f') = gf + gf'.$$

**Proposition 4.1.5** *Let $\mathsf{L}$ be a linear category. Then:*

*a) for any object $A$ in $\mathsf{L}$, $\mathrm{Hom}_{\mathsf{L}}(A, A)$ is an associative ring, with composition as product and $\mathsf{id}_A$ as unit;*

*b) if $\mathsf{L}$ has a zero object, then the zero morphism $0_{A,B} \in \mathrm{Hom}_{\mathsf{L}}(A, B)$ is the zero element of the abelian group $\mathrm{Hom}_{\mathsf{L}}(A, B)$;*

*c) for a zero object 0, $\mathsf{id}_0 : 0 \longrightarrow 0$ is the zero element of $\mathrm{Hom}_{\mathsf{L}}(0, 0)$. Conversely, any object $A$ for which $\mathsf{id}_A = 0$, is a zero object.*

The proof of this Proposition is left to you as Exercise *a*) below.

**Definition 4.1.6** The *kernel* of a map $f\colon A \longrightarrow B$ is a morphism $i\colon K \longrightarrow A$ such that $fi = 0_{K,B}$ and for any other map $i'\colon K' \longrightarrow A$ such that $fi' = 0_{K',B}$ there exists a unique morphism $e\colon K' \longrightarrow K$ such that $ie = i'$:

$$\begin{tikzcd} K' \arrow[d, dotted, "e"'] \arrow[dr, bend left, "i'"] & & \\ K \arrow[r, "i"'] & A \arrow[r, "f"'] & B. \end{tikzcd}$$

We denote the kernel of $f$ by $\ker(f) \xrightarrow{i} A$, when it exists.

Dually, the *cokernel* of a morphism $f$ as above is a morphism $j\colon B \longrightarrow L$ such that $jf = 0_{A,L}$ and for any other map $j'\colon B \longrightarrow L'$ such that $j'f = 0_{A,L'}$, there is a unique morphism $e\colon L \longrightarrow L'$ such that $ej = j'$:

$$\begin{tikzcd} A \arrow[r, "f"] & B \arrow[r, "j"] \arrow[dr, bend right, "j'"'] & L \arrow[d, dotted, "e"] \\ & & L'. \end{tikzcd}$$

We denote the cokernel of $f$ by $B \xrightarrow{j} \operatorname{coker}(f)$, when it exists.

The following basic properties are left to you to prove (as Exercise *b*) below).

**Proposition 4.1.7** *Let* $\mathsf{L}$ *be a linear category and* $f\colon A \longrightarrow B$ *a map in it. Then:*

*a) the kernel (cokernel) of* $f$*, when it exists, is unique up to isomorphism;*

*b) the sequences*

$$0 \longrightarrow \operatorname{Hom}_{\mathsf{L}}(C, \ker(f)) \xrightarrow{i_*} \operatorname{Hom}_{\mathsf{L}}(C, A) \xrightarrow{f_*} \operatorname{Hom}_{\mathsf{L}}(C, B)$$

*and*

$$\operatorname{Hom}_{\mathsf{L}}(A, C) \xleftarrow{f^*} \operatorname{Hom}_{\mathsf{L}}(B, C) \xleftarrow{j^*} \operatorname{Hom}_{\mathsf{L}}(\operatorname{coker}(f), C) \leftarrow 0$$

*are exact sequences of abelian groups;*

*c) the morphism* $i\colon \ker(f) \longrightarrow A$ *is monic;*

*d) the morphism* $j\colon B \longrightarrow \operatorname{coker}(f)$ *is epic.*

**Proposition 4.1.8** *In a linear category* $\mathsf{L}$*, the sum* $A \oplus B$ *of the objects* $A$ *and* $B$ *(with injections* $i\colon A \longrightarrow A \oplus B$ *and* $j\colon B \longrightarrow A \oplus B$*) is also the product of* $A$ *and* $B$*.*

**Proof.** Consider the morphisms $\mathsf{id}_A\colon A \longrightarrow A$ and $0_{B,A}\colon B \longrightarrow A$. By the definition of sum, they determine a morphism $p\colon A \oplus B \longrightarrow A$ such that $pi = \mathsf{id}_A$ and $pj = 0_{B,A}$. Similarly, there exists a morphism $q\colon A \oplus B \longrightarrow B$ such that $qi = 0_{A,B}$ and $qj = \mathsf{id}_B$.

Now, let $C$ be an object in $\mathsf{L}$ with maps $f\colon C \longrightarrow A$ and $g\colon C \longrightarrow B$. Then, by putting $h\colon = if + jg$, we get

$$\begin{aligned} ph &= p(if + jg) = (pi)f + (pj)g = f, \\ qh &= q(if + jg) = (qi)f + (qj)g = g; \end{aligned}$$

Hence, $h$ fits in the commutative diagram

$$\begin{tikzcd} & C \arrow[dl, bend right, "f"'] \arrow[d, "h"] \arrow[dr, bend left, "g"] & \\ A & A \oplus B \arrow[l, "q"] \arrow[r, "p"'] & B. \end{tikzcd}$$

To show that $h$ is the unique map with this property, notice that, by the universal property of sums, the map $ip+jq: A\oplus B \longrightarrow A\oplus B$ is the identity morphism, since $(ip+jq)i = i$ and $(ip+jq)j = j$. Therefore, if $k: C \longrightarrow A \oplus B$ is any morphism which satisfies $pk = f$ and $qk = g$, then

$$\begin{aligned} k &= \mathsf{id}_{A\oplus B} k = (ip+jq)k \\ &= i(pk) + j(qk) = if + jg = h. \end{aligned}$$

□

When, in a linear category $\mathsf{L}$, products and sums coincide as in Proposition 4.1.8, it is common to call the object $A \oplus B$ the *direct sum* of $A$ and $B$.

**Definition 4.1.9** A linear category $\mathsf{A}$ is called *additive* if it has a zero object 0 and a direct sum $A \oplus B$ for any two objects $A$ and $B$ in $\mathsf{A}$.

A functor $F: \mathsf{A} \longrightarrow \mathsf{B}$ between linear categories is called *additive* if for all $A, A'$ in $\mathsf{A}$, the function $F_{A,A'}: \mathrm{Hom}_{\mathsf{A}}(A, A') \longrightarrow \mathrm{Hom}_{\mathsf{B}}(F(A), F(A'))$ is a group homomorphism.

**Proposition 4.1.10** *Every additive functor $F: \mathsf{A} \longrightarrow \mathsf{B}$ between additive categories preserves the zero object and direct sums.*

**Proof.** This Proposition is true, essentially because the sum/product $A \oplus B$ together with its inclusions $i$ and $j$ and its projections $p$ and $q$ as in the proof of Proposition 4.1.8, is characterised by equations (such as $pi = \mathsf{id}_A$, etc.), of which the universal properties are a consequence. We leave a detailed proof to you as Exercise *c*) below. □

In an additive category, we can use zero maps to detect mono- and epimorphisms.

**Remark 4.1.11** Let $\mathsf{A}$ be an additive category and $f: A \longrightarrow B$ a map in it. Then, $f$ is monic if and only if, for any map $g: C \longrightarrow A$, $fg = 0$ implies $g = 0$. Analogously, $f$ is epic if and only if, for any map $h: B \longrightarrow D$, $hf = 0$ implies $h = 0$.

**Definition 4.1.12** An *abelian category* is an additive category $\mathsf{A}$ in which

*a)* every morphism has a kernel and a cokernel;

*b)* every monomorphism $f$ is the kernel of its cokernel; i.e. the unique morphism $e$ filling the following diagram is an isomorphism:

$$\begin{tikzcd} A \arrow[d, dotted, "e"'] \arrow[dr, tail, bend left, "f"] & & \\ \ker(p) \arrow[r, "i"'] & B \arrow[r, "p"'] & \mathrm{coker}(f); \end{tikzcd}$$

*c)* every epimorphism $f$ is the cokernel of its kernel; i.e. the unique morphism $e$ filling the diagram below is an isomorphism:

$$\begin{tikzcd}
\ker(f) \arrow[r, "i"] & A \arrow[r, "p"] \arrow[dr, two heads, bend right, "f"'] & \mathrm{coker}(i) \arrow[d, dotted, "e"] \\
& & B.
\end{tikzcd}$$

An evident but important feature of this definition is that it is *self-dual*, in the following sense.

**Proposition 4.1.13** *If* $\mathsf{A}$ *is an abelian category, so is* $\mathsf{A}^{\mathrm{op}}$*.*

Notice in particular that sums in $\mathsf{A}$ are products in $\mathsf{A}^{\mathrm{op}}$, kernels in $\mathsf{A}$ are cokernels in $\mathsf{A}^{\mathrm{op}}$, etc. Abelian categories also have the following important property:

**Proposition 4.1.14** *A map in an abelian category is an isomorphism if and only if it is both monic and epic.*

**Proof.** See Exercise *d*) below. □

**Definition 4.1.15** The *image* of a morphism $f\colon A \longrightarrow B$ is a factorisation

$$\begin{tikzcd}
A \arrow[r, two heads, "e"] \arrow[dr, bend right, "f"'] & \mathrm{im}(f) \arrow[d, tail, "i"] \\
& B.
\end{tikzcd}$$

where $i$ is monic, $e$ is epic and $f = ie$.

Next, we want to show that every abelian category $\mathsf{A}$ has images. In fact, given such a category and a map $f\colon A \longrightarrow B$ in it, we can form its cokernel $p\colon B \longrightarrow \mathrm{coker}(f)$ and the kernel of this, $i\colon \ker(p) \longrightarrow B$. These fit in the following diagram:

$$\begin{tikzcd}
A \arrow[d, dotted, "e"'] \arrow[dr, bend left, "f"] & & \\
\ker(p) \arrow[r, "i"'] & B \arrow[r, "p"'] & \mathrm{coker}(f).
\end{tikzcd} \tag{4.1}$$

Since $pf = 0$, there is a unique morphism $e\colon A \longrightarrow \ker(p)$ with $ie = f$. We will show that $e$ is an epimorphism, thus proving that in every abelian category images exist; but first, we need the following result.

**Lemma 4.1.16** *For a given morphism* $f\colon A \longrightarrow B$ *in an abelian category* $\mathsf{A}$*, form maps* $i$ *and* $e$ *as in (4.1). Then,* $i$ *is the "smallest" monomorphism through which* $f$ *factors. More precisely, for any factorisation* $f = mg$ *of* $f$ *as a morphism* $g\colon A \longrightarrow I$ *followed by a monomorphism* $m\colon I \longrightarrow B$*, there is a morphism* $v\colon \ker(p) \longrightarrow I$ *such that* $i = mv$*.*

**Proof.** Since $m$ is monic, by condition *b)* in Definition 4.1.12, there is an object $C$ in $\mathsf{A}$ and a morphism $\varphi\colon B \longrightarrow C$ such that $m$ is the kernel of $\varphi$. From $\varphi m = 0$ we obtain $\varphi f = 0$, so there is a morphism $\psi\colon \operatorname{coker}(f) \longrightarrow C$ with $\psi p = \varphi$:

$$\begin{tikzcd}
A \arrow[r, "g"] & I \arrow[r, "m"] & B \arrow[r, "\varphi"] \arrow[d, "p"] & C. \\
 & \ker(p) \arrow[u, dotted, "v"] \arrow[ur, bend left, "i"'] & \operatorname{coker}(f) \arrow[ur, dotted, "\psi"'] &
\end{tikzcd}$$

Thus, $\varphi i = 0$ and by definition of kernel, there is a morphism $v\colon \ker(p) \longrightarrow I$ with $mv = i$. $\square$

**Proposition 4.1.17** *Let $\mathsf{A}$ be an abelian category. Then, every morphism $f\colon A \longrightarrow B$ has an image, which is unique up to isomorphism; that is, for any two image factorisations $A \xrightarrow{e} I \xrightarrow{i} B$ and $A \xrightarrow{e'} I' \xrightarrow{i'} B$ of $f$, there is an isomorphism $\delta\colon I \longrightarrow I'$ such that $\delta e = e'$ and $i'\delta = i$.*

**Proof.** Let $p$, $i$ and $e$ be as in (4.1), so that $f = ie$. If we show that $e$ is epic, then we have given an image of $f$. For this, we shall use the characterisation of Remark 4.1.11; so, let $g\colon \ker(p) \longrightarrow T$ be a morphism with $ge = 0$, and let $k\colon \ker(g) \longrightarrow \ker(p)$ be its kernel. Since $ge = 0$, there is a morphism $w\colon A \longrightarrow \ker(g)$ such that $kw = e$:

$$\begin{tikzcd}
 & \ker(g) \arrow[d, "k"] & & \\
A \arrow[ur, dotted, "w"] \arrow[r, "e"'] & \ker(p) \arrow[r, "i"] \arrow[d, "g"] & B \arrow[r, "p"] & \operatorname{coker}(f) \\
 & T. & &
\end{tikzcd}$$

We then have $(ik)w = i(kw) = ie = f$, and $ik$ is a monomorphism. Therefore, by Lemma 4.1.16, there is a morphism $v\colon \ker(p) \longrightarrow \ker(g)$ with $(ik)v = i$. Since $i$ is monomorphism, $kv = \mathsf{id}_{\ker(p)}$ and hence $g = g(kv) = (gk)v = 0$.

