The p-torsion invariants of non-ordinary Jacobians and Prym varieties

Rachel Pries
joint work with Jeff Achter and Ekin Ozman

Colorado State University
pries@math.colostate.edu

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Abstract

One theme of the research program of Frans Oort is to understand stratifications of moduli spaces of abelian varieties in positive characteristic by invariants such as the Newton polygon or Ekedahl Oort type.

In the first half of the talk, I will describe the current state of knowledge about how the Torelli locus intersects these strata in the moduli space of principally polarized abelian varieties of dimension g.

In the second half of the talk, I will discuss recent research about p-ranks of Prym varieties.

Thanks: organizers, Oort, Goren, Faber/Van der Geer 2004

Outline

1. Introduction for the non-experts: supersingular elliptic curves, the main question for $g > 1$

2. Survey of results:
intersection of Torelli locus with p-torsion stratifications,
a result in codimension 4 (joint with Jeff Achter)

3. p-ranks of Prym varieties: (joint with Ekin Ozman)
a. generalization of a result of Nakajima,
b. existence of unramified double covers with non-ordinary Pryms

4. Applications to torsion points in theta divisors
Related to work of Raynaud, Pop/Saidi

5. Proofs

Introduction: elliptic curves

Let k be an algebraically closed field of characteristic $p > 0$.

Let E/k be an elliptic curve and let ℓ be prime.

The kernel $E[\ell]$ of $[\ell] : E \to E$ is the ℓ-torsion group scheme.

$$\#E[\ell](k) = \begin{cases} \ell^2 & \text{if } \ell \neq p \\ \ell & \text{if } \ell = p, E \text{ ordinary} \\ 1 & \text{if } \ell = p, E \text{ supersingular} \end{cases} .$$

Let $E' \to E$ be a degree ℓ isogeny.
Then E' supersingular iff E supersingular.

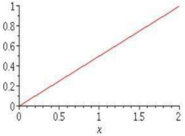
Introduction: supersingular elliptic curves

Let E be a smooth elliptic curve over $k = \overline{k}$, with $\mathrm{char}(k) = p$.
Let $E[p]$ be the kernel of the inseparable multiplication-by-p morphism.

E is **supersingular** if it satisfies the following equivalent conditions:

A. The only p-torsion point is the identity: $E[p](k) = \{\mathrm{id}\}$.

B. The Newton polygon of E is a line segment of slope $\frac{1}{2}$.



C. The group scheme $E[p]$ contains 1 copy of α_p, the kernel of Frobenius on $\mathbb{G}_a$.

For all p, there exists a supersingular elliptic curve $E/\mathbb{F}_{p^2}$ (Igusa).

Intro: group scheme of supersingular elliptic curve

C. $E[p]$ contains 1 copy of α_p, the kernel of Frobenius on $\mathbb{G}_a$.

As a k-scheme, $\alpha_p \simeq \mathrm{Spec}(k[x]/x^p)$ with co-multiplication $m^*(x) = x \otimes 1 + 1 \otimes x$ and co-inverse $\mathrm{inv}^*(x) = -x$.

Let $I_{1,1}$ denote unique local-local BT_1 group scheme of rank p^2.
There is a non-split exact sequence

$$0 \to \alpha_p \to I_{1,1} \to \alpha_p \to 0.$$

The image of α_p is the kernel of F (Frobenius) and V (Verschiebung).

The mod p Dieudonné module of $I_{1,1}$ is $D_{1,1} := \mathbb{E}/\mathbb{E}(F+V)$.

Here $\mathbb{E} = k[F,V]$ is the non-commutative ring generated by semi-linear operators F and V with relations $FV = VF = 0$ and $F\lambda = \lambda^p F$ and $\lambda V = V\lambda^p$ for all $\lambda \in k$.

Let $L = \mathbb{Z}/p \oplus \mu_p$ (occurs as p-torsion for ordinary elliptic curve).

Introduction: different properties when $g > 1$

Let A be a p.p. abelian variety of dimension g over $k = \overline{k}$, $\mathrm{char}(k) = p$. Let $A[p]$ be the kernel of the inseparable multiplication-by-p morphism.

The following conditions are all different for $g \geq 3$.

A. ***p*-rank** 0 - The only p-torsion point is the identity: $A[p](k) = \{\mathrm{id}\}$.

B. supersingular - The Newton polygon of A is a line of slope $\frac{1}{2}$.

C. superspecial - The group scheme $A[p]$ contains g copies of α_p, the kernel of Frobenius on $\mathbb{G}_a$.

Then $C \Rightarrow B \Rightarrow A$ but $A \overset{g \geq 3}{\not\Rightarrow} B \overset{g \geq 2}{\not\Rightarrow} C$

Question: if $g \geq 2$, do these occur for Jacobian of smooth k-curve?

