Representations of Algebraic Groups

Lecture 1, Sept. 6, 2016

Throughout: $\mathbb{C}$ is our ground field

Most of the theory works without any changes over any $k = \bar{k}$, $\mathrm{char}(k) = 0$.

More difficult: $k \subsetneq \bar{k}$ or $\mathrm{char}(k) > 0$.

Unless stated otherwise: vector spaces have dim $< \infty$.

V: $\mathbb{C}$-vector space

$\mathrm{End}(V) = \{ \mathbb{C}\text{-lin. } f: V \to V \}$ this is a ring

$GL(V)$ = group of units in $\mathrm{End}(V)$ $= \{ f: V \xrightarrow{\sim} V \text{ bijective } \mathbb{C}\text{-linear} \}$

Choice of (ordered) basis for V $\hat{=}$ choice of isomorphism $\mathbb{C}^n \xrightarrow{\sim} V$
$(n = \dim(V))$

Such a choice induces
$$\mathrm{End}(V) \xrightarrow[\sim]{\varphi} M_n(\mathbb{C})$$
$$GL(V) \xrightarrow{\sim} GL_n(\mathbb{C}).$$

But: we want to understand things in a coordinate-free way, as much as possible.

Let $x_{ij}: M_n(\mathbb{C}) \longrightarrow \mathbb{C}$ be the function given by
$$A \longmapsto A_{ij} = \text{coefficient in position } (i,j)$$

Proposition Let $f: \mathrm{End}(V) \to \mathbb{C}$ be a function. Then the following are equivalent:

(1) For some choice of basis for V, the function $f \circ \varphi^{-1}: M_n(\mathbb{C}) \to \mathbb{C}$ is a polynomial in the functions x_{ij}.

(2) For every choice of basis for V, the function $f \circ \varphi^{-1}$ is a polynomial in the functions x_{ij}.

Proof : (2) $\Rightarrow$ (1) is obvious, so we must show that (1) $\Rightarrow$ (2). We have a basis $e : \mathbb{C}^n \xrightarrow{\sim} V$, and $\varphi : \mathrm{End}(V) \xrightarrow{\sim} M_n(\mathbb{C})$ is the induced isomorphism. If $e' : \mathbb{C}^n \xrightarrow{\sim} V$ is another basis and $\psi : \mathrm{End}(V) \xrightarrow{\sim} M_n(\mathbb{C})$ is the induced isomorphism, there is an invertible matrix $Q \in GL_n(\mathbb{C})$ such that $e' = e \circ Q$, and then $\psi = \mathrm{Inn}(Q^{-1}) \circ \varphi$. So

$$f \circ \psi^{-1} = (f \circ \varphi^{-1}) \circ \mathrm{Inn}(Q).$$

Assumption (2) says that $(f \circ \varphi^{-1})$ is a polynomial in the x_{ij}. On the other hand, $x_{ij} \circ \mathrm{Inn}(Q)$ is a linear combination of the functions $x_{k\ell}$, for every (i,j). So $f \circ \psi^{-1}$ is again a polynomial in the $x_{k\ell}$.

This proves (1) $\rightarrow$ (2) $\boxtimes$

<u>Remark</u> : Instead of asking that $f \circ \varphi^{-1}$ is a polynomial in the x_{ij}, we could ask that it is an (everywhere defined!) rational expression in the x_{ij}. This turns out to be equivalent.

<u>Definition</u> We call $f : \mathrm{End}(V) \to \mathbb{C}$ algebraic if the equivalent conditions (1) and (2) are satisfied. Notation:

$$\mathcal{O}(\mathrm{End}(V)) = \left\{ f : \mathrm{End}(V) \to \mathbb{C} \;\middle|\; f \text{ is algebraic} \right\}.$$

Note: this is a commutative ring.

The choice of a basis for V induces an isomorphism of rings

$$\mathcal{O}(\mathrm{End}(V)) \cong \mathbb{C}[x_{ij} \;;\; 1 \le i,j \le n].$$

Example: $\det \in \mathcal{O}(\mathrm{End}(V))$.

Note: $GL(V) = \{ A \in \mathrm{End}(V) \mid \det(A) \neq 0 \}$.

Definition We call $f : GL(V) \to \mathbb{C}$ algebraic if f can be written as $f = \frac{g}{\det^m}$ with $m \in \mathbb{Z}$ and g an algebraic function on $\mathrm{End}(V)$.