To prove uniqueness of the image, suppose we are given another factorisation $A \xrightarrow{e'} I' \xrightarrow{i'} B$. Then, by Lemma 4.1.16, there is a morphism $\delta\colon \ker(p) \longrightarrow I'$ such that $i'\delta = i$:

$$\begin{tikzcd}
 & \ker(p) \arrow[d, dotted, "\delta"] \arrow[dr, bend left, "i"] & \\
A \arrow[ur, bend left, "e"] \arrow[r, "e'"'] & I' \arrow[r, "i'"'] & B.
\end{tikzcd}$$

We then have $i'(\delta e) = (i'\delta)e = ie = f = i'e'$, and since $i'$ is monomorphism, $\delta e = e'$. The morphism $\delta$ is epic because $e'$ is, and it is monic because $i$ is a monomorphism. Therefore, $\delta$ is an isomorphism by Proposition 4.1.14, and the proof is closed. $\square$

Having recovered the notion of kernel, image and zero map, we can now reformulate in any abelian category the definition of an exact sequence. Formally, this is the same as Definition 1.6.2. So, we shall say that a (possibly infinite) sequence

$$\cdots \longrightarrow A_{i+1} \xrightarrow{\phi_{i+1}} A_i \xrightarrow{\phi_i} A_{i-1} \longrightarrow \cdots$$

of objects and maps in an abelian category $\mathsf{A}$ is *exact at* $A_i$ if $\ker(\phi_i) = \operatorname{im}(\phi_{i+1})$, and that it is *exact* if it is exact at $A_i$ for all $i$. Likewise, a *short exact sequence* will be an exact sequence of the special form

$$0 \longrightarrow A' \xrightarrow{\phi} A \xrightarrow{\psi} A'' \longrightarrow 0.$$

The notions of *left exact*, *right exact* and *exact* functor as introduced in Definition 1.6.9 readily translate in the present context, for any additive functor between abelian categories. Notice that the notion of exactness is self-dual, in the sense that a sequence in $\mathsf{A}$ is exact if and only if the same sequence is exact in $\mathsf{A}^{\mathrm{op}}$.

Many arguments which we saw in Chapter 1 can be recovered in this more general context, just by chasing arrows in a diagram, instead of elements.

**Example 4.1.18** Just as for categories of modules, we can characterise mono- and epimorphisms in terms of exact sequences, in any abelian category:

*a)* the sequence $0 \longrightarrow A \xrightarrow{\varphi} B$ is exact if and only if $\varphi$ is monic;

*b)* the sequence $A \xrightarrow{\varphi} B \longrightarrow 0$ is exact if and only if $\varphi$ is epic.

**Example 4.1.19** If $T$ is any object in an abelian category $\mathsf{A}$, then the functors $\operatorname{Hom}_{\mathsf{A}}(T, -)\colon \mathsf{A} \longrightarrow \mathsf{Ab}$ and $\operatorname{Hom}_{\mathsf{A}}(-, T)\colon \mathsf{A}^{\mathrm{op}} \longrightarrow \mathsf{Ab}$ are left exact.

The contravariant Hom functor is also useful in detecting exactness of sequences, as made precise by the following result.

**Lemma 4.1.20** *If a sequence* $A \xrightarrow{f} B \xrightarrow{g} C$ *in an abelian category* $\mathsf{A}$ *has the property that for every object* $T$*,* $\operatorname{Hom}_{\mathsf{A}}(-, T)$ *sends it to the exact sequence*

$$\operatorname{Hom}_{\mathsf{A}}(C, T) \xrightarrow{g^*} \operatorname{Hom}_{\mathsf{A}}(B, T) \xrightarrow{f^*} \operatorname{Hom}_{\mathsf{A}}(A, T)$$

*of abelian groups, then the sequence is itself exact.*

**Proof.** Take $T = C$, then the sequence

$$\operatorname{Hom}_{\mathsf{A}}(C, C) \xrightarrow{g^*} \operatorname{Hom}_{\mathsf{A}}(B, C) \xrightarrow{f^*} \operatorname{Hom}_{\mathsf{A}}(A, C)$$

is exact, so $f^*g^* = 0$ and hence $gf = 0$, because $gf = f^*g^*(\mathsf{id}_C)$. Therefore, $\operatorname{im}(f) \subseteq \ker(g)$ as subobjects of $B$, i.e. the inclusion of $\operatorname{im}(f)$ in $B$ factors through the monomorphism $j\colon \ker(g) \longrightarrow B$. For the opposite inclusion, take $T = \operatorname{coker}(f)$. This determines an exact sequence

$$\operatorname{Hom}_{\mathsf{A}}(C, \operatorname{coker}(f)) \xrightarrow{g^*} \operatorname{Hom}_{\mathsf{A}}(B, \operatorname{coker}(f)) \xrightarrow{f^*} \operatorname{Hom}_{\mathsf{A}}(A, \operatorname{coker}(f));$$

thus, $\ker(f^*) = \mathrm{im}(g^*)$. Consider the following diagram:

$$\begin{tikzcd}
 & \mathrm{im}(f) = \ker(p) \arrow[d, "i"] & \\
A \arrow[ur, bend left, "e"] \arrow[r, "f"'] & B \arrow[r, "g"] \arrow[d, "p"'] & C \arrow[dl, dotted, "t"] \\
 & \mathrm{coker}(f). &
\end{tikzcd}$$

By definition of cokernel, we have $0 = pf = f^*(p)$. So, $p \in \ker(f^*) = \mathrm{im}(g^*)$, i.e. there exists a morphism $t\colon C \longrightarrow \mathrm{coker}(f)$ such that $g^*(t) = p$. This means that $\ker(g) \subseteq \ker(p) = \mathrm{im}(f)$. $\square$

**Theorem 4.1.21** *Suppose $L\colon \mathsf{A} \longrightarrow \mathsf{B}$ and $R\colon \mathsf{B} \longrightarrow \mathsf{A}$ are additive functors between abelian categories, with $L \dashv R$ (i.e. $L$ is left adjoint to $R$). Then, $L$ is right exact and $R$ is left exact.*

**Proof.** Let

$$0 \longrightarrow B' \xrightarrow{f} B \xrightarrow{g} B'' \longrightarrow 0$$

be a short exact sequence in $\mathsf{B}$ and $A$ an object in $\mathsf{A}$. We have the commutative diagram, where the first row is exact by *b*) of Proposition 4.1.7:

$$\begin{tikzcd}
\mathrm{Hom}_{\mathsf{B}}(L(A), B') \arrow[r, "f_*"] \arrow[d, "\tau"'] & \mathrm{Hom}_{\mathsf{B}}(L(A), B) \arrow[r, "g_*"] \arrow[d, "\tau"'] & \mathrm{Hom}_{\mathsf{B}}(L(A), B'') \arrow[d, "\tau"] \\
\mathrm{Hom}_{\mathsf{A}}(A, R(B')) \arrow[r, "R(f)_*"'] & \mathrm{Hom}_{\mathsf{A}}(A, R(B)) \arrow[r, "R(g)_*"'] & \mathrm{Hom}_{\mathsf{A}}(A, R(B'')),
\end{tikzcd}$$

and since $\tau$ is bijection, the bottom row is exact too, for any $A$ in $\mathsf{A}$. Then, by Lemma 4.1.20, the sequence

$$R(B') \xrightarrow{R(f)} R(B) \xrightarrow{R(g)} R(B'')$$

is exact. Now, apply the same argument to the exact sequence $0 \longrightarrow B' \xrightarrow{f} B$ to show the exactness of $0 \longrightarrow R(B') \xrightarrow{R(f)} R(B)$. Therefore, $R$ is left exact, and by duality we deduce at once that $L$ is right exact. $\square$

**Remark 4.1.22** We close this Section by noticing that, having introduced the notions of kernel, image, and zero map, we can read many definitions given in Chapter 1 in the context of any abelian category. We shall henceforth speak of *chain complex*, *cochain complex*, *homotopy* between them, *homology*, *cohomology* etc. Most of the results concerning them are still valid in this more general setting. In what follows, we shall feel free to appeal to Proposition 1.6.4 on splicing of exact sequences, Proposition 1.6.6, the 5-lemma 1.6.8, Proposition 1.9.8 or Proposition 1.10.6. However, we shall not give any proof here. Part of the motivation for this choice is that the arguments are easy formal adaptations of the original proofs in the case of modules (see exercise *h*) below). Moreover, such proofs are made unnecessary by Freyd's embedding theorem [1, 5], to which we refer any interested reader.

### Exercises

*a)* Prove the properties of linear categories expressed in Proposition 4.1.5.

*b)* Prove the properties of kernel and cokernel maps stated in Proposition 4.1.7.

*c)* Prove Proposition 4.1.10.

*d)* Prove Proposition 4.1.14.

*e)* Prove *a*) and *b*) of Example 4.1.18.

*f)* Show functoriality and exactness of the Hom functors in Example 4.1.19.

*g)* Show that any abelian category $\mathsf{A}$ admits *pullbacks*, i.e. that, for any diagram of the form

$$\begin{tikzcd} & B \arrow[d, "f"] \\ A \arrow[r, "g"'] & C \end{tikzcd}$$

in $\mathsf{A}$, there exists an object $P$ and maps $i$ and $j$ fitting in a commutative square

$$\begin{tikzcd} P \arrow[r, "i"] \arrow[d, "j"'] & B \arrow[d, "f"] \\ A \arrow[r, "g"'] & C, \end{tikzcd}$$

with the universal property that, for any other pair of maps $j' \colon X \longrightarrow A$ and $i' \colon X \longrightarrow B$ with $fi' = gj'$, there is a unique morphism $e \colon X \longrightarrow P$ such that $je = j'$ and $ie = i'$. (Hint: build $P$ as the kernel of an appropriate map from $A \oplus B$ to $C$.)

*h)* In light of Remark 4.1.22, give a categorical proof of some of the results in Chapter 1 concerning exact sequences and (co)homology, such as the 5-lemma or Proposition 1.10.6.

## 4.2 Derived functors

In this Section, we shall study the notion of derived functors. Given a left exact functor $F \colon \mathsf{A} \longrightarrow \mathsf{X}$ between abelian categories, and a short exact sequence $0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$ in $\mathsf{A}$, we know that there is an exact sequence

$$0 \longrightarrow FA \longrightarrow FB \longrightarrow FC, \tag{4.2}$$

but the rightmost map need not be an epimorphism. The right derived functors of $F$ provide a suitable extension of (4.2) to a long exact sequence. Left derived functors provide the dual extension for right exact functors.

We shall henceforth assume all functors to be additive.

### 4.2.1 Left derived functors

As made clear in Remark 4.1.22, most notions introduced for modules in Chapter 1 can be formulated in any abelian category. This is the case also for the notion of a projective module (Definition 1.7.1).

**Definition 4.2.1** An object $P$ in an abelian category $\mathsf{A}$ is *projective* if the functor $\mathrm{Hom}_{\mathsf{A}}(P, -)\colon \mathsf{A} \longrightarrow \mathsf{Ab}$ is (right) exact. Equivalently, $P$ is projective if, for any epimorphism $f\colon A \longrightarrow B$, every map $g\colon P \longrightarrow B$ factors through $f$:

$$\begin{tikzcd} & A \arrow[d, two heads, "f"] \\ P \arrow[ur, dotted, bend left] \arrow[r, "g"'] & B. \end{tikzcd}$$

We say that $\mathsf{A}$ has *enough projectives* if every $A$ in $\mathsf{C}$ can be covered by a projective; that is, if there exists an epimorphism $f\colon P \longrightarrow A$, where $P$ is a projective object.

**Example 4.2.2** For any ring $R$, the category $R$-$\mathsf{mod}$ of left $R$-modules has enough projectives, by Lemma 1.7.5.

**Remark 4.2.3** The arguments given for Lemma 1.10.12 and Proposition 1.10.13 readily extend to any abelian category $\mathsf{A}$, showing that if $\mathsf{A}$ has enough projectives, then every object admits a projective resolution, which is unique up to homotopy of chain complexes.

**Definition 4.2.4** Let $\mathsf{A}$ and $\mathsf{X}$ be abelian categories such that $\mathsf{A}$ has enough projectives. Let $F\colon \mathsf{A} \longrightarrow \mathsf{X}$ be a right exact functor. Then, the *left derived* functors of $F$ are the functors $L_i(F)\colon \mathsf{A} \longrightarrow \mathsf{X}$ ($i \in \mathbb{N}$) which, for an object $A$ in $\mathsf{A}$ with a projective resolution $0 \longleftarrow A \longleftarrow P_\star$, are defined as

$$L_i(F)(A) = H_i(F(P_\star)) \quad \text{for} \quad i \geq 0.$$

Given a morphism $f\colon A \longrightarrow B$ in $\mathsf{A}$, where $0 \longleftarrow A \longleftarrow P_\star$ and $0 \longleftarrow B \longleftarrow Q_\star$ are two projective resolutions of $A$ and $B$, respectively, there exists a (unique, up to homotopy) chain complex map $f_\star\colon P_\star \longrightarrow Q_\star$ which extends $f$. The action of the derived functor $L_i(F)$ on $f$ is then defined as

$$L_i(F)(f) = H_i(F(f_\star)). \quad (i \geq 0)$$

**Remark 4.2.5** The next proposition will make the properties of left derived functors clear. However, before moving forward, we have to understand what we mean by homology group in Definition 4.2.4. Given a chain complex $A_\star$ in an abelian category $\mathsf{A}$, the condition $dd = 0$ implies that, for every $i \in \mathbb{N}$, there exists a monomorphism $\mathrm{im}(d_{i+1}) \longrightarrow \ker(d_i)$. The *i-th homology group* $H_i(A_\star)$ of $A_\star$ is defined as the cokernel of this monomorphism.

**Proposition 4.2.6** *Let $\mathsf{A}$ be an abelian category with enough projectives and $F$ a right exact functor. Then:*

*a) the object $L_i(F)(A)$ is independent (up to a canonical isomorphism) of the projective resolution $P_\star$ of $A$, and the image along $L_i(F)(-)$ of a map $f\colon A \longrightarrow B$ is independent of the chain map $f_\star$ lifting $f$. Moreover, $L_i(F)(-)$ is a covariant functor;*

*b) for any $A$ in $\mathsf{A}$, we have $L_0(F)(A) \cong FA$. More precisely there is a natural isomorphism $L_0(F) \cong F$;*

*c) any short exact sequence in $\mathsf{A}$*

$$0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0$$

*gives rise to a long exact sequence in $\mathsf{X}$*

$$\cdots \longrightarrow L_1(F)(C) \xrightarrow{\delta} L_0(F)(A) \longrightarrow L_0(F)(B) \longrightarrow L_0(F)(C) \longrightarrow 0;$$

*d) If $A$ is projective in $\mathsf{A}$, then $L_i(F)(A) = 0$ for $i > 0$.*

**Proof.** *a*) Consider a morphism $f\colon A \longrightarrow B$ in $\mathsf{A}$, and suppose two projective resolutions $P_\star \to A$ and $Q_\star \to B$ are given. Then, by Lemma 1.10.12 we know that $f$ lifts to a morphism of exact sequences $f_\star\colon P_\star \longrightarrow Q_\star$. Moreover, $f_\star$ is unique up to homotopy. Therefore, it induces a unique map in homology:

$$H_i(F(f_\star))\colon H_i(F(P_\star)) \longrightarrow H_i(F(Q_\star)).$$

This shows that the action of $L_i(F)$ on maps is well-defined. Moreover, by taking $f = \mathsf{id}_A$, we show that, up to isomorphism, $L_i(F)(A) = H_i(F(P_\star))$ does not depend on the choice of the projective resolution $P_\star$ of $A$. Functoriality now follows immediately.