Introduction: stratifications

Let $\mathcal{A}_g = \mathcal{A}_g \otimes \mathbb{F}_p$ be moduli space of p.p. abelian varieties of dim. g.

There are stratifications of $\mathcal{A}_g$ by:

A. ***p*-rank** f - $\#A[p](k) = p^f$ for some $0 \leq f \leq g$.

B. Newton polygon - p-divisible group $A[p^\infty] \sim \oplus_\lambda G_{c,d}^{m_\lambda}$.
If $A/\mathbb{F}_q$, NP is lower convex hull of p-adic valuations of coefficients of L-polynomial of A

line segments from $(0,0)$ to $(2g,g)$ - slope $\lambda = c/(c+d)$ with horizontal length m_λ

C. Ekedahl-Oort type - isomorphism class of BT_1 group scheme $A[p]$.

Question: Which of these strata intersect open Torelli locus?

:) Test yourself! Which cases are known?

Introduction: incomplete history of stratifications of $\mathcal{A}_g$

***p*-rank:** Grothendieck EGA, 1967
purity: Norman/Oort *Moduli of abelian varieties*, 1980

Newton polygon:
semi-continuity: Katz *Slope filtration of F-crystals*, 1979
purity: de Jong/Oort *Purity of stratification by Newton polygons*, 2000
irreducibility: Chai/Oort *Monodromy and irreducibility of leaves*, 2011

Ekedahl-Oort type:
Kraft *Kommutative algebraische p-Gruppen*, 1975
Oort *A stratification of a moduli space of abelian varieties*, 1999
Moonen/Wedhorn *Discrete invariants of varieties in pos. char.*, 2004
Ekedahl/Van der Geer *Cycle classes of the E-O stratification...*, 2009

For $\mathcal{M}_g$: no deformation theory (Serre-Tate and Dieudonné theory); no Hecke operators; and morphism $T : \overline{\mathcal{M}}_g \to \tilde{\mathcal{A}}_g$ is not flat.
(Fibers have positive dimension on boundary components.)

Example - dimension $g = 2$:

Torelli locus $\mathcal{T}_2^0 = T(\mathcal{M}_2)$ is open and dense in $\mathcal{A}_2$.

All 3 Newton polygons and all 4 EO types occur for Jacobians of smooth k-curves, except...

Ibukiyama/Katsura/Oort 1986:
count number of smooth curves X with $\mathrm{Jac}_X[p] \simeq (I_{1,1})^2$;
there exists superspecial smooth k-curve of genus 2 iff $p \geq 5$

Name	cod	f	a	ν	μ	Dieudonné module
L^2	0	2	0	$[1,2]$	$\emptyset$	$D(L)^2$
$L \oplus I_{1,1}$	1	1	1	$[1,1]$	$\{1\}$	$D(L) \oplus D_{1,1}$
$I_{2,1}$	2	0	1	$[0,1]$	$\{2\}$	$\mathbb{E}/\mathbb{E}(F^2+V^2)$
$(I_{1,1})^2$	3	0	2	$[0,0]$	$\{2,1\}$	$(D_{1,1})^2$

A supersingular iff EO type is $(I_{1,1})^2$ or $I_{2,1}$.

Example - dimension $g = 3$:

Torelli locus $\mathcal{T}_3^0 = T(\mathcal{M}_3)$ is open and dense in $\mathcal{A}_3$.

All 5 Newton polygons and all 8 EO types occur for Jacobians of smooth k-curves, except...
Oort 1991: there exists superspecial smooth curve X iff $p \geq 3$;
If $p \geq 3$, there exists smooth k-curve X with $\mathrm{Jac}_X[p] \simeq I_{1,1} \oplus I_{2,1}$.

Name	cod	f	a	ν	μ	Dieudonné module
L^3	0	3	0	$[1,2,3]$	$\emptyset$	$D(L)^3$
$L^2 \oplus I_{1,1}$	1	2	1	$[1,2,2]$	$\{1\}$	$D(L)^2 \oplus D_{1,1}$
$L \oplus I_{2,1}$	2	1	1	$[1,1,2]$	$\{2\}$	$D(L) \oplus \mathbb{E}/\mathbb{E}(F^2+V^2)$
$L \oplus (I_{1,1})^2$	3	1	2	$[1,1,1]$	$\{2,1\}$	$D(L) \oplus (D_{1,1})^2$
$I_{3,1}$	3	0	1	$[0,1,2]$	$\{3\}$	$\mathbb{E}/\mathbb{E}(F^3+V^3)$
$I_{3,2}$	4	0	2	$[0,1,1]$	$\{3,1\}$	$(F^2+V) \oplus (V^2+F)$
$I_{1,1} \oplus I_{2,1}$	5	0	2	$[0,0,1]$	$\{3,2\}$	$D_{1,1} \oplus \mathbb{E}/\mathbb{E}(F^2+V^2)$
$(I_{1,1})^3$	6	0	3	$[0,0,0]$	$\{3,2,1\}$	$(D_{1,1})^3$

Main question

Fix $g \geq 4$ and p prime.