*b*) Let $A \longleftarrow P_\star$ be a projective resolution of $A$. Then, by right exactness of $F$, we get the exact sequence $F(P_1) \xrightarrow{F(d_1)} F(P_0) \xrightarrow{F(d_0)} F(A) \longrightarrow 0$, showing that $F(A) \cong \operatorname{coker}(F(d_1))$. However, we know by the definition of the left derived functor that $L_0(F)(A) = H_0(F(P_\star)) = \operatorname{coker}(F(d_1))$, hence $L_0(F)(A) \cong F(A)$.

*c*) By Lemma 4.2.7 below, we can derive projective resolutions $P_\star$, $Q_\star$ and $R_\star$ for $A$, $B$ and $C$, respectively, with the sequence $0 \longrightarrow P_\star \longrightarrow Q_\star \longrightarrow R_\star \longrightarrow 0$ exact and pointwise split. Since additive functors preserve split exact sequences, it follows that $0 \longrightarrow F(P_\star) \longrightarrow F(Q_\star) \longrightarrow F(R_\star) \longrightarrow 0$ is also exact (and pointwise split). Hence, by Proposition 1.9.8, there is a long exact sequence in homology:

$$\cdots \longrightarrow H_1(F(R_\star)) \xrightarrow{\delta} H_0(F(P_\star)) \longrightarrow H_0(F(Q_\star)) \longrightarrow H_0(F(R_\star)) \longrightarrow 0.$$

*d*) If $A$ is a projective object, then $0 \longrightarrow A \xrightarrow{\mathsf{id}_A} A \longrightarrow 0$ is a projective resolution of $A$; so, this part of the statement follows immediately from *a*). □

**Lemma 4.2.7** *Let $0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0$ be a short exact sequence in an abelian category $\mathsf{A}$, and suppose $A \xleftarrow{d} P_\star$ and $C \xleftarrow{l} R_\star$ are two projective resolutions of $A$ and $C$, respectively. Then, there is a projective resolution $B \xleftarrow{e} Q_\star$ and two chain maps $f_\star\colon P_\star \longrightarrow Q_\star$ and $g_\star\colon Q_\star \longrightarrow R_\star$ (lifting $f$ and $g$, respectively), such that the following sequence is exact and poinwise split:*

$$0 \longrightarrow P_\star \xrightarrow{f_\star} Q_\star \xrightarrow{g_\star} R_\star \longrightarrow 0. \tag{4.3}$$

**Proof.** Since $g\colon B \longrightarrow C$ is epic and $R_0$ is projective, there exists a morphism $\alpha\colon R_0 \longrightarrow B$ such that $g\alpha = l$. Consider the direct sum $Q_0 = P_0 \oplus R_0$, with injections $i$ and $j$, and projections $p$ and $q$ from and to $P_0$ and $R_0$, respectively. Then, the composite $fd$ and the map $\alpha$ determine, via the universal property of $Q_0$, a unique map $e\colon Q_0 \longrightarrow B$ such that $ei = fd$ and $ej = \alpha$:

$$\begin{tikzcd}
0 \arrow[r] & A \arrow[r, "f"] & B \arrow[r, "g"] & C \arrow[r] & 0 \\
0 \arrow[r] & P_0 \arrow[r, "i"] \arrow[u, "d"] & P_0 \oplus R_0 \arrow[u, dotted, "e"] \arrow[r, shift left, "q"] & R_0 \arrow[r] \arrow[u, "l"'] \arrow[ul, "\alpha"'] \arrow[l, shift left, "j"] & 0.
\end{tikzcd} \tag{4.4}$$

We have $ge = ge(\mathsf{id}_{Q_0}) = (ge)(ip + jq) = g(ei)p + g(ej)q = gfd + (g\alpha)q = lq$; moreover, the map $e$ is epic. To see this, suppose $h\colon B \longrightarrow E$ is a morphism such that $he = 0$; we then have $0 = hei = hfd$, and $hf = 0$, because $d$ is an epimorphism. Therefore, $h$ factors through a map $y\colon C \longrightarrow E$, so that $yg = h$, and we have $0 = he = yge = ylq$. Since $l$ and $q$ are epic, we get $y = 0$, and therefore $h = 0$.

Now, form the kernel of the maps $d$, $e$ and $l$, with inclusions $i_1$, $i_2$ and $i_3$, respectively. Then, the maps $i$ and $q$ restrict to maps $i'$ and $q'$, which form the sequence

$$0 \longrightarrow \ker(d) \xrightarrow{i'} \ker(e) \xrightarrow{q'} \ker(l) \longrightarrow 0. \tag{4.5}$$

The map $i'$ is clearly monic, since $i$ and $i_1$ are. Moreover, $q'$ is epic. To see this, notice first that there is an obvious epimorphism $l'\colon R_1 \longrightarrow \ker(l)$. Then, we have $geji_3l' = 0$, hence the composite $eji_3l'$ factors as $f\gamma$, for some map $\gamma\colon R_1 \longrightarrow A$, and because $R_1$ is projective, this factors in turn as $\gamma = d\beta$, for a map $\beta\colon R_1 \longrightarrow P_0$. Now, consider the map $\phi = ji_3l' - i\beta$. We have that $q\phi = qji_3l' - qi\beta = i_3l'$; also, $e\phi = eji_3l' - ei\beta = eji_3l' - fd\beta = eji_3l' - f\gamma = 0$, hence $\phi = i_2\psi$, for some $\psi\colon R_1 \longrightarrow \ker(e)$. Therefore, $i_3q'\psi = qi_2\psi = q\phi = i_3l'$, and since $i_3$ is monic, we have $l' = q'\psi$. This shows that $q'$ is epic, because $l'$ is. This proves exactness of (4.5) at $\ker(d)$ and $\ker(l)$.

To prove that it is exact also at $\ker(e)$, notice that $i_3q'i' = qi_2i' = qii_1 = 0$, and because $i_3$ is monic, $q'i' = 0$, which implies $\mathrm{im}(i') \subseteq \ker(q')$. Conversely, given a map $h\colon X \longrightarrow Q_0$ such that $eh = 0$ (i.e. $h$ factors through $\ker(e)$) and $qh = 0$, we have $h = id_{Q_0}h = (ip + jq)h = iph$, hence $fdph = eiph = eh = 0$. Because $f$ is monic, we get that $dph = 0$, so we have factored $h$ through $i$ via $ph$, which in turn factors through $\ker(d)$. This shows that $\ker(q') \subseteq \mathrm{im}(i')$, hence (4.5) is an exact sequence. We then have the following diagram in which $d', l'$ are epimorphisms and the map $e'$ is determined in the same way as the map $e$ in (4.4) above:

$$\begin{tikzcd}
0 \arrow[r] & P_0 \arrow[r, "i"] & P_0 \oplus R_0 \arrow[r, "q"] & R_0 \arrow[r] & 0 \\
0 \arrow[r] & \ker(d) \arrow[r, "i'"] \arrow[u, "i_1"] & \ker(e) \arrow[r, "q'"] \arrow[u, "i_2"] & \ker(l) \arrow[r] \arrow[u, "i_3"] & 0 \\
0 \arrow[r] & P_1 \arrow[r, "i"'] \arrow[u, "d'"] & P_1 \oplus R_1 \arrow[r, "q"'] \arrow[u, dotted, "e'"] & R_1 \arrow[r] \arrow[u, "l'"] & 0.
\end{tikzcd}$$

We now set $Q_1 = P_1 \oplus R_1$ and $e = i_2 e'$, and proceed by induction in defining the objects $Q_i = P_i \oplus R_i$ and the morphism $e$. It is clear by construction that the chain complex $B \longleftarrow Q_\star$ is a projective resolution of $B$, and the maps $i$ and $q$ form a pointwise split exact sequence. $\square$

**Remark 4.2.8** Notice that the sequence (4.3) in the previous statement is a short exact sequence of chain complexes, which is pointwise split. However, this does not mean that it is itself a split sequence of chain complexes.

**Definition 4.2.9** Let $\mathsf{A}$ and $\mathsf{X}$ be abelian categories. A *(homological) $\delta$-functor* is a sequence of functors $G_n\colon \mathsf{A} \longrightarrow \mathsf{X}$ $(n \geq 0)$ such that, for any short exact sequence $0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$, there exist morhisms $\delta\colon G_n(C) \longrightarrow G_{n-1}(A)$ $(n \geq 0)$ which satisfy the following properties:

*a)* there is a long exact sequence

$$\cdots \longrightarrow G_1(C) \xrightarrow{\delta} G_0(A) \longrightarrow G_0(B) \longrightarrow G_0(C) \longrightarrow 0;$$

*b)* the morphisms $\delta$ are natural; that is, for any commutative diagram

$$\begin{tikzcd} 0 \arrow[r] & A \arrow[r] \arrow[d] & B \arrow[r] \arrow[d] & C \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & A' \arrow[r] & B' \arrow[r] & C' \arrow[r] & 0 \end{tikzcd}$$

of short exact sequences, the following diagram is commutative for all $n > 0$:

$$\begin{tikzcd} G_n(C) \arrow[r, "\delta"] \arrow[d] & G_{n-1}(A) \arrow[d] \\ G_n(C') \arrow[r, "\delta'"'] & G_{n-1}(A'). \end{tikzcd}$$

**Example 4.2.10** For any right exact functor $F\colon \mathsf{A} \longrightarrow \mathsf{X}$, the derived functors and the connecting morphism described in Proposition 4.2.6 *c)* form a (homological) $\delta$-functor $(L_i(F), \delta)$.

**Definition 4.2.11** A *morphism* $\phi\colon (G, \delta) \longrightarrow (G', \delta')$ between $\delta$-functors is a sequence of natural transformations $\phi_n\colon G_n \longrightarrow G'_n$ which are compatible with $\delta$'s, that is, for any short exact sequence $0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$, the following square commutes:

$$\begin{tikzcd} G_n(C) \arrow[r, "\delta"] \arrow[d, "\phi_n"'] & G_{n-1}(A) \arrow[d, "\phi_{n-1}"] \\ G'_n(C) \arrow[r, "\delta'"'] & G'_{n-1}(A). \end{tikzcd}$$

Now, we are ready to give a universal characterisation of left derived functors.

**Theorem 4.2.12 (Universal property of left derived functors)** *Let $F$ be a right exact functor $\mathsf{A} \longrightarrow \mathsf{X}$ between abelian categories, where $\mathsf{A}$ has enough projectives, and let $(G, \delta')\colon \mathsf{A} \longrightarrow \mathsf{X}$ be a (homological) $\delta$-functor. Then:*

*a) any natural transformation $\phi_0: G_0 \longrightarrow L_0(F)$ extends uniquely to a morphism $\phi: (G, \delta') \longrightarrow (L_\star(F), \delta)$ of homological $\delta$-functors;*

*b) if $G_n(P) = 0$ for all $n > 0$ and all projective objects $P$, and $\phi_0$ is a (natural) isomorphism, then so is each $\phi_i$.*

**Proof.** *a*) Since $\mathsf{A}$ has enough projectives, we can build a short exact sequence $0 \longrightarrow K \longrightarrow P \longrightarrow A \longrightarrow 0$, with $P$ projective. Given $\phi_0: G_0 \longrightarrow L_0(F) = F$, we define $\phi_n: G_n \longrightarrow L_n(F)$ by induction on $n$, as follows. Consider the following diagram:

$$\begin{tikzcd}
\cdots \arrow[r] & L_1(F)(P) \arrow[r] & L_1(F)(A) \arrow[r, "\delta"] & L_0(F)(K) \arrow[r, "d_0"] & L_0(F)(P) \\
\cdots \arrow[r] & G_1(P) \arrow[r] & G_1(A) \arrow[r, "\delta'"'] \arrow[u, dotted, "\phi_1"] & G_0(K) \arrow[r, "d_0'"'] \arrow[u, "\phi_0"] & G_0(P). \arrow[u, "\phi_0"']
\end{tikzcd} \tag{4.6}$$

Since $P$ is projective, $L_1(F)(P) = 0$ by part *d*) of Proposition 4.2.6. Therefore, $\delta = \ker(d_0)$ and because $d_0\phi_0\delta' = 0$, the composite $\phi_0\delta'$ factors through a unique map $\phi_1: G_1(A) \longrightarrow L_1(F)(A)$ such that $\delta\phi_1 = \phi_0\delta'$. Now, suppose we have constructed $\phi_1, \phi_2, \cdots, \phi_i$; Then, we construct $\phi_{i+1}$ analogously, from the following diagram:

$$\begin{tikzcd}
\cdots \arrow[r] & L_{i+1}(F)(P) \arrow[r] & L_{i+1}(F)(A) \arrow[r, "\delta"] & L_i(F)(K) \arrow[r, "d_i"] & L_i(F)(P) \\
\cdots \arrow[r] & G_{i+1}(P) \arrow[r] & G_{i+1}(A) \arrow[r, "\delta'"'] \arrow[u, dotted, "\phi_{i+1}"] & G_i(K) \arrow[r, "d_i'"'] \arrow[u, "\phi_i"] & G_i(P). \arrow[u, "\phi_i"']
\end{tikzcd}$$

We leave to the reader the proof of the fact that this definition of the $\phi_i$'s does not depend on the chosen projective $P$ (or on the object $K$) and that they actually define a morphism of homological $\delta$-functors.

*b*) Consider diagram (4.6) above. Since $G_1(P) = 0$, we have $\delta' = \ker(d_0')$. Hence, there is a morphism $\phi_1': L_1(F)(A) \longrightarrow G_1(A)$ with $\delta'\phi_1 = \phi_0^{-1}\delta$, because $\phi_0$ is an isomorphism. By the definition of the kernel, it follows at once that $\phi_1$ and $\phi_1'$ are mutually inverse, hence $\phi_1$ is an isomorphism. The same holds for all $\phi_i$, by induction. (Alternatively, one can use the 5-lemma, Proposition 1.6.8.) $\square$

### 4.2.2 Right derived functors

The notion of right derived functor is dual to that of a left derived one. In particular, we shall start with a left exact functor, and we shall need injective objects instead of projective ones. The following results are also dual to the ones in the previous section, therefore we shall omit their proofs (see exercise *c*) at the end of this Section).

**Definition 4.2.13** An object $I$ in an additive category $\mathsf{A}$ is *injective* if the functor $\mathrm{Hom}_{\mathsf{C}}(-, I): \mathsf{C} \longrightarrow \mathsf{Ab}$ is exact. Equivalently, $I$ is injective if, for all monomorphisms $f\colon A \longrightarrow B$ and maps $g\colon A \longrightarrow I$ there is an extension of $g$ along $f$:

$$\begin{tikzcd} A \arrow[r, "g"] \arrow[d, "f"'] & I. \\ B \arrow[ur, dotted] & \end{tikzcd}$$

We say that $\mathsf{A}$ has *enough injectives* if, for any object $A$ in $\mathsf{A}$, there exists a monomorphism $f\colon A \longrightarrow I$, where $I$ is an injective object.

**Definition 4.2.14 (Dual of Definition 4.2.4)** Let $\mathsf{A}$ and $\mathsf{X}$ be abelian categories such that $\mathsf{A}$ has enough injectives. Let $F\colon \mathsf{A} \longrightarrow \mathsf{X}$ be a left exact functor. Then, the *right derived* functors of $F$ are the functors $R^i(F)\colon \mathsf{A} \longrightarrow \mathsf{X}$ $(i \in \mathbb{N})$ which, for an object $A$ in $\mathsf{A}$ with an injective resolution $0 \longrightarrow A \longrightarrow I^\star$, are defined as

$$R^i(F)(A) = H^i(F(I^\star)) \quad \text{for} \quad i \geq 0.$$

Given a morphism $f\colon A \longrightarrow B$ in $\mathsf{A}$, where $0 \longrightarrow A \longrightarrow I^\star$ and $0 \longrightarrow B \longrightarrow J^\star$ are two injective resolutions of $A$ and $B$, respectively, there exists a (unique, up to homotopy) chain complex map $f^\star\colon I^\star \longrightarrow J^\star$ which extends $f$. The action of the derived functor $R^i(F)$ on $f$ is then defined as

$$R^i(F)(f) = H^i(F(f^\star)).$$

**Lemma 4.2.15 (Dual of Lemma 4.2.7)** *Let* $0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0$ *be a short exact sequence in an additive category* $\mathsf{A}$*. Suppose* $A \longrightarrow I^\star$ *and* $C \longrightarrow K^\star$ *are two injective resolutions of* $A$ *and* $C$*, respectively. Then, there is an injective resolution* $B \longrightarrow J^\star$ *and two chain maps* $f^\star\colon I^\star \longrightarrow J^\star$ *and* $g^\star\colon J^\star \longrightarrow K^\star$ *such that the sequence*

$$0 \longrightarrow I^\star \xrightarrow{f^\star} J^\star \xrightarrow{g^\star} K^\star \longrightarrow 0$$

*is exact and pointwise split.*

**Proposition 4.2.16 (Dual of Proposition 4.2.6)** *Let* $\mathsf{A}$ *be an abelian category with enough injectives and* $F$ *a left exact functor. Then:*

*a) the object* $R^i(F)(A)$ *is independent of the injective resolution* $I^\star$ *of* $A$*, and the image along* $R^i(F)(-)$ *of a map* $f\colon A \longrightarrow B$ *is independent of the chain map* $f^\star$ *lifting* $f$*. Moreover,* $R^i(F)(-)$ *is a covariant functor;*

*b) for any* $A$ *in* $\mathsf{A}$*, we have* $R_0(F)(A) \cong FA$*. More precisely there is a natural isomorphism* $R_0(F) \cong F$*;*

*c) any short exact sequence in* $\mathsf{A}$

$$0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0$$

*gives rise to a long exact sequence in* $\mathsf{X}$

$$0 \longrightarrow R_0(F)(A) \longrightarrow R_0(F)(B) \longrightarrow R_0(F)(C) \xrightarrow{\delta} R_1(F)(A) \longrightarrow \cdots;$$

*d) if* $A$ *is injective in* $\mathsf{A}$*, then* $R^i(F)(A) = 0$ *for* $i > 0$*.*

**Definition 4.2.17** A *(cohomological) $\delta$-functor* between abelian categories $\mathsf{A}$ and $\mathsf{X}$ is a sequence of functors $G^n\colon \mathsf{A} \longrightarrow \mathsf{X}$ $(n > 0)$ such that for any short exact sequence $0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$, there exist morphisms $\delta\colon G^n(C) \longrightarrow G^{n+1}(A)$ satisfying the following properties:

*a)* there is a long exact sequence

$$0 \longrightarrow G^0(A) \longrightarrow G^0(B) \longrightarrow G^0(C) \xrightarrow{\delta} G^1(A) \longrightarrow \cdots ;$$

*b)* the morphism $\delta$ is natural; that is, for any commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & A \arrow[r] \arrow[d] & B \arrow[r] \arrow[d] & C \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & A' \arrow[r] & B' \arrow[r] & C' \arrow[r] & 0
\end{tikzcd}$$

of short exact sequences, the following square commutes for all $n > 0$:

$$\begin{tikzcd}
G^n(C) \arrow[r, "\delta"] \arrow[d] & G^{n+1}(A) \arrow[d] \\
G^n(C') \arrow[r, "\delta'"'] & G^{n+1}(A').
\end{tikzcd}$$

**Example 4.2.18** For any left exact functor $F\colon \mathsf{A} \longrightarrow \mathsf{X}$, the right derived functors and the connecting morphisms of Proposition 4.2.16 *c)* form a (cohomological) $\delta$-functor $(R^i(F), \delta)$.

**Definition 4.2.19** Let $(G, \delta)$ and $(G', \delta')$ be two (cohomological) $\delta$-functors. A *morphism* $\phi\colon (G, \delta) \longrightarrow (G', \delta')$ consists of a sequence of natural transformations $\phi^n\colon G^n \longrightarrow G'^n$ which are compatible with $\delta$'s; that is, for every short exact sequence $0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$, the following commutes:

$$\begin{tikzcd}
G^n(C) \arrow[r, "\delta"] \arrow[d, "\phi^n"'] & G^{n+1}(A) \arrow[d, "\phi^{n+1}"] \\
G'^n(C) \arrow[r, "\delta'"'] & G'^{n+1}(A).
\end{tikzcd}$$

**Theorem 4.2.20 (Universal property of right derived functors)** *Let $F$ be a left exact functor $\mathsf{A} \longrightarrow \mathsf{X}$ between abelian categories, where $\mathsf{A}$ has enough injectives, and $(G, \delta')\colon \mathsf{A} \longrightarrow \mathsf{X}$ a (cohomological) $\delta$-functor. Then:*

*a) any natural transformation $\phi^0\colon R^0(F) \longrightarrow G^0$ extends uniquely to a morphism $\phi\colon (R^\star(F), \delta) \longrightarrow (G, \delta')$ of cohomological $\delta$-functors;*

*b) if $G^n(I) = 0$ for all $n > 0$ and all injective objects $I$, and $\phi^0$ is a (natural) isomorphism, then so is each $\phi^i$.*

## Exercises

*a)* Give an argument for the statement of Remark 4.2.3.

*b)* Reformulate Theorem 4.2.12 *a*) in terms of an adjunction between the category of right exact functors from $\mathsf{A}$ to $\mathsf{X}$ and that of homological $\delta$-functors between the same categories.

*c)* Check that the definition and results of Section 4.2.2 are reformulations of those of Section 4.2.1 for the dual category $\mathsf{A}^{\mathrm{op}}$, hence do not need to be proved again.

*d)* Reformulate Theorem 4.2.20 *a*) in terms of an adjunction between the category of left exact functors from $\mathsf{A}$ to $\mathsf{X}$ and that of cohomological $\delta$-functors between the same categories.

## 4.3 Cohomology of Small Categories

In this Section, we shall apply the machinery of derived functors to the study of cohomology of small categories. Starting with a fixed small category $\mathsf{C}$, we shall associate an abelian category $\mathsf{Ab}(\mathsf{C})$ to it, and a left exact functor $\Gamma\colon \mathsf{Ab}(\mathsf{C}) \longrightarrow \mathsf{Ab}$. The right derived functors $R^n\Gamma$ will then describe the cohomology of $\mathsf{C}$, as obtained from its nerve.

So, we start by defining the category $\mathsf{Ab}(\mathsf{C})$ of *presheaves of abelian groups on* $\mathsf{C}$. This is the functor category $[\mathsf{C}^{\mathrm{op}}, \mathsf{Ab}]$, whose objects are contravariant functors from $\mathsf{C}$ to $\mathsf{Ab}$ and whose arrows are natural transformations (see Remark A.0.13 in the Appendix).

For an arrow $u\colon C \longrightarrow D$ in $\mathsf{C}$ and a presheaf $A\colon \mathsf{C}^{\mathrm{op}} \longrightarrow \mathsf{Ab}$ in $\mathsf{Ab}(\mathsf{C})$, the arrow $A(u)\colon A(D) \longrightarrow A(C)$ will be sometimes denoted by $u^*$. Recall that the naturality condition in the definition of morphisms in $\mathsf{Ab}(\mathsf{C})$ reads as commutativity of the following diagram:

$$\begin{tikzcd} A(D) \arrow[r, "A(u)"] \arrow[d, "f_D"'] & A(C) \arrow[d, "f_C"] \\ B(D) \arrow[r, "B(u)"'] & B(C). \end{tikzcd}$$

For every object $C$ in $\mathsf{C}$, there is an evaluation functor $ev_C\colon \mathsf{Ab}(\mathsf{C}) \longrightarrow \mathsf{Ab}$, defined on a presheaf $A$ as $ev_C(A) = A(C)$, and on a presheaf morphism $f\colon A \longrightarrow B$ as $ev_C(f) = f_C$.

**Example 4.3.1**

*a)* Consider a group $G$ as a category $\mathsf{G}$ with just one object $\star$ and one arrow $g\colon \star \to \star$ for every element $g \in G$. Then, $\mathsf{Ab}(\mathsf{G}) = \mathsf{mod}\text{-}G = \mathsf{mod}\text{-}\mathbb{Z}[G]$;

*b)* Consider the category $\Delta$, whose objects are ordered initial segments of the natural numbers $[n] = \{0, 1, ..., n\}$, $n \geq 0$, and arrows are non decreasing functions. Then, $\mathsf{Ab}(\Delta)$ is the abelian category of simplicial abelian groups.

*c)* Let $X$ be a topological space with a basis $\mathcal{B}$ of open sets. View $\mathcal{B}$ as a category whose arrows are inclusions. Many "geometric structures" on $X$ give rise to objects of $\mathsf{Ab}(\mathcal{B})$; for example, if $E \longrightarrow X$ is a vector bundle, then there is a functor $\Gamma(E)$ in $\mathsf{Ab}(\mathcal{B})$, where $\Gamma(E)(U) = \Gamma(U, E)$, the sections of $E|_U$, for $U \in \mathcal{B}$.

**Proposition 4.3.2** *Let* $\mathsf{C}$ *be a small category. Then:*

*a)* $\mathsf{Ab}(\mathsf{C})$ *is an abelian category;*

*b)* *for each $C$ in $\mathsf{C}$, $ev_C$ is an exact functor;*

*c)* $\mathsf{Ab}(\mathsf{C})$ *has enough projectives.*

*The category $\mathsf{Ab}(\mathsf{C})$ also has enough injectives, but we will not prove this here.*

**Proof.** In order to prove *a)* we need to construct finite direct sums, kernels and cokernels of functors. We do this pointwise; for example, if $A$ and $B$ are presheaves in $\mathsf{Ab}(\mathsf{C})$, put $(A \oplus B)(C) := A(C) \oplus B(C)$, for a morphism $f: A \longrightarrow B$ in $\mathsf{Ab}(\mathsf{C})$, put $(\ker f)(C) := \ker(A(C) \longrightarrow B(C))$ and so on. We may take these as a definition and check that they have the required universal properties. As an illustration, we derive the existence of cokernels: let $(\operatorname{coker} f)(C) := \operatorname{coker}(A(C) \longrightarrow B(C))$ and, for an arrow $u: C \longrightarrow D$ in $\mathsf{C}$, $(\operatorname{coker} f)(u)$ is the unique arrow $\operatorname{coker} f(D) \longrightarrow \operatorname{coker} f(C)$ making the following commute:

$$\begin{tikzcd} A(D) \arrow[r, "f_D"] \arrow[d, "A(u)"'] & B(D) \arrow[r, "P_D"] \arrow[d, "B(u)"] & (\operatorname{coker} f)(D) \arrow[d, dotted] \\ A(C) \arrow[r, "f_C"'] & B(C) \arrow[r, "P_C"'] & (\operatorname{coker} f)(C). \end{tikzcd} \tag{4.7}$$

The mappings $C \mapsto (\operatorname{coker} f)(C)$ and $u \mapsto (\operatorname{coker} f)(u)$ determine a contravariant functor from $\mathsf{C}$ to $\mathsf{Ab}$, i.e. an object in $\mathsf{Ab}(\mathsf{C})$, which we denote by $\operatorname{coker} f$. Moreover, the collection of maps $P_C$ ($C$ in $\mathsf{C}$) form a natural transformation; that is, an arrow in $\mathsf{Ab}(\mathsf{C})$. It is immediate from (4.7) that the object $\operatorname{coker} f$ and the arrow $P$ form the cokernel of $f$ in $\mathsf{Ab}(\mathsf{C})$.

It is now immediate that *b)* holds, since we just saw that kernels and cokernels in $\mathsf{Ab}(\mathsf{C})$ are computed pointwise.

The proof of *c)* goes via a reduction to the category of sets. First of all, notice that, similarly to the definition of $\mathsf{Ab}(\mathsf{C})$, we may define $\mathsf{Set}(\mathsf{C})$ as the category of contravariant functors $S: \mathsf{C}^{\mathrm{op}} \longrightarrow \mathsf{Set}$ and natural transformations between them. As before, we have for each $C$ in $\mathsf{C}$ an evaluation functor $ev_C: \mathsf{Set}(\mathsf{C}) \longrightarrow \mathsf{Set}$. The important category of *simplicial sets* is of this type: it is the category $\mathsf{Set}(\Delta)$. Now, we need a couple of results.