Question: geometric

How does the Torelli locus $\mathcal{T}_g^0 = T(\mathcal{M}_g)$ intersect the stratifications of $\mathcal{A}_g$ by p-rank, Newton polygon and Ekedahl-Oort type?

For a given invariant η, consider the intersection $\mathcal{T}_g^0 \cap \mathcal{A}_g[\eta]$.

* is it non-empty?

* does it have the expected dimension?

* is it irreducible?

Problem: it is often easy to construct a *singular* curve of type η. In general, it is not possible to deform this to a smooth curve with type η.

Recall: The morphism $T : \overline{\mathcal{M}}_g \to \tilde{\mathcal{A}}_g$ is not flat. On boundary components, fibers have positive dimension.

A. Smooth curves of given genus and p-rank

Let $g \in \mathbb{N}$, $0 \leq f \leq g$ and p prime.

The moduli space $\mathcal{M}_g$ of curves of genus g can be stratified by p-rank into strata $\mathcal{M}_g^f$ whose points represent curves of genus g and p-rank f.

Theorem: Faber/Van der Geer 2004

Every component S of $\overline{\mathcal{M}}_g^f$ has dimension $2g-3+f$ (codim $g-f$); there exists a smooth curve over $\overline{\mathbb{F}}_p$ with genus g and p-rank f.

Also: S intersects boundary component Δ_i for each $1 \leq i \leq g-1$.

Similarly, can stratify the moduli space $\mathcal{H}_g$ of hyperelliptic curves by p-rank

Theorem: Glass/P (p odd), P/Zhu ($p=2$)
Every component of $\mathcal{H}_g^f$ has dimension $g-1+f$ (codim $g-f$);
there exists a smooth hyp. curve over $\overline{\mathbb{F}}_p$ with genus g and p-rank f.

In most cases, it is not known whether $\mathcal{M}_g^f$ and $\mathcal{H}_g^f$ are irreducible.

Do all NPs occur for Jacobians? Guess - unlikely?

Observation (Oort 2005) $\dim(\mathcal{A}_g) = g(g+1)/2$ and
the dimension of the supersingular locus $\mathcal{A}_g[\sigma_g]$ is $\lfloor g^2/4 \rfloor$.

The difference δ_g is length of longest chain of NPs connecting the supersingular NP σ_g to the ordinary NP ν_g.

If $g \geq 9$, then $\delta_g > 3g - 3 = \dim(\mathcal{M}_g)$.

Either (i) $\mathcal{M}_g$ does not admit a perfect stratification by NP
(i.e., there are two NPs ξ_1 and ξ_2 such that $\mathcal{A}_g[\xi_1]$ is in the closure of $\mathcal{A}_g[\xi_2]$ but $\mathcal{M}_g[\xi_1]$ is not in the closure of $\mathcal{M}_g[\xi_2]$.)

or (ii) some NPs do not occur for Jacobians of smooth curves.

Test case: $g = 11$ with NP $G_{5,6} \oplus G_{6,5}$ having slopes of $5/11, 6/11$.

Do all NPs occur for Jacobians? Evidence?

The only non-existence results are for curves with automorphisms:

Bouw 2001: Not all p-ranks occur for cyclic degree $d > 2$ covers

Especially, not all NPs occur for wildly ramified covers:

Deuring-Shafarevich formula restricts p-rank.

Oort 1991: If $p = 2$, there does not exist a hyperelliptic supersingular curve of genus 3.

Scholten/Zhu 2002: If $p = 2$ and $n \geq 2$, then there does not exist a hyperelliptic supersingular curve of genus $2^n - 1$.

(for odd p, generalized for Artin-Schreier covers $X \stackrel{\mathbb{Z}/p}{\to} \mathbb{P}^1$ by Blache)

But.....

Van der Geer & Van der Vlugt 1994: If $p = 2$, then there exists a supersingular curve of every genus.

Do all EOs occur for Jacobians? Evidence?

If $g \geq 4$, lengths of chains of EO types exceed $\dim(\mathcal{M}_g)$.

Not all EO types occur for wildly ramified covers:

Elkin/P: $p = 2$, complete classification of EO types for hyp. curves:
If $y^2 + y = h(x)$, $\mathrm{div}_\infty(h(x)) = \sum_{i=0}^{f} c_i P_i$, then $\{c_i\}$ determines EO type.

Number of EO types that occur $\sim$ number of partitions of $g+1$.