**Lemma 4.3.3** *An arrow $f: S \longrightarrow T$ in $\mathsf{Set}(\mathsf{C})$ is an epimorphism if and only if $f_C: S(C) \longrightarrow T(C)$ is surjective for all $C$ in $\mathsf{C}$.*

**Proof.** See Exercise *a)* below. □

Now, notice that there is an obvious adjunction $\mathbb{Z}[-] \dashv U$ between the forgetful functor $U: \mathsf{Ab} \longrightarrow \mathsf{Set}$ and the free abelian group functor $\mathbb{Z}[-]: \mathsf{Set} \longrightarrow \mathsf{Ab}$. Composition with these two functors induces an adjunction

$$\begin{tikzcd} \mathsf{Ab}(\mathsf{C}) \arrow[r, shift left, "U"] \arrow[r, phantom, "\top"] & \mathsf{Set}(\mathsf{C}), \arrow[l, shift left, "\mathbb{Z}[-]"] \end{tikzcd} \tag{4.8}$$

where, by an abuse of notation, we denote again by $U$ the functor which pointwise forgets the abelian group structure, and by $\mathbb{Z}[-]$ the functor which pointwise takes the free abelian group; in other words, $\mathbb{Z}[S](C) = \mathbb{Z}[S(C)]$ for any functor $S: \mathsf{C}^{\mathrm{op}} \longrightarrow \mathsf{Set}$ and any $C$ in $\mathsf{C}$.

From Lemma 4.3.3, it then follows (see Exercise *b)* below) that:

**Remark 4.3.4**

*a)* For any $A$ in $\mathsf{Ab}(\mathsf{C})$, the $A$-th component of the counit

$$\epsilon_A \colon \mathbb{Z}[U(A)] \longrightarrow A$$

is epic;

*b)* by the dual of Lemma 1.8.12, since $\mathbb{Z}[-]$ is left-adjoint to $U$ and the latter clearly preserves epimorphisms, $\mathbb{Z}[-]$ preserves projectives;

*c)* if $P \longrightarrow S$ is an epimorphism in $\mathsf{Set}(\mathsf{C})$, then $\mathbb{Z}[P] \longrightarrow \mathbb{Z}[S]$ is epic in $\mathsf{Ab}(\mathsf{C})$ (see exercise *e*) in the Appendix).

Now, in order to prove that $\mathsf{Ab}(\mathsf{C})$ has enough projectives, it is enough to show that $\mathsf{Set}(\mathsf{C})$ does; for, if this is the case, then, given any presheaf $A \colon \mathsf{C}^{\mathrm{op}} \longrightarrow \mathsf{Ab}$, we can cover its image $U(A)$ in $\mathsf{Set}(\mathsf{C})$ with a projective $P$, and then, by Remark 4.3.4, the composite

$$\mathbb{Z}[P] \longrightarrow \mathbb{Z}[U(A)] \xrightarrow{\ \epsilon\ } A$$

is a covering of $A$ by an epimorphism from the projective $\mathbb{Z}[P]$.

To show that $\mathsf{Set}(\mathsf{C})$ has enough projectives, recall from the Appendix the notion of *representable* functor in $\mathsf{Set}(\mathsf{C})$ and Yoneda's lemma A.0.15. Then, the following result closes the proof of Proposition 4.3.2. □

**Proposition 4.3.5** *For a small category* $\mathsf{C}$*:*

*a) each representable presheaf in* $\mathsf{Set}(\mathsf{C})$ *is projective;*

*b)* $\mathsf{Set}(\mathsf{C})$ *has enough projectives.*

**Proof.**

*a)* Let $B$ in $\mathsf{C}$ and consider the diagram in $\mathsf{Set}(\mathsf{C})$

$$\begin{tikzcd} & T \arrow[d, "g"] \\ \mathrm{Hom}_{\mathsf{C}}(-,B) \arrow[r, "f"'] & S, \end{tikzcd}$$

where $g$ is an epimorphism. Then, by Yoneda's Lemma, a natural transformation $\mathrm{Hom}_{\mathsf{C}}(-,B) \longrightarrow T$ corresponds to an element of $T(B)$. Since $g_B$ is surjective, we may take any element in the preimage of $f_B(\mathsf{id}_B)$.

*b)* Let $S$ in $\mathsf{Set}(\mathsf{C})$ and consider the coproduct $P = \coprod_{B \in \mathsf{C}, s \in S(B)} \mathrm{Hom}_{\mathsf{C}}(-,B)$. Then, by *a)* $P$ is projective in $\mathsf{Set}(\mathsf{C})$. For each $B$ in $\mathsf{C}$ and $s \in S(B)$, Yoneda's Lemma gives an arrow $\mathrm{Hom}_{\mathsf{C}}(-,B) \longrightarrow S$, so by the universal property of coproducts we obtain an arrow $P \longrightarrow S$, which is epic by Lemma 4.3.3.

□

Now that we have established the fact that $\mathsf{Ab}(\mathsf{C})$ is an abelian category and has enough projectives, we can turn our attention to describing a left exact functor from $\mathsf{Ab}(\mathsf{C})$ to $\mathsf{Ab}$.

This will be the functor $\Gamma := \mathrm{Hom}_{\mathsf{Ab}(\mathsf{C})}(\tilde{\mathbb{Z}}, -) \colon \mathsf{Ab}(\mathsf{C}) \longrightarrow \mathsf{Ab}$, where $\tilde{\mathbb{Z}}$ is the constant functor mapping every object of $\mathsf{C}$ to $\mathbb{Z}$.

Recall that, for every abelian group $B$, we have $\mathrm{Hom}_{\mathsf{Ab}}(\mathbb{Z}, B) \simeq B$; therefore, for $A$ in $\mathsf{Ab}(\mathsf{C})$ and a natural transformation $\alpha$ in $\mathrm{Hom}_{\mathsf{Ab}(\mathsf{C})}(\tilde{\mathbb{Z}}, A)$, we can identify the $C$-th component of $\alpha$ (for $C$ in $\mathsf{C}$) with an element in $A(C)$. It follows that $\Gamma(A)$ is the set of functions $\alpha$ (also called *sections*) assigning to each object $C \in \mathsf{C}$ an element $\alpha(C) \in A(C)$ such that, for each arrow $u \colon D \longrightarrow C$, $u^*(\alpha(C)) = \alpha(D)$.

**Remark 4.3.6** The functor $\Gamma$ has a left adjoint $K \colon \mathsf{Ab} \longrightarrow \mathsf{Ab}(\mathsf{C})$, sending an abelian group to the corresponding constant presheaf. In particular, by Theorem 4.1.21, $\Gamma$ is left exact.

**Example 4.3.7** Looking back at Example 4.3.1, we have:

*a)* When $\mathsf{G}$ is the category associated to a group $G$, we recover $(-)^G$ as $\Gamma$; indeed, we have already seen in Example 4.3.1 *a*) that $\mathsf{Ab}(\mathsf{G}) = G\text{-}\mathsf{mod}$. Moreover, the functor $K$ above is just the trivial module functor. Hence, by exercise *b*) of Section 2.4 and uniqueness of adjoints, $(-)^G$ and $\Gamma$ are naturally isomorphic;

*b)* for $\mathsf{C} = \Delta$, we see that $\Gamma(A) \simeq A([0]) = A_0$ (the set of vertices).

## 4.3.1 The nerve of a category

Our next goal is to give an explicit description of $R^n\Gamma$, the right derived functors of $\Gamma$. To this end, we recall from Example 3.1.3 *b*) that the *nerve* of a (small) category $\mathsf{C}$ is the simplicial set $N_\star(\mathsf{C})$ defined as $N_0(\mathsf{C}) = |\mathsf{C}|$ and for $n \geq 1$ as $N_n(\mathsf{C}) = \{C_0 \xleftarrow{u_1} \cdots \xleftarrow{u_n} C_n\}$, with maps $d_i \colon N_n(\mathsf{C}) \longrightarrow N_{n-1}(\mathsf{C})$ which "omit $C_i$"; that is:

$$d_i(C_0 \xleftarrow{u_1} \cdots \xleftarrow{u_n} C_n) = \begin{cases} C_1 \xleftarrow{u_2} \cdots \xleftarrow{u_n} C_n, & i = 0 \\ C_0 \leftarrow \cdots C_{i-1} \xleftarrow{u_i u_{i+1}} C_{i+1} \cdots \leftarrow C_n, & 0 < i < n \\ C_0 \xleftarrow{u_1} \cdots \xleftarrow{u_{n-1}} C_{n-1}, & i = n \end{cases}$$

The $d_i$'s satisfy the simplicial identities $d_i d_{j+1} = d_j d_i$, for $i \leq j$.

For a presheaf $A$ in $\mathsf{Ab}(\mathsf{C})$, we can build a cochain complex $C^\star(\mathsf{C}, A)$ using the nerve $N_\star(\mathsf{C})$, by putting

$$C^n(\mathsf{C}, A) := \prod_{C_0 \xleftarrow{u_1} \cdots \xleftarrow{u_n} C_n} A(C_n),$$

with boundary maps

$$d = \sum_{i=0}^{n} (-1)^i \delta_i \colon C^{n-1}(\mathsf{C}, A) \longrightarrow C^n(\mathsf{C}, A),$$

where $\delta_i$ is defined on a tuple $\alpha = \{a_{C_0 \xleftarrow{u_1} \cdots \xleftarrow{u_{n-1}} C_{n-1}}\} \in C^{n-1}(\mathsf{C}, A)$ as

$$\delta_i(\alpha)_{C_0 \xleftarrow{u_1} \cdots \xleftarrow{u_n} C_n} = \begin{cases} a_{d_i(C_0 \xleftarrow{u_1} \cdots \xleftarrow{u_n} C_n)}, & i < n \\ u_n^*(a_{d_n(C_0 \xleftarrow{u_1} \cdots \xleftarrow{u_n} C_n)}), & i = n. \end{cases}$$

The cohomology of this cochain complex gives an explicit description of the right derived functors of $\Gamma$:

**Theorem 4.3.8** *For any object $A$ in $\mathsf{Ab(C)}$, there is a natural isomorphism $R^n\Gamma(A) \simeq H^n(C^\star(\mathsf{C}, A))$. We denote this group by $H^n(\mathsf{C}, A)$.*

**Proof.** By definition, $R^n\Gamma(A) = H^n(\mathrm{Hom}_{\mathsf{Ab(C)}}(\tilde{\mathbb{Z}}, I^\star))$, where $I^\star$ is an injective resolution of $A$ in $\mathsf{Ab(C)}$. But, by the double complex lemma 1.11.5, the cohomology can also be computed as $H^n(\mathrm{Hom}_{\mathsf{Ab(C)}}(P_\star, A))$, for any projective resolution

$$\cdots \to P_0 \to \tilde{\mathbb{Z}} \to 0 \tag{4.9}$$

of $\tilde{\mathbb{Z}}$ in $\mathsf{Ab(C)}$. In particular, the following is a projective resolution of $\tilde{\mathbb{Z}}$: define

$$P_n = \coprod_{C_0 \xleftarrow{u_1} \cdots \xleftarrow{u_n} C_n} \mathbb{Z}[\mathrm{Hom}_{\mathsf{C}}(-, C_n)],$$

where $\mathbb{Z}[-]$ is as in (4.8). The maps $d \colon P_n \longrightarrow P_{n-1}$ of this complex are defined as

$$d = \sum_{i=0}^{n} (-1)^i \delta_i,$$

where $\delta_i \colon P_n \longrightarrow P_{n-1}$ takes the summand $\mathbb{Z}[\mathrm{Hom}_{\mathsf{C}}(-, C_n)]$ indexed by the chain $C_0 \xleftarrow{u_1} \cdots \xleftarrow{u_n} C_n$ to the summand indexed by $d_i(C_0 \xleftarrow{u_1} \cdots \xleftarrow{u_n} C_n)$, for $i = 0, \ldots, n$.

Clearly, by Remark 4.3.4 $b$) and Proposition 4.3.5 $a$), the $P_n$'s are projective. Once we show that they form an exact sequence, Yoneda lemma proves the theorem, since

$$\begin{aligned}
\mathrm{Hom}_{\mathsf{Ab(C)}}(P_n, A) &\cong \prod_{C_0 \leftarrow \cdots \leftarrow C_n} \mathrm{Hom}_{\mathsf{Ab(C)}}(\mathbb{Z}[\mathrm{Hom}_{\mathsf{C}}(-, C_n)], A) \\
&\cong \prod_{C_0 \leftarrow \cdots \leftarrow C_n} \mathrm{Hom}_{\mathsf{Set(C)}}(\mathrm{Hom}_{\mathsf{C}}(-, C_n), U(A)) \\
&\cong \prod_{C_0 \leftarrow \cdots \leftarrow C_n} U(A(C_n)) = C^n(\mathsf{C}, A).
\end{aligned}$$

So, all we have to do is to show exactness of (4.9) in $\mathsf{Ab(C)}$, and for this it is enough to show the exactness of $\cdots \to P_0(B) \to \mathbb{Z} \to 0$ for each $B$ in $\mathsf{C}$. But first, we make some general remarks.

If $\mathsf{B}$ is a small category, then $N_\star(\mathsf{B})$ gives a chain complex

$$\cdots \to \mathbb{Z}[N_n(\mathsf{B})] \xrightarrow{\partial} \mathbb{Z}[N_{n-1}(\mathsf{B})] \to \cdots$$

where $\partial_n = \sum_{i=0}^n (-1)^i d_i$, the $d_i$'s being as in the definition of the nerve of a category, above.

**Lemma 4.3.9** *If $\mathsf{B}$ has an initial object, then $\mathbb{Z}[N_\star(\mathsf{B})]$ is a contractible chain complex (hence its homology is zero).*

**Proof.** Recall that a chain complex $C_\star$ is contractible if $\mathsf{id}_{C_\star}$ is homotopic to the zero map of complexes; that is, if the equality $\mathsf{id} = \partial h + h\partial$ holds for opportune maps $h_n: \mathbb{Z}[N_{n-1}(\mathsf{B})] \longrightarrow \mathbb{Z}[N_n(\mathsf{B})]$. Let $I$ be the initial object of $\mathsf{B}$. Then, the maps defined by

$$h_n(C_0 \leftarrow \cdots \leftarrow C_{n-1}) = (-1)^n (C_0 \leftarrow \cdots \leftarrow C_{n-1} \leftarrow I)$$

satisfy the condition $d_i h_{n+1} = -h_n d_i$ for $i = 0, \ldots, n$ and $d_n h_n = (-1)^n \mathsf{id}$. Therefore, $\mathsf{id} = \partial h + h\partial$. □

Now, fix an object $B$ in $\mathsf{C}$. The *slice category* $B/\mathsf{C}$ has as objects all pairs $(u, C)$ with $C$ in $\mathsf{C}$ and $u: B \longrightarrow C$, and as arrows $f: (u, C) \longrightarrow (v, C')$ all those arrows $f: C \longrightarrow C'$ for which $fu = v$. The definition of arrows in $B/\mathsf{C}$ allows us to identify $N_n(B/\mathsf{C})$ with the set $S_n$ of strings $C_0 \leftarrow \cdots \leftarrow C_n \leftarrow B$, hence $S_\star$ is exactly the nerve $N_\star(B/\mathsf{C})$. Also, the sum of all abelian groups $\mathbb{Z}[\mathrm{Hom}_{\mathsf{C}}(B, C_n)]$, indexed over the strings $C_0 \leftarrow \cdots \leftarrow C_n$, is isomorphic to $\mathbb{Z}[S_n]$, hence $P_n(B) \cong \mathbb{Z}[S_n]$. From this we derive (see Exercise *c*) below)

**Proposition 4.3.10** *The complexes $P_\star(B)$ and $\mathbb{Z}[N_\star(B/\mathsf{C})]$ are isomorphic.*

Now, since $(\mathsf{id}_B, B)$ is an initial object for $B/\mathsf{C}$, Lemma 4.3.9 and Proposition 4.3.10 imply the exactness of:

$$\cdots \to P_1(B) \to P_0(B) \to \mathbb{Z} \to 0.$$

This finishes the proof of Theorem 4.3.8. □

## Exercises

*a)* For an object $C$ in $\mathsf{C}$, show that the evaluation functor $ev_C: \mathsf{Set}(\mathsf{C}) \longrightarrow \mathsf{Set}$ has a right adjoint defined by

$$R_C(S) = \mathrm{Hom}_{\mathsf{Set}}(\mathrm{Hom}_{\mathsf{C}}(C, -), S);$$

Use this adjunction to prove Lemma 4.3.3.

*b)* Give a proof of *a*), *b*) and *c*) in Remark 4.3.4.

*c)* Prove Proposition 4.3.10 in detail.

# 4.4 Sheaf cohomology

For a fixed topological space $X$, we shall construct an abelian category Sh(X) of *sheaves on* $X$, together with a left exact functor $\Gamma: \mathrm{Sh(X)} \longrightarrow \mathsf{Ab}$. Then, we shall define the *sheaf cohomology* of the space $X$ as $H^n(X, A) = (R^n\Gamma)(A)$, for $A$ in Sh(X).

As a starting point, consider the partial order $\mathcal{O}(X)$, the set of open subsets of $X$ as a small category. We can form the abelian category $\mathsf{Ab}(\mathcal{O}(X))$ of *presheaves of abelian groups on* $X$, just like in Section 4.3. Its objects are contravariant functors from $\mathcal{O}(X)$ to $\mathsf{Ab}$; for an inclusion $U \subseteq V$ of open subsets of $X$, such a presheaf $A$ in $\mathsf{Ab}(\mathcal{O}(X))$ determines a map $A(V) \longrightarrow A(U)$, called a *restriction morphism.* The image of an element $a \in A(V)$ along such a restriction morphism will be denoted by $a|_U$.

Just as in Section 4.3, for any open set $U \in \mathcal{O}(X)$, there is an exact evaluation functor $ev_U: \mathsf{Ab}(\mathcal{O}(X)) \longrightarrow \mathsf{Ab}$. The definition of the functor $\Gamma$ stands also in this context, and one can check that, for any presheaf $A$ in $\mathsf{Ab}(\mathcal{O}(X))$, $\Gamma(A) = ev_X(A) = A(X)$.

Now, we define Sh(X) as a full subcategory of $\mathsf{Ab}(\mathcal{O}(X))$ and, with an abuse of notation, we still write $\Gamma$ for the restriction $\Gamma: \mathrm{Sh(X)} \longrightarrow \mathsf{Ab}$. It is important to note that this restriction will not be exact; in particular, the inclusion functor $i: \mathrm{Sh(X)} \longrightarrow \mathsf{Ab}(\mathcal{O}(\mathrm{X}))$ is not exact.

**Definition 4.4.1** Let $A \in \mathsf{Ab}(\mathcal{O}(X))$ be a presheaf of abelian groups on $X$, $U \in \mathcal{O}(X)$ and consider an open cover $\{U_i\}_{i\in I}$ of $U$. A family $\{a_i \in A(U_i)\}_{i\in I}$ is called *compatible* if $a_i|_{U_{ij}} = a_j|_{U_{ij}}$ for all $i, j$, where $U_{ij} = U_i \cap U_j$. An *amalgamation*, or *glueing*, of the family $\{a_i\}$ is an $a \in A(U)$ with $a|_{U_i} = a_i$ for all $i$.

We say that $A$ is a *sheaf* if for every open cover $\{U_i\}$ of an open $U \in \mathcal{O}(X)$, every compatible family has a unique amalgamation.

**Example 4.4.2** Classical examples of sheaves are:

*a)* Define $\tilde{\mathbb{R}}(U) := C^0(U, \mathbb{R})$, the set of continuous functions from $U$ to $\mathbb{R}$, and for $U \subseteq V$ let the map $\tilde{\mathbb{R}}(V) \to \tilde{\mathbb{R}}(U)$ be the usual restriction of functions. Then, $\tilde{\mathbb{R}}$ is a sheaf.

*b)* For any topological abelian group $A$, we have a similar sheaf $\tilde{A}$, where $\tilde{A}(U) = C^0(U, A)$. If $A$ has the discrete topology, we also denote the sheaf $\tilde{A}$ by $K(A)$, and in this case $K(A)(U)$ is the set of locally constant functions $U \to A$.

*c)* Let $p: E \longrightarrow X$ be a vector bundle on $X$. We define a sheaf $\tilde{E}$ by

$$\tilde{E}(U) := \Gamma(U, E) = \{s: U \to E \mid s \text{ continuous and } ps = id\}.$$

This construction makes sense for any bundle of abelian groups $A \to X$.

### 4.4.1 Germs of functions

Let $U$ be an open subset of $X$, $f: U \longrightarrow \mathbb{R}$ a continuous map and $x \in U$. The germ of $f$ at $x$, $germ_x(f)$, is the equivalence class of all those functions whose behaviour in a neighbourhood of $x$ is equal to that of $f$. More precisely, we define the set $Germ(x)$ of *germs of continuous functions at* $x$ as the set of equivalence classes of pairs $(f, U)$, where $f \in C^0(U, \mathbb{R})$ and $x \in U$, modulo the equivalence relation defined by $(f, U) \sim (g, V)$ iff there is $W \in \mathcal{O}(X)$ with $x \in W \subseteq U \cap V$ such that $f|_W = g|_W$.

Using the group structure of $\mathbb{R}$, we can induce on the set $Germ(x)$ an abelian group structure, by letting

$$[(f, U)] + [(f', U')] = [(f|_{U\cap U'} + f'|_{U\cap U'}, U \cap U')].$$

Categorically, we can present this construction as a colimit: if $U$ is an open neighbourhood of $x$, then

$$Germ(x) = \varinjlim_{U\subseteq X} C^0(U, \mathbb{R}) = \varinjlim_{x\in U} \tilde{\mathbb{R}}(U).$$

In fact, this expression makes sense for any sheaf $A$:

**Definition 4.4.3** The *stalk of $A$ at the point $x \in X$* is the abelian group

$$A_x = \varinjlim_{x \in U} A(U).$$

In other words, the elements of $A_x$ are equivalence classes of pairs $(a, U)$, where $x \in U$ and $a \in A(U)$, under the relation $(a, U) \sim (b, V)$ iff $a|_W = b|_W$ for some $W \in \mathcal{O}(X)$ with $x \in W \subset U \cap V$. These classes add up as:

$$[(a, U)] + [(b, V)] = [(a|_W + b|_W, W)] \tag{4.10}$$

for some $W \in \mathcal{O}(X)$ with $x \in W \subseteq U \cap V$.

We now have all the elements to state the following result, which collects all the properties of Sh(X) necessary in order to define and study sheaf cohomology.

**Theorem 4.4.4** *Let $X$ be a topological space. Then:*

*a)* Sh(X) *is an abelian category;*

*b) the inclusion functor $i$:* Sh(X) $\hookrightarrow \mathsf{Ab}(\mathcal{O}(\mathrm{X}))$ *is additive and left exact;*

*c) for each $U \in \mathcal{O}(X)$, the functor $ev_U$:* Sh(X)$\longrightarrow\mathsf{Ab}$ *is additive and left exact;*

*d) For each $x \in X$, there is an exact functor $(-)_x$:* Sh(X)$\longrightarrow\mathsf{Ab}$*, taking each sheaf to its stalk at $x$;*

*e)* Sh(X) *has enough injectives (hence right derived functors like $R^n\Gamma$ are defined).*

**Proof.** We shall give an admittedly sketchy proof, here, and advise the reader to fill in the details. First of all, note that *b*) and *c*) are equivalent, by Proposition 4.3.2 *b*).

For *c*) to hold, we must be able to compute the sum of $f, g \colon A \longrightarrow B$ pointwise, as $(f+g)_U = f_U + g_U \colon A(U) \longrightarrow B(U)$. Likewise, kernels and direct sums must be defined as $(\ker f)(U) = \ker(f_U)$ and $(A \oplus B)(U) = A(U) \oplus B(U) \simeq A(U) \prod B(U)$. Indeed, it is easy to check that these operations define a linear structure on Sh(X). So, we have kernels and direct sums in Sh(X). We now focus on cokernels. Let $f \colon A \longrightarrow B$ and $U \in \mathcal{O}(X)$. Say that $b \in B(U)$ is *locally in the image of $f$* if there is a cover $U = \cup U_i$ and elements $a_i \in A(U_i)$ such that $b|_{U_i} = f_{U_i}(a_i)$. For an open cover $U = \cup U_i$, we call a family $\{b_i \in B(U_i)\}$ *locally $f$-compatible* if, for all $i$, $j$, the element $b_i|_{U_{ij}} - b_j|_{U_{ij}}$ is locally in the image of $f$. We define $\operatorname{coker} f(U)$ as the set of equivalence classes of pairs $(\{U_i\}, \{b_i\})$, where the $U_i$ form an open cover of $U$ and $\{b_i \in B(U_i)\}$ is a locally $f$-compatible family, two such pairs $(\{U_i\}, \{b_i\})$ and $(\{V_j\}, \{c_j\})$ being equivalent if and only if $b_i|_{U_i \cap V_j} - c_j|_{U_i \cap V_j}$ is locally in the image of $f$; that is, $(\{U_i\}, \{b_i\}) \sim (\{V_j\}, \{c_j\})$ if and only if $(\{U_i\} \cup \{V_j\}, \{b_i\} \cup \{c_j\})$ is still locally $f$-compatible. The group structure on $\operatorname{coker} f(U)$ is

$$[(\{U_i\}, \{b_i\})] + [(\{V_j\}, \{c_j\})] = [(\{U_i \cap V_j\}, \{b_i|_{U_i \cap V_j} + c_j|_{U_i \cap V_j}\})].$$

For an inclusion $V \subseteq U$ of open subsets in $X$, the action of the restriction morphism $\operatorname{coker} f(U) \longrightarrow \operatorname{coker} f(V)$ making $\operatorname{coker} f$ into a presheaf is given by

$$[(\{U_i\}, \{b_i\})] \mapsto [(\{U_i \cap V\}, \{b_i|_{U_i \cap V}\})].$$

There is an obvious map $q\colon B \longrightarrow \operatorname{coker} f$ in $\mathsf{Ab}(\mathcal{O}(\mathsf{X}))$, whose components are the maps $q_U\colon B(U) \longrightarrow \operatorname{coker} f(U)$ defined as $q_U(b) = [(\{U\}, \{b\})]$.

One should now check that $\operatorname{coker} f$ as defined above is indeed a sheaf, and that the pair $(\operatorname{coker} f, q)$ has the required universal property. This then proves *b*) and *c*). With these explicit constructions of kernels, cokernels, etc. *a*) also follows.

We now prove *e*), leaving *d*) aside. For this, we look at the category $\prod_{x\in X} \mathsf{Ab}$, whose objects are families $\{A_x\}_{x\in X}$ of abelian groups and arrows are families of groups homomorphisms. Note that, if $X$ has the discrete topology, then $\mathrm{Sh(X)} \simeq \prod_{\mathrm{x}\in \mathrm{X}} \mathsf{Ab}$. In the more general case, there is only an adjunction

$$\begin{tikzcd} \mathrm{Sh(X)} \arrow[r, bend left, "\mathrm{Stalk}"] \arrow[r, phantom, "\perp"] & \prod_{x\in X} \mathsf{Ab}, \arrow[l, bend left, "G"] \end{tikzcd}$$

where $\mathrm{Stalk}(A)_x = A_x$ and $G(\{A_x\})(U) = \prod_{x\in U} A_x$. To see that $G$ is well defined, let $\{U_i\}$ be an open cover of $U$ and $\{a_i = \{a^i_x\} \in \prod_{x\in U_i} A_x\}_i$ be a compatible family; this means that for all $i, j$, $a^i_x = a^j_x$ if $x \in U_i \cap U_j$. Therefore, $a = \{a_x\}_{x\in U}$ defined by $a_x = a^i_x$ for any $i$ with $x \in U_i$ is well defined, and is an amalgamation of $\{a_i\}$.