Not all EO types occur for small p:

Ekedahl 1987

If $X/\overline{\mathbb{F}}_p$ is a superspecial curve of genus g, then $g \leq p(p-1)/2$.

Upper bound realized by Hermitian curve $X_p : y^p + y = x^{p+1}$.

Re: non-existence results for curves with high a-number, etc.

Reduce existence question to case where p-rank $f = 0$

General inductive strategy: reduce question to $f = 0$.

Prop. [P] Let ν_r be a NP or EO type with p-rank 0 occurring in dim. r.

Let $c_r = \mathrm{codim}(\mathcal{A}_g[\nu_r], \mathcal{A}_g)$.

For $g \geq r$, let ν_g be the NP or EO type with p-rank $g - r$ 'containing' ν_r

($\nu_g = (G_{0,1} \oplus G_{1,0})^{g-r} \oplus \nu_r$ or $\nu_g = L^{g-r} \oplus \nu_r$)

If $\exists$ a component S_r of $\mathcal{M}_r[\nu_r]$ s.t. $\mathrm{codim}(S_r, \mathcal{M}_r) = c_r$,

then, for all $g \geq r$, $\exists$ a component S_g of $\mathcal{M}_g[\nu_g]$ s.t. $\mathrm{codim}(S_g, \mathcal{M}_g) = c_r$.

Application: when $g \geq 3$ and $f = g - 3$, both NPs and all 4 EO types occur (for $p \geq 3$).

A result in codimension 4

Let $\nu_4 = G_{1,3} \oplus G_{3,1}$ (slopes $1/4$, $3/4$)
or let $I_{4,1}$ (unique BT_1 rank 8, p-rank 0, a-number 1)

EO type $[0,1,2,3]$, Dieudonné module $\mathbb{E}/\mathbb{E}(F^4+V^4)$, Young type $\{4\}$.

Application (Achter/P): Let $g \geq 4$ and p prime.

There exists smooth curve of genus g with p-rank $g-4$ with NP $(G_{0,1} \oplus G_{1,0})^{g-4} \oplus \nu_4$ and EO type $L^{g-4} \oplus I_{4,1}$.

To produce $[X] \in \mathcal{M}_4^0$ with NP ν_4 and EO type $I_{4,1}$: consider moduli space $\mathcal{W}$ of p.p. abelian 4-folds with action by $\mathbb{Z}[\zeta_3]$ of signature $(3,1)$. Then $\dim(\mathcal{W}) = 3$ and $\mathcal{W}$ is irreducible since $\mathbb{Z}[\zeta_3]$ has class number 1.

Dim 3 moduli space of curves $C_f : y^3 = f(x)$ (square-free $f(x)$ degree 6). So $T(\mathcal{M}_4)$ contains an open, dense subspace of $\mathcal{W}$.

There exists a point of $\mathcal{W}$ with NP ν_4^0: Mantovan 2004 if p splits in $\mathbb{Q}(\zeta_3)$, Bültel/Wedhorn 06 if p inert in $\mathbb{Q}(\zeta_3)$, SAGE if $p=3$.

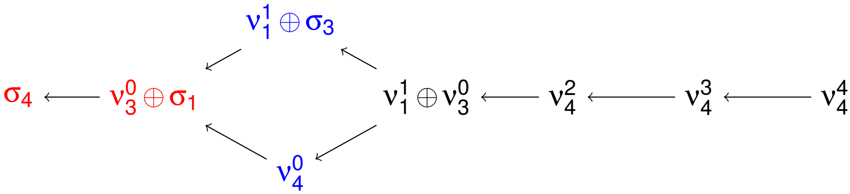
Question (problem list): If $g \geq 5$, does there exist a smooth curve with p-rank 0 s.t. NP has slopes $1/g$ and $(g-1)/g$ and EO type has a-number 1? (This is generic in every component in $\mathcal{A}_g^0$.)

Open questions for $g = 4$

Is the p-rank 0 stratum of $\mathcal{M}_4$ irreducible?

If not, does generic geometric point of each component have EO type $I_{4,1}$ and NP ν_4^0 (slopes $\{1/4, 3/4\}$)?

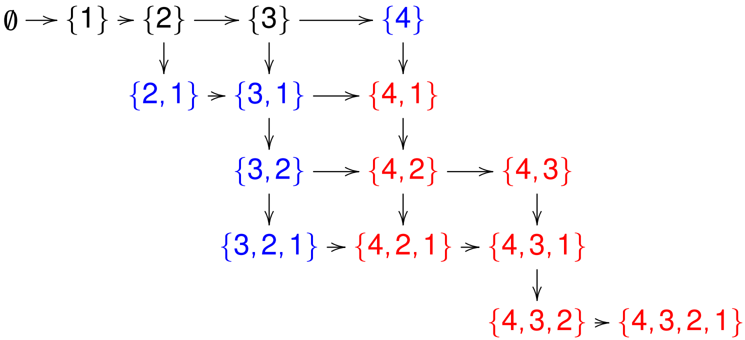
For all p, does there exist a smooth curve of genus 4
which is supersingular? or whose NP is $(G_{1,2} \oplus G_{2,1}) \oplus G_{1,1}$?