To establish the adjunction, let $A$ and $B = \{B_x\}$ be objects in $\mathrm{Sh(X)}$ and $\prod_{x\in X} \mathsf{Ab}$, respectively. Then, for any open $U \subseteq X$, a map from $\mathrm{Stalk}(A)$ to $B$, specifies a family of maps $A_x \to B_x$ (for $x \in U$). These compose with the canonical maps $\tau_{U,x}\colon A(U) \longrightarrow A_x$ to give a map $A(U) \to \prod_{x\in U} B_x$, which is natural in $U$. Conversely, for a presheaf morphism $f\colon A \longrightarrow G(B)$ and an element $x \in X$, application of the functor $\mathrm{Stalk}(-)_x$ gives a map

$$A_x \xrightarrow{f_x} G(B)_x = \varinjlim_{x\in U} (\prod_{y\in U} B_y).$$

For each $U \in \mathcal{O}(X)$ containing $x$, we have a map $p^U_x\colon \prod_{y\in U} B_y \longrightarrow B_x$, and these form a compatible family. Therefore, by the universal property of the colimit, we obtain a map $G(B)_x \longrightarrow B_x$, hence a map $A_x \to B_x$. It is easy to see that these associations are inverse to each other.

Let now $A$ be a sheaf on $X$. Since $\mathsf{Ab}$ has enough injectives, for each $x \in X$ we can find an injective object $I_x \in \mathsf{Ab}$ and a monomorphism $f_x\colon A_x \longrightarrow I_x$. Then, for each $U \in \mathcal{O}(X)$, we obtain, as above, a map $g_U\colon A(U) \longrightarrow G(\{I_x\})(U)$. Since Stalk preserves monomorphisms, $G$ preserves injectives (see Lemma 1.8.12). So, we only have to show that the morphism $g\colon A \longrightarrow G(\{I_x\})$ is monic, and by part *b*), This amounts to show that $g_U$ is injective for any open $U \in \mathcal{O}(X)$: if $g_U(a) = 0$, then $\tau_{U,x}(a) = 0$, hence there is $V_x \in \mathcal{O}(X)$ with $x \in V_x \subseteq U$, such that $a|_{V_x} = 0$. The sets $V_x$ form an open cover of $U$, so $a = 0$ since $A$ is a sheaf.

This finishes the proof of the theorem. $\square$

## Exercises

*a)* Show that $A(\emptyset) = \{0\}$ for any sheaf $A$ on a topological space $X$.

*b)* Verify that the operation (4.10) in Definition 4.4.3 defines on the stalk $A_x$ the structure of an abelian group.

*c)* Give a proof of *d*) in Theorem 4.4.4.

## 4.5 Additional exercises

1. Assume that an abelian category $\mathsf{A}$ has enough injectives.

   i) Prove the following sharpening of Lemma 4.1.20: A sequence

   $$A \longrightarrow B \longrightarrow C$$

   is exact if and only if for every injective object $I$, the sequence:

   $$\mathrm{Hom}_{\mathsf{A}}(C, I) \longrightarrow \mathrm{Hom}_{\mathsf{A}}(B, I) \longrightarrow \mathrm{Hom}_{\mathsf{A}}(A, I)$$

   is an exact sequence of abelian groups.

   ii) Formulate the dual statement.

2. Let $\mathsf{sAb}$ be the category of simplicial abelian groups. It is an abelian category of the form $\mathsf{Ab}(\mathsf{C})$, discussed in Section 4.3, hence has enough projectives.

   i) Prove that every projective object $P$ is up to isomorphism of the form $P \cong \oplus P_i$, where $P_i = \mathbb{Z}[\Delta[n_i]]$ is the free abelian group on the standard simplex $\Delta[n_i]$ (cf. Example 3.1.3 *d*)).

   ii) Let $H_k\colon \mathsf{sAb} \longrightarrow \mathsf{Ab}$ be the homology functor defined in Remark 3.1.6. Prove that the $H_k$ form a homological $\delta$-functor.

   iii) Prove that $H_k$ vanishes on projectives if $k > 0$. (Hint: Use Additional exercise 2 *iii*) from Chapter 3.)

   iv) Conclude that $H_*(A) \cong H_*(\Delta^{op}, A)$. Here we use the following definition of the homology of small categories: let

   $$\varinjlim\colon \mathsf{Ab}(\mathsf{C}^{\mathrm{op}}) \longrightarrow \mathsf{Ab}$$

   be the colimit functor, mapping a functor $A\colon \mathsf{C} \longrightarrow \mathsf{Ab}$ to the cokernel of the morphism

   $$\bigoplus_{f:C\to D} A(C) \xrightarrow{\alpha-\beta} \bigoplus_{E\in \mathsf{C}} A(E)$$

   where $\alpha_f(a) = a \in A(C)$ and $\beta_f(a) = A(f)(a) \in A(D)$. Check that $\varinjlim$ is right exact, and define

   $$H_i(\mathsf{C}, A) \cong L_i(\varinjlim)(A).$$

# A. Categories and Functors

We have collated in this Appendix all the background definitions about category theory which provide the language to express and formalise the arguments in the course. For further reading on this topic, we refer to Mac Lane's classic book [4].

The very basic observation lying behind the definition of a category, is that whenever we study some class of mathematical objects and their interactions, we have to restrict our attention to the "right kind of maps" between them. So, for example, when we compare groups, it is not interesting to look at any function between their underlying sets. Rather, we have to focus on the maps that do preserve the group structure. Starting from this remark, we formulate the following.

**Definition A.0.1** A *category* $\mathsf{C}$ consists of the following data:

*a)* a class $|\mathsf{C}|$ whose elements are the *objects* of $\mathsf{C}$;
*b)* a set $\mathrm{Hom}_{\mathsf{C}}(A, B)$ of *morphisms* from $A$ to $B$ for every ordered pair $(A, B)$ of objects in $\mathsf{C}$;
*c)* a *composition function*

$$\mathrm{Hom}_{\mathsf{C}}(B, C) \times \mathrm{Hom}_{\mathsf{C}}(A, B) \longrightarrow \mathrm{Hom}_{\mathsf{C}}(A, C)$$

for any objects $A$, $B$ and $C$ in $\mathsf{C}$;
*d)* an *identity morphism* $\mathsf{id}_A$ in $\mathrm{Hom}_{\mathsf{C}}(A, A)$ for every object $A$ in $\mathsf{C}$;

We use the words *arrow*, *map*, or *morphism* to denote elements of the Hom sets. We write $f \colon A \longrightarrow B$ to indicate that $f$ is a morphism in $\mathrm{Hom}_{\mathsf{C}}(A, B)$ (or a map *from $A$ to $B$*), and we write $g \circ f$ (or even $gf$, when there is no ambiguity) for the composite of two morphisms $f \colon A \longrightarrow B$ and $g \colon B \longrightarrow C$.

These data will have to satisfy the following conditions:

*a)* identity morphisms are left and right units for composition; that is, for any $f$ in $\mathrm{Hom}_{\mathsf{C}}(A, B)$,

$$\mathsf{id}_B \circ f = f = f \circ \mathsf{id}_A;$$

*b)* composition of morphisms is associative; namely, for any maps $f$ in $\mathrm{Hom}_{\mathsf{C}}(A, B)$, $g$ in $\mathrm{Hom}_{\mathsf{C}}(B, C)$ and $h$ in $\mathrm{Hom}_{\mathsf{C}}(C, D)$, one has

$$h \circ (g \circ f) = (h \circ g) \circ f$$

in $\mathrm{Hom}_{\mathsf{C}}(A, D)$.

In a category $\mathsf{C}$, we say that two objects $A$ and $B$ are *isomorphic* when there are maps $f\colon A \longrightarrow B$ and $g\colon B \longrightarrow A$ such that $gf = \mathsf{id}_A$ and $fg = \mathsf{id}_B$.

**Example A.0.2** Examples of categories come mainly in two forms. On the one hand, we have categories whose objects are sets with some structure, and arrows are functions between the underlying sets that preserve the structures; on the other, there are some mathematical structures that carry within themselves the inherent structure of a category.

The most obvious example of a category of the first kind, of course, is $\mathsf{Set}$ itself. Its objects are sets, its morphisms are functions between sets, composition of morphisms is the usual composition of functions, and the identity morphisms are given by the identity function on each set. Another example is the category $\mathsf{Ab}$ of abelian groups, whose objects are abelian groups, morphisms are group homomorphisms, composition and identities are those in $\mathsf{Set}$. Likewise, for rings $R$ and $S$ we have seen the categories $R$-$\mathsf{mod}$, $\mathsf{mod}$-$S$, $R$-$\mathsf{mod}$-$S$ of left $R$-modules, right $S$-modules, and $(R, S)$-bimodules, and their homomorphisms.

However, it is restrictive to think of objects in a category as sets with some structure, for there are examples of categories where this is not true. Examples of this second kind are the following. Every partially ordered set $(A, \leq)$ can be viewed as a category, where the objects are the elements of $A$ and there is precisely one arrow between $a$ and $b$ when $a \leq b$. Conversely, one can show that any (small) category where $\mathrm{Hom}(a, b)$ has at most one element for any two objects $a$ and $b$ is a partially ordered set. Also, given a unary ring $R$, we can define a category $\mathsf{R}$ as follows. $\mathsf{R}$ has only one object: $\bullet$. The set $\mathrm{Hom}_{\mathsf{R}}(\bullet, \bullet)$ is defined to be $R$. Composition is given by the product operation in $R$, and the identity is defined to be the unit 1.

The gain in abstracting the notion of category, has a price in that, not dealing with sets, we can no longer formulate certain properties. For example, it makes no sense to say that a map $f\colon A \longrightarrow B$ is injective any more. This is because $A$ is no longer a set, and therefore does not have elements.

One of the major challenges in category theory is that of defining notions that capture in this new and abstract formalism all the classical set-theoretic notions that are needed in developing a certain argument.

For example, it is not hard to see that injective functions between sets can be equivalently characterised as those maps $f\colon A \longrightarrow B$ such that, for any two maps $g, h\colon X \longrightarrow A$, if $fg = fh$ then $g = h$. Now, this is a property that makes sense in *any* category, and therefore we take this as a definition.

**Definition A.0.3** A map $f\colon A \longrightarrow B$ in $\mathsf{C}$ is called *monic*, or a *mono(morphism)*, when for any pair of parallel maps $g, h\colon X \longrightarrow A$, if $fg = fh$ then $g = h$. We sometimes express this by saying that $f$ has the *left cancellation property*. The map $f$ is called *epic*, or an *epi(morphism)*, if for any pair of parallel maps $d, e\colon B \longrightarrow Y$, $d = e$ whenever $df = ef$. In this case, we shall also say that $f$ has the *right cancellation property*.

**Example A.0.4** It is an easy verification that monic and epic maps in $\mathsf{Set}$ are precisely injective and surjective functions. Likewise, in $\mathsf{Ab}$ monomorphisms are precisely injective homomorphisms of abelian groups. This holds more generally for the category $R$-$\mathsf{mod}$, for any ring $R$.

In $R$-$\mathsf{mod}$ it is obvious that every surjective module homomorphism is also an epimorphism. The converse is also true, since for any module homomorphism $f\colon A \longrightarrow B$ the two composites in

$$A \xrightarrow{\ f\ } B \overset{p}{\underset{0}{\rightrightarrows}} B/f(A)$$

are both the null map 0; hence, whenever $f$ is epic, $p = 0$, proving that $f(A) = B$ and $f$ is surjective.

In the category $\mathsf{Rng}$ of unary rings and ring homomorphisms (preserving units), however, the inclusion $\mathbb{Z} \longrightarrow \mathbb{Q}$ is epic but clearly not surjective.

**Remark A.0.5** In $\mathsf{Set}$ it is clearly the case that if a map is injective and surjective, then it is an isomorphism. Note that this is not true in general. In a partially ordered set, considered as a category, every map is trivially monic and epic (the conditions being vacuously satisfied), but this does not mean that every two elements are equal!

As for all other mathematical structures, we can identify the right notion of morphism between categories as well.

**Definition A.0.6** Let $\mathsf{C}$ and $\mathsf{D}$ be two categories. By a *covariant functor* $F\colon \mathsf{C} \longrightarrow \mathsf{D}$ we mean a rule that assigns to each object $A$ in $|\mathsf{C}|$ an object $F(A)$ in $|\mathsf{D}|$, and to each map $f\colon A \longrightarrow B$ in $\mathsf{C}$ a morphism $F(f)\colon F(A) \longrightarrow F(B)$ in $\mathsf{D}$, such that the following conditions hold:

*a)* For each object $A$ in $|\mathsf{C}|$, $F(\mathsf{id}_A) = \mathsf{id}_{F(A)}$.

*b)* For each $f$ in $\mathrm{Hom}_{\mathsf{C}}(A, B)$ and $g$ in $\mathrm{Hom}_{\mathsf{C}}(B, C)$, $F(g \circ f) = F(g) \circ F(f)$.

**Remark A.0.7** Given two functors $F\colon \mathsf{C} \longrightarrow \mathsf{D}$ and $G\colon \mathsf{D} \longrightarrow \mathsf{E}$, we can define their composite to be the functor $G \circ F\colon \mathsf{C} \longrightarrow \mathsf{E}$ taking an object $A$ in $\mathsf{C}$ to the object $G(F(A))$ in $\mathsf{E}$, and a map $f$ in $\mathrm{Hom}_{\mathsf{C}}(A, B)$ to the map $G(F(f))$ in $\mathrm{Hom}_{\mathsf{E}}(G(F(A)), G(F(B)))$. This composite operation is clearly associative. Moreover, there is an obvious identity functor $\mathsf{Id}_{\mathsf{C}}\colon \mathsf{C} \longrightarrow \mathsf{C}$ for any category $\mathsf{C}$. So, we can consider the category $\mathsf{Cat}$, whose objects are categories, arrows are functors, and composition and identities are the ones just defined.

**Definition A.0.8** The *opposite category* $\mathsf{C}^{\mathrm{op}}$ of a category $\mathsf{C}$ has as objects the same as $\mathsf{C}$, but arrows "reversed"; more precisely, the set $\mathrm{Hom}_{\mathsf{C}^{\mathrm{op}}}(A, B)$ is the same as $\mathrm{Hom}_{\mathsf{C}}(B, A)$. Identities are the same, and composition is itself "reversed": if $f\colon A \longrightarrow B$ and $g\colon B \longrightarrow C$ are maps in $\mathsf{C}^{\mathrm{op}}$, then they are maps in $\mathsf{C}$ from $B$ to $A$ and from $C$ to $B$, respectively. Therefore, we can form the composite $f \circ g\colon C \longrightarrow A$ in $\mathsf{C}$, and this is a map from $A$ to $C$ in $\mathsf{C}^{\mathrm{op}}$, which we define as the composite $g \circ f$. It is easy to verify that the necessary conditions hold; in fact, they are inherited from $\mathsf{C}$.

**Definition A.0.9** A *contravariant functor* $F\colon \mathsf{C} \longrightarrow \mathsf{D}$ is a covariant functor from $\mathsf{C}^{\mathrm{op}}$ to $\mathsf{D}$. Naively, we can think of it as a functor that "reverses arrows". More formally, it maps objects of $\mathsf{C}$ to objects of $\mathsf{D}$, and an takes an arrow $f$ in $\mathrm{Hom}_{\mathsf{C}}(A, B)$ to a map $F(f)$ in $\mathrm{Hom}_{\mathsf{D}}(FB, FA)$. For composition, we have that $F(g \circ f) = F(f) \circ F(g)$; identities are preserved as usual.

Notice that the composite of two contravariant functors is covariant. Generally, we refer to covariant functors simply as functors, and specify explicitly only the contravariant case.

**Example A.0.10** The inner structure of any category $\mathsf{C}$ determines both a covariant and a contravariant Hom functor. For a fixed object $A$ in $\mathsf{C}$, the functor $\mathrm{Hom}_{\mathsf{C}}(A,-)\colon \mathsf{C}\longrightarrow\mathsf{Set}$ maps an object $B$ to the set $\mathrm{Hom}_{\mathsf{C}}(A,B)$ of maps in $\mathsf{C}$ between $A$ and $B$. For a given map $f\colon B\longrightarrow B'$ in $\mathsf{C}$, the $\mathsf{Set}$-function $\mathrm{Hom}_{\mathsf{C}}(A,f)\colon \mathrm{Hom}_{\mathsf{C}}(A,B)\longrightarrow \mathrm{Hom}_{\mathsf{C}}(A,B')$ is defined just as composition with $f$, and takes a map $g\colon A\longrightarrow B$ to the composite $fg\colon A\longrightarrow B'$. Functoriality of $\mathrm{Hom}_{\mathsf{C}}(A,-)$ is determined by associativity of composition and the properties of the identity maps.

Likewise, $A$ determines a contravariant functor $\mathrm{Hom}_{\mathsf{C}}(-,A)\colon \mathsf{C}\longrightarrow\mathsf{Set}$, taking an object $B$ in $\mathsf{C}$ to the set $\mathrm{Hom}_{\mathsf{C}}(B,A)$ and a map $f\colon B'\longrightarrow B$ to the $\mathsf{Set}$-function $\mathrm{Hom}_{\mathsf{C}}(f,A)\colon \mathrm{Hom}_{\mathsf{C}}(B,A)\longrightarrow \mathrm{Hom}_{\mathsf{C}}(B',A)$ that is defined by precomposition with $f$.

**Example A.0.11** We saw in Section 1.2 how the Hom sets between $R$-modules can be given different algebraic structures, therefore determining different Hom functors relating $R$-$\mathsf{mod}$ to $\mathsf{Ab}$, or other categories, according to the structured considered. We also saw the tensor product functor of Section 1.3 and the (co)homology functors of Section 1.9.

It is also possible to introduce a notion of morphism between functors.

**Definition A.0.12** Given two parallel functors $F,G\colon \mathsf{C}\longrightarrow\mathsf{D}$, a *natural transformation* $\alpha\colon F\longrightarrow G$ is a collection of maps

$$(\alpha_A\colon F(A)\longrightarrow G(A))_{A \text{ in } |\mathsf{C}|}$$

in $\mathsf{D}$, indexed by the objects of $\mathsf{C}$ and such that, for any map $f\colon A\longrightarrow B$ in $\mathsf{C}$, the following square commutes in $\mathsf{D}$:

$$\begin{tikzcd} F(A) \arrow[r, "\alpha_A"] \arrow[d, "F(f)"'] & G(A) \arrow[d, "G(f)"] \\ F(B) \arrow[r, "\alpha_B"'] & G(B). \end{tikzcd}$$

If all the components $\alpha_A$ of a natural transformation $\alpha\colon F\longrightarrow G$ are isomorphisms, then we say that $\alpha$ is a *natural isomorphism* between $F$ and $G$.

**Remark A.0.13** We can also define the *(vertical) composition* of two natural transformation. That is, for any three parallel functors $F,G,H\colon \mathsf{C}\longrightarrow\mathsf{D}$ and two natural transformations $\alpha\colon F\longrightarrow G$ and $\beta\colon G\longrightarrow H$ there is a natural transformation $\beta\circ\alpha\colon F\longrightarrow H$ whose component on an object $A$ in $\mathsf{C}$ is the map $\beta_A\circ\alpha_A\colon F(A)\longrightarrow H(A)$ in $\mathsf{D}$.

It easy to show that composition of natural transformations is associative and that the collection $(\mathsf{id}_{F(A)})_{A \text{ in } \mathsf{C}}$ defines the identity natural transformation from $F$ to itself. Therefore, we can form the *category of functors*, denoted $[\mathsf{C},\mathsf{D}]$, whose objects are functors from $\mathsf{C}$ to $\mathsf{D}$, and arrows are natural transformations between them, with the composition and identities just defined.

It is easy to see that, when $\alpha\colon F\longrightarrow G$ is a natural isomorphism in $[\mathsf{C},\mathsf{D}]$, then the collection $(\alpha^{-1}\colon G(A)\longrightarrow F(A))_{A\text{in}|\mathsf{C}|}$ defines a natural transformation from $G$ to $F$, which is inverse to $\alpha$. Therefore, natural isomorphisms between $F$ and $G$ are precisely isomorphisms in the functor category $[\mathsf{C},\mathsf{D}]$.

When we work in the particular functor category $[\mathsf{C}^{\mathrm{op}}, \mathsf{Set}]$ of contravariant functors from a category $\mathsf{C}$ to $\mathsf{Set}$, we can identify a special collection of objects.

**Definition A.0.14** A functor $F\colon \mathsf{C}^{\mathrm{op}} \longrightarrow \mathsf{Set}$ is called *representable* if it is (up to isomorphism) of the form $\mathrm{Hom}_{\mathsf{C}}(-, B)$ for a fixed object $B$ in $\mathsf{C}$.

The *Yoneda embedding* functor $Y\colon \mathsf{C} \longrightarrow [\mathsf{C}^{\mathrm{op}}, \mathsf{Set}]$ maps an object $A$ to the functor $\mathrm{Hom}_{\mathsf{C}}(-, A)$, and a morphism $f\colon A \longrightarrow B$ to the natural transformation $f_*\colon \mathrm{Hom}_{\mathsf{C}}(-, A) \longrightarrow \mathrm{Hom}_{\mathsf{C}}(-, B)$.

**Lemma A.0.15 (Yoneda Lemma)** There is an isomorphism

$$\mathrm{Hom}_{[\mathsf{C}^{\mathrm{op}}, \mathsf{Set}]}(Y(A), F) \simeq F(A)$$

natural in $A$ in $\mathsf{C}$ and $F$ in $[\mathsf{C}^{\mathrm{op}}, \mathsf{Set}]$.

When working in $\mathsf{Cat}$, the notion of isomorphism is often too strict. In fact, it is often the case that the composites of two functors $F\colon \mathsf{C} \longrightarrow \mathsf{D}$ and $G\colon \mathsf{D} \longrightarrow \mathsf{C}$ are not exactly the identity functors; rather they may be just isomorphic to them in the functor categories $[\mathsf{C}, \mathsf{C}]$ and $[\mathsf{D}, \mathsf{D}]$. In this case, we say that the categories $\mathsf{C}$ and $\mathsf{D}$ are *equivalent*. More concretely, this means that there are natural isomorphisms $\eta\colon \mathsf{Id}_{\mathsf{C}} \longrightarrow GF$ and $\epsilon\colon FG \longrightarrow \mathsf{Id}_{\mathsf{D}}$.

The notion of equivalence of categories can be further relaxed, by asking for $\eta$ and $\epsilon$ to be merely natural transformations.

**Definition A.0.16** Let $\mathsf{C}$ and $\mathsf{D}$ be two categories. Two functors $L\colon \mathsf{C} \longrightarrow \mathsf{D}$ and $R\colon \mathsf{D} \longrightarrow \mathsf{C}$ are said to be *adjoint* if there are two natural transformations:

$$\eta\colon \mathsf{Id}_{\mathsf{C}} \longrightarrow RL \quad \text{and} \quad \epsilon\colon LR \longrightarrow \mathsf{Id}_{\mathsf{D}}$$

satisfying the triangular identities:

$$\begin{tikzcd} & RLR \arrow[dr, "R\epsilon"] & \\ R \arrow[ur, "\eta R"] \arrow[rr, "\mathsf{id}_R"'] & & R, \end{tikzcd} \qquad \begin{tikzcd} & LRL \arrow[dr, "\epsilon F"] & \\ L \arrow[ur, "L\eta"] \arrow[rr, "\mathsf{id}_L"'] & & L. \end{tikzcd}$$

When this is the case we shall say that $L$ is *left adjoint* to $R$ (denoted $L \dashv R$), and vice versa that $R$ is *right adjoint* to $L$. Finally, the maps $\eta$ and $\epsilon$ are called the *unit* and *counit* of the adjunction, respectively.

**Proposition A.0.17** *The following are equivalent for a functor $L\colon \mathsf{C} \longrightarrow \mathsf{D}$:*

*a) $L$ has a right adjoint $R$;*

*b) there is a functor $R\colon \mathsf{D} \longrightarrow \mathsf{C}$ and, for any object $A$ in $\mathsf{C}$ and $B$ in $\mathsf{D}$ there are isomorphisms*

$$\tau_{A,B}\colon \mathrm{Hom}_{\mathsf{D}}(L(A), B) \longrightarrow \mathrm{Hom}_{\mathsf{C}}(A, R(B));$$

*moreover, these are natural in $A$ and $B$, in the sense that for all $f$ in $\mathrm{Hom}_{\mathsf{C}}(A, A')$ and $g$ in $\mathrm{Hom}_{\mathsf{D}}(B, B')$ the following diagram commutes:*

$$\begin{tikzcd} \mathrm{Hom}_{\mathsf{D}}(L(A'), B) \arrow[r, "-\circ L(f)"] \arrow[d, "\tau_{A',B}"'] & \mathrm{Hom}_{\mathsf{D}}(L(A), B) \arrow[r, "g\circ -"] \arrow[d, "\tau_{A,B}"'] & \mathrm{Hom}_{\mathsf{D}}(L(A), B') \arrow[d, "\tau_{A,B'}"'] \\ \mathrm{Hom}_{\mathsf{C}}(A', R(B)) \arrow[r, "-\circ f"'] & \mathrm{Hom}_{\mathsf{C}}(A, R(B)) \arrow[r, "R(g)\circ -"'] & \mathrm{Hom}_{\mathsf{C}}(A, R(B')); \end{tikzcd}$$

*c) for any $B$ in $\mathsf{D}$ there is an object $R(B)$ in $\mathsf{C}$ and a map $\epsilon_B\colon L(R(B))\longrightarrow B$ with the universal property that, for any $A$ in $\mathsf{C}$ and any map $f\colon L(A)\longrightarrow B$ in $\mathsf{D}$, there is a unique morphism $\widehat{f}\colon A\longrightarrow R(B)$ making the following commute:*

$$\begin{tikzcd} A \arrow[d, dotted, "\widehat{f}"'] & L(A) \arrow[d, "L(\widehat{f})"'] \arrow[dr, bend left, "f"] & \\ R(B) & L(R(B)) \arrow[r, "\epsilon_B"'] & B. \end{tikzcd}$$

**Remark A.0.18** Dually to point *c*) in Proposition A.0.17, one can describe an adjunction in terms of the right adjoint $R\colon \mathsf{D}\longrightarrow\mathsf{C}$ and, for any $A$ in $\mathsf{C}$, an object $L(A)$ in $\mathsf{D}$ and a map $\eta_A\colon A\longrightarrow R(L(A))$ with the universal property that, for any other map $f\colon A\longrightarrow RB$ there is a unique morphism $\widehat{f}\colon L(A)\longrightarrow B$ making the following commute:

$$\begin{tikzcd} A \arrow[r, "\eta_A"] \arrow[dr, bend right, "f"'] & R(L(A)) \arrow[d, "R(\widehat{f})"] & L(A) \arrow[d, dotted, "\widehat{f}"] \\ & R(B) & B. \end{tikzcd}$$

## Exercises

*a)* Given a category $\mathsf{C}$ and a functor $F\colon \mathsf{D}\longrightarrow\mathsf{E}$, show that post-composition with $F$ determines a functor

$$F\circ -\colon [\mathsf{C},\mathsf{D}]\longrightarrow[\mathsf{C},\mathsf{E}].$$

*b)* Prove the Yoneda lemma A.0.15. (Hint: define the isomorphism in one direction by taking $\alpha\in \mathrm{Hom}_{[\mathsf{C}^{\mathrm{op}},\mathsf{Set}]}(Y(A),F)$ to $f_A(\mathsf{id}_A)$; for its inverse, map an element $x\in F(A)$ to the natural transformation defined by $\alpha_C(f)=F(f)(x)$, where $f\colon C\longrightarrow A$ is an element of $\mathrm{Hom}_{\mathsf{C}}(C,B)$)

*c)* Prove Proposition A.0.17.

*d)* Show that the left and right adjoints to a functor $F\colon \mathsf{C}\longrightarrow\mathsf{D}$, when they exist, are unique up to natural isomorphism.

*e)* Show that a map $f\colon A\longrightarrow B$ is monic in $\mathsf{C}$ if and only if it is epic in $\mathsf{C}^{\mathrm{op}}$, and deduce from this the dual statement, that $f$ is epic in $\mathsf{C}$ if and only if it is monic in $\mathsf{C}^{\mathrm{op}}$.

*f)* Show that right adjoint functors preserve monics and left adjoint preserve epics.

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