$*$: don't know if this NP occurs for Jacobian of smooth curve for all p
$*$: this NP occurs but some components may have problems
$*$: each component has good geometric properties.

Open questions for genus 4 continued

Do the EO types of codim ≥ 5 occur (for all but finitely many p)?



$*$: don't know if this EO occurs for Jacobian of smooth curve for all p

$*$: this EO occurs but some components may have problems

$*$: each component has good geometric properties.

New topic: Prym varieties

Let X/k be a curve of genus $g \geq 2$. Let $\ell \neq p$ be prime.

There is a bijection between:
the $\ell^{2g} - 1$ points of order ℓ on $J_X = \mathrm{Jac}(X)$
and the unramified $\mathbb{Z}/\ell$-covers $\pi : Y \to X$ with Y connected.

Then $\mathrm{Jac}(Y) \sim J_X \oplus P_\pi$ where P_π is the Prym variety of π
(connected component containing 0 of norm map on Jacobians).

If $\sigma \in \mathrm{End}(J_X)$ is induced by the chosen generator of $\mathrm{Gal}(Y/X)$, then
$P_\pi = \mathrm{Im}(1 - \sigma) = \mathrm{Ker}(1 + \sigma + \ldots + \sigma^{\ell-1})^0$.

Here P_π is an abelian variety of dimension $(\ell - 1)(g_X - 1)$.
If $\ell = 2$, then P_π is principally polarized.

Question:

What is the interaction between the p-ranks f_X of X and f_π of P_π?

Previous work

Nakajima 1983: if X is the generic curve of genus g, then P_π is ordinary for all unramified $\mathbb{Z}/\ell$ covers $\pi : Y \to X$.

Raynaud 1982, 2000:
for any genus g curve X, there exists an unramified
(i) $\mathbb{Z}/\ell$-cover $\pi : Y \to X$ s.t. P_π is ordinary if $\ell \geq (p-1)g-1$.
(ii) solvable cover $Z \to X$ s.t. new part of $\mathrm{Jac}(Z)$ is not ordinary.

Pop/Saidi 2003: If X is non-ordinary or if $\mathrm{Jac}(X)$ is simple,
then, for infinitely many ℓ,
$\exists$ an unramified $\mathbb{Z}/\ell$-cover $\pi_\ell : Y_\ell \to X$ s.t. P_{π_ℓ} is not ordinary.

Results on p-ranks of Pryms: Ozman/P

Let $g \geq 2$ and let $0 \leq f \leq g$. Let $\mathcal{M}_g^f$ be the p-rank f stratum of $\mathcal{M}_g$.

Theorem 1: Let $\ell \neq p$ and $(g, f) \neq (2, 0)$.

For each component S of $\mathcal{M}_g^f$, the locus of points $[X]$ for which all unramified $\mathbb{Z}/\ell$-covers $\pi : Y \to X$ have an ordinary Prym P_π is open and dense in S.

Theorem 2: Let $\ell = 2$ and $p \geq 5$.

For each component S of $\mathcal{M}_g^f$, the locus of points $[X]$ for which there exists an unramified double cover $\pi : Y \to X$ with P_π almost ordinary ($f_\pi = g - 2$) is non-empty with codimension 1 in S.

Corollary 3: Let $\ell = 2$, $p \geq 5$, and $g \geq 4$. Let $\frac{g}{2} - 1 \leq f' \leq g - 1$.

There exists a smooth curve $X/\overline{\mathbb{F}}_p$ of genus g and p-rank f having an unramified double cover $\pi : Y \to X$ for which P_π has p-rank $f_\pi = f'$.

The arithmetic theta divisor

Let X' be base change of X by absolute Frobenius.
Let $F : X \to X'$ be the relative Frobenius.
Define B as image of differential $F_* d : F_*(\mathcal{O}_X) \to F_*(\Omega^1_X)$,

$$0 \to \mathcal{O}_{X'} \to F_*(\mathcal{O}_X) \to B \to 0.$$

B is the *sheaf of locally exact differentials*:

$$0 \to B \to F_*(\Omega^1_X) \stackrel{C}{\to} \Omega^1_{X'} \to 0.$$

B is a vector bundle on X' of rank $p-1$ and slope = deg/rank = $g-1$;
B is algebraically equivalent to $(p-1)\theta'_{\mathrm{cl}}$.

Intuition: Let $J_X^{[g-1]}$ be the torsor for J_X rep. line bundles of degree $g-1$. Recall

$$\mathrm{Sym}^{g-1}(X) \to J_X^{[g-1]},\ (x_1, \ldots, x_{g-1}) \mapsto \mathcal{O}_X(x_1 + \cdots x_{g-1}).$$

The image is a translate of the classical theta divisor θ_{cl}.

Raynaud: theta divisor and *p*-rank of Prym

A point $\alpha \in J_{X'}[\ell](k)$ determines an unramified $\mathbb{Z}/\ell$ cover $\pi_\alpha : Y \to X$ and an invertible sheaf $L_\alpha \in \mathrm{Pic}^0(X')$ of order ℓ.

Recall

$$0 \to \mathcal{O}_{X'} \to F_*(\mathcal{O}_X) \to B \to 0.$$

B admits a theta divisor, a positive Cartier divisor Θ_X on $J_{X'}$.

α is in the support of Θ_X if and only if $H^0(X', B \otimes L_\alpha) \neq 0$.

It measures extent to which F is an isomorphism on $\mathbb{Z}/\ell$-eigenspaces.

$0 \in \Theta_X$ iff X ordinary.
P_{π_α} ordinary iff $\{i\alpha \mid i \in (\mathbb{Z}/\ell)^*\}$ does not intersect Θ_X.

Proposition [P]: More generally, get information about *a*-number of P_π.

Results: Ozman/P on torsion points in theta divisors

(geometry of theta divisor Θ_X) $\Leftrightarrow$ (p-ranks of Prym varieties)

$\Rightarrow$ used by Raynaud and Pop/Saidi. $\Leftarrow$ used by Ozman/P.

Let $g \geq 2$ and let $0 \leq f \leq g$. Let $\mathcal{M}_g^f$ be the p-rank f stratum of $\mathcal{M}_g$.

Theorem 1: Let $\ell \neq p$ and $(g, f) \neq (2, 0)$.

For each component S of $\mathcal{M}_g^f$, the locus of points $[X]$ for which Θ_X does not contain any point of order ℓ is open and dense in S.

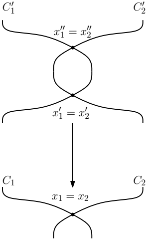
Theorem 2: Let $\ell = 2$ and $p \geq 5$.

For each component S of $\mathcal{M}_g^f$, the locus of points $[X]$ for which Θ_X contains a point of order 2 is non-empty with codimension 1 in S.

Proof of Theorem 1 - naive version

Goal: produce unramified $\mathbb{Z}/\ell$-cover $\pi : Y \to X$ s.t. X has genus g and p-rank f and P_π is ordinary.

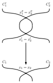
Build a cover of singular curves (a point of boundary component $\Delta_{i,j}$):



$$\mathrm{BLR}: 1 \to \mathbb{T} \to P_\pi \to P_{\pi_1} \oplus P_{\pi_2} \to 1,$$

where $\mathbb{T}$ is a torus of rank $\ell - 1$.

Proof - naive version page 2



Inductive hyp: choose C_1, C_2 with p-ranks f_1, f_2 s.t. $f_1 + f_2 = f$, for which $\exists\ \pi_1 : C_1' \to C_1$ and $\pi_2 : C_2' \to C_2$ s.t. P_{π_1} and P_{π_2} are ordinary. Then

$$\begin{aligned} f_\pi &= f_{\pi_1} + f_{\pi_2} + (\ell - 1) \\ &= (\ell-1)(g_1 - 1) + (\ell - 1)(g_2 - 1) + (\ell - 1) \\ &= (\ell - 1)(g - 1). \end{aligned}$$

So P_π is ordinary too.

Technical complications

* When deforming to smooth curve, the p-rank of X could go up.

* It is not obvious, for each component of $\mathcal{M}_g^f$, that points of $\Delta_{i,j}$ occur as degeneration of associated families of unramified $\mathbb{Z}/\ell$-covers.

* or that there is no restriction on the p-ranks of the Pryms P_{π_1} and P_{π_2} occurring in such a degeneration.

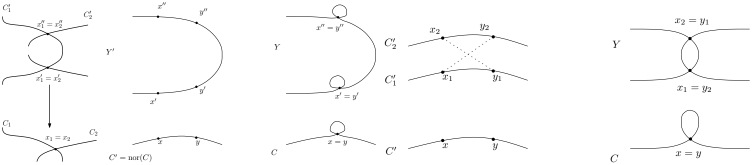
Geometric set-up for proof

$R_{g,\ell}$ moduli space of unramified $\mathbb{Z}/\ell$-covers $\pi : Y \to X$ with $g_X = g$.

There is a finite flat morphism $\Pi : R_{g,\ell} \to \mathcal{M}_g$ taking $\pi \mapsto X$.

Consider the compactifications $\Pi : \overline{R}_{g,\ell} \to \overline{\mathcal{M}}_g$.

The boundary of $\overline{R}_{g,\ell}$ is well-understood, Chiodo/Farkas/et al.



We analyze the p-rank stratification of the boundary components.

Proof of Theorem 1

Let S be an irreducible component of $\overline{\mathcal{M}}_g^f$. Let $\tilde{S} = \Pi^{-1}(S) \subset \overline{R}_{g,\ell}$.

Want to show:
(T1) a generic point $\tilde{\eta}$ of $\tilde{S}$ represents a cover whose Prym is ordinary.

Step 1: $\tilde{\eta}$ represents an unramified $\mathbb{Z}/\ell$-cover of a smooth curve.

Step 2: Every generic point of boundary of $\tilde{S}$ is in $\Pi_\ell^{-1}(\overline{\mathcal{M}}_g^f - \overline{\mathcal{M}}_g^{f-1})$.

Step 3: need to show that $\tilde{S}$ intersects $\Delta_{i,j}$

It is not sufficient to say that S intersects Δ_i.

If the restriction of π above component of $[X] \in \Delta_i$ is not connected then the inductive step breaks down: $f_\pi = f_{\pi_1} + (\ell - 1)f_2 < \dim(P_\pi)$.

Proof of Theorem 1: Good degenerations

Let S be a component of $\overline{\mathcal{M}}_g^f$. Let $\tilde{S} = \Pi^{-1}(S) \subset \overline{R}_{g,\ell}$.

Step 2: need to show that $\tilde{S}$ intersects $\Delta_{i,j}$

Luckily...we can consider the $\mathbb{Z}/\ell$-monodromy, which measures the representation of fundamental group $\pi_1(S)$ on $J_X[\ell]$.

Theorem Achter/P: p prime, $g \geq 2$, $0 \leq f \leq g$, $(g,f) \neq (2,0)$

The $\mathbb{Z}/\ell$-monodromy of every component S of $\overline{\mathcal{M}}_g^f$ is $\mathrm{Sp}_{2g}(\mathbb{Z}/\ell)$.

Let $S_{[\ell]} := \mathrm{Isom}((\mathrm{Pic}^0(\mathcal{C}/S)[\ell], \langle\cdot,\cdot\rangle_\lambda), ((\mathbb{Z}/\ell)_S^{2g}, \langle\cdot,\cdot\rangle_{\mathrm{std}}))$.

Monodromy $= \mathrm{Sp}_{2g}(\mathbb{Z}/\ell)$ iff $S_{[\ell]}$ is irreducible.
Now $\tilde{S} = \Pi_\ell^{-1}(S)$ dominated by $S_{[\ell]}$ so also irreducible.

So $\tilde{S}$ intersects $\Delta_{i,g-i}$ (fiber above point $[X]$ of Δ_i contains point where restriction of π to each component of X is connected).

On to Theorem 2. Let $\ell = 2$.

If $\ell = 2$, then $R_{g,2} \to \mathcal{A}_{g-1}$, with $(\pi : Y \to X) \mapsto P_\pi$.

Define $R_g^{(f,f')}$ stratum representing covers $\pi : Y \to X$ s.t. X has p-rank f and P_π has p-rank f'.

Applying purity to p-rank strata of $[Y] \in \mathcal{M}_{2g-1}$ yields

Proposition:

Every component of $R_g^{(f,f')}$ has dimension at least $g - 2 + f + f'$.

For Theorem 2, set $f' = g - 2$, S component of $\overline{\mathcal{M}}_g^f$, and $\tilde{S} = \Pi^{-1}(S)$.
(T2) Prym is almost ordinary under non-empty codim 1 condition on $\tilde{S}$.
Key point: need $R_g^{f,g-2} \cap \tilde{S}$ non-empty.

Theorem 2: Low genus base cases

Let S be component of $\overline{\mathcal{M}}_g^f$ and $\tilde{S} = \Pi^{-1}(S)$.
(T2) Prym is almost ordinary under non-empty codim 1 condition on $\tilde{S}$.

Remark: (T2) is non-trivial even for $g_X = 2$, where there is an explicit fiber product construction of $\pi : Y \to X$, because equations for p-rank are unwieldly and not useful for studying this problem for large p.

Faber/Van der Geer: if $\ell = 2$, $g_X = 2$, $\exists \pi : Y \stackrel{\mathbb{Z}/2}{\to} X$ s.t. $f_Y = 0$ iff $p \geq 5$.

Ekin/P: for base case $g = 3$ and $0 \leq f \leq 3$, prove $R_{3,2}^{f,1}$ non-empty,

i.e., $\exists \pi : Y \stackrel{\mathbb{Z}/2}{\to} X$ s.t. X has genus 3 and p-rank f and P_π has p-rank 1.

Note $\mathcal{M}_3^f$ is irreducible for $0 \leq f \leq 3$.

Theorem 2: inductive step

Let S be component of $\overline{\mathcal{M}}_g^f$. Let $\tilde{S} = \Pi_\ell^{-1}(S)$.

Key point: need $R_g^{f,g-2} \cap \tilde{S}$ non-empty.

Degenerate S to Δ_3. Then S contains $\kappa_{i:g-i}(B \times C)$
for some component B of $\overline{\mathcal{M}}_{i;1}^{f_1}$ and component C of $\overline{\mathcal{M}}_{g-i;1}^{f_2}$.

$\Pi_2^{-1}(C)$ is irreducible, and generic Prym is ordinary, p-rank $g-i-1$.
Base case, there exists a point of $\Pi^{-1}(B)$ s.t. Prym has p-rank 1.

Then $\kappa_{i:g-i}(\Pi_2^{-1}(B) \times \Pi_2^{-1}(C)) \subset \Pi_2^{-1}(S)$ contains point whose Prym has p-rank $g-2$, almost ordinary.

Conclusion

1st half: existence of Jacobians with given NP and EO type.

2nd half: existence of curves fixed p-rank having Prym of given p-rank.

Both halves: require precise information about intersection of p-torsion strata of curves with boundary components of $\overline{\mathcal{M}}_g$.

To generalize either half: need better control of variation of p-torsion strata in deformation of curves

Question: under what conditions can a singular curve with a-number 2 be deformed to smooth curve with a-number 2?

Not always: Example $\kappa_{i:g-i}(\overline{\mathcal{M}}_{i;1}^{i-1} \times \overline{\mathcal{M}}_{g-i}^{g-i-1})$ codim 3 in $\overline{\mathcal{M}}_g$.
Component of a-number 2 locus of $\overline{\mathcal{M}}_g$ which is fully contained in $\partial\mathcal{M}_g$.

Thanks for your attention.

The p-rank

Let A/k be an abelian variety of dimension g.

E.g. $A = \mathrm{Jac}(X)$ for a (smooth) proj. conn. curve X/k of genus g.

Then $A[p]$ is a group scheme of rank p^{2g}.

Fact/Def:

Then $\#A[p](k) = p^f$ for some integer $0 \leq f \leq g$ called the p-rank of A.

Def: A is *ordinary* if $f = g$ and *almost ordinary* if $f = g - 1$.

If $X/\mathbb{F}_q$, then f is the length of the slope 0 portion of $\mathrm{NP}(X)$, where $NP(X)$ is the Newton polygon of the L-polynomial of X.

X is supersingular if all slopes of $\mathrm{NP}(X)$ equal $1/2$.
X supersingular implies X has p-rank 0 but converse false for $g \geq 3$.

B. Definition of Newton polygon

Let X be a smooth projective curve defined over $\mathbb{F}_q$.
Zeta function of X is $Z(X/\mathbb{F}_q, t) = L(X/\mathbb{F}_q, t)/(1-t)(1-qt)$

where $L(X/\mathbb{F}_q, t) = \prod_{i=1}^{2g}(1 - w_i t) \in \mathbb{Z}[t]$ and $|w_i| = \sqrt{q}$.

The Newton polygon of X is the NP of the L-polynomial $L(t)$.
Find p-adic valuation v_i of coefficient of t^i in $L(t)$.
Draw lower convex hull of $(i, v_i/a)$ where $q = p^a$.

Facts: The NP goes from $(0,0)$ to $(2g, g)$.
NP line segments break at points with integer coefficients;
If slope λ occurs with length m_λ, so does slope $1-\lambda$.

Definition

$X/\mathbb{F}_q$ is *supersingular* if the Newton polygon of $L(X/\mathbb{F}_q, t)$ is a line segment of slope $1/2$.

B. Definition of Newton polygon

Let A be a p.p. abelian variety of dimension g over k.

Manin: for c, d relatively prime s.t. $\lambda = \frac{c}{d} \in \mathbb{Q} \cap [0, 1]$,
define a p-divisible group $G_{c,d}$ of dimension c and height d.

The Dieudonné module D_λ for $G_{c,d}$ is a $W(k)$-module.
Over $\mathrm{Frac}(W(k))$, there is a basis $x_1, \ldots, x_d$ for D_λ s.t. $F^d x_i = p^c x_i$.

There is an isogeny of p-divisible groups $A[p^\infty] \sim \oplus_\lambda G_{c,d}^{m_\lambda}$.

Newton polygon: lower convex hull - line segments slope λ, length m_λ.

Definition: A *supersingular* iff $\lambda = \frac{1}{2}$ is the only slope.

There is a partial ordering on NPs; the supersingular NP is 'smallest'.