Notation:

$$\mathcal{O}(GL(V)) = \{ f : GL(V) \to \mathbb{C} \mid f \text{ is algebraic} \}$$
$$= \mathcal{O}(\mathrm{End}(V)) \left[\frac{1}{\det} \right].$$

The choice of a basis for V induces an isomorphism of rings

$$\mathcal{O}(GL(V)) \cong \mathbb{C}\left[x_{ij} ; \frac{1}{\det} \right]$$

Note that the actual expression of "det" as a polynomial in the x_{ij} is independent of the choice of basis for V:

$$\det = \sum_{\sigma \in S_n} (-1)^{\varepsilon(\sigma)} \cdot \prod_{i=1}^{n} x_{i\sigma(i)}.$$

Definition A subgroup $G \subset GL(V)$ is called an algebraic subgroup if there exist algebraic functions f_i $(i \in I)$ such that

$$G = \mathcal{Z}(\{f_i\}_{i \in I})$$
$$\stackrel{\text{def}}{=} \{ g \in GL(V) \mid f_i(g) = 0 \text{ for all } i \in I \}$$

$(\mathcal{Z} = \text{"zero locus"})$

Example Let $W \subset V$ be a linear subspace, and consider

$$G = \mathrm{Stab}(W) = \{ g \in GL(V) \mid g(W) \subseteq W \}$$
$$= \{ g \in GL(V) \mid g(W) = W \}.$$

Then G is an algebraic subgroup. To see this: choose a basis $e_1, \dots, e_n$ for V such that $W = \mathrm{Span}(e_1, \dots, e_d)$ for some $d \leq n$. Then:

$$\begin{array}{ccc} GL(V) & \xrightarrow{\sim} & GL_n(\mathbb{C}) \\ \cup & & \cup \\ G & \xrightarrow{\sim} & \left\{ \begin{array}{cc} \begin{pmatrix} * & * \\ 0 & * \end{pmatrix} & \begin{array}{c} d \\ n-d \end{array} \\ \begin{array}{cc} d & n-d \end{array} & \end{array} \right\} \end{array}$$

We see: G is defined by the polynomial (even linear) equations

$$x_{ij} = 0 \quad \text{for} \quad \begin{cases} 1 \leq j \leq d \\ d+1 \leq i \leq n \end{cases}.$$

Example $SL(V) \subset GL(V)$ is defined by $\det = 1$; hence it is an algebraic subgroup.

Bilinear forms. To prepare for the next two (very important) examples, let us quickly review some basic notions from the theory of bilinear forms. As before, V is a complex vector space.

Let $\varphi : V \times V \to \mathbb{C}$ be a bilinear form. For $v \in V$, $\varphi(-, v) : V \to \mathbb{C}$ is a linear map, so an element of the dual vector space $V^\vee$. The map $\Phi : V \to V^\vee$ given by $v \mapsto \varphi(-, v)$ is a linear map. We can recover φ from Φ by $\varphi(x, y) = \Phi(y)(x)$.

In what follows we fix φ. For $W \subseteq V$, define
$$W^{\perp} := \{ y \in V \mid \varphi(w,y) = 0 \text{ for all } w \in W \} \quad ;$$
this is again a linear subspace of V.

<u>Lemma.</u> The following are equivalent:

(1) Φ is injective ;

(2) Φ is an isomorphism ;

(3) $V^{\perp} = (0)$

(4) for every $0 \neq y \in V$ there exists an $x \in V$ with $\varphi(x,y) \neq 0$.

The proof is easy and is left as an exercise.

If the equivalent conditions (1) – (4) are satisfied, we say that the form φ is non-degenerate.

<u>Matrix of a bilinear form.</u> If we choose an ordered basis $e_1, \ldots, e_n$ (equiv: an isom $e : \mathbb{C}^n \xrightarrow{\sim} V$) then we can express φ using a matrix, as follows. Define an $n \times n$ matrix B by

$$B_{ij} = \varphi(e_i, e_j) \qquad : \text{ the matrix of } \varphi \text{ w.r.t. the basis } e.$$

Then $\psi = \varphi \circ (e \times e) : \mathbb{C}^n \times \mathbb{C}^n \longrightarrow \mathbb{C}$ is given by

$$\psi(x,y) = {}^t x \cdot B \cdot y = \sum_{i,j} B_{ij}\, x_i y_j \, .$$

<u>Note</u>: If $e' : \mathbb{C}^n \longrightarrow V$ is another choice of basis, there is an invertible matrix $Q \in GL_n(\mathbb{C})$ s.t. $e' = e \circ Q$, and then the matrix of φ with regard to the basis e' is ${}^t Q \cdot B \cdot Q$.

<u>Important</u>: This is very different from the "change of basis"-formula

for the matrix expressing a linear transformation. You should not mistake the above matrix B for the matrix of a linear transformation — that's really something else.

<u>Remark</u>. Let $\varphi : V \times V \to \mathbb{C}$ be a bilinear form. Then

φ is non-degenerate

$\iff$ for some choice of basis, the matrix B satisfies $\det(B) \neq 0$

$\iff$ for every ——— " ———

Try to prove this yourself!

A bilinear form φ as above is

symmetric if $\varphi(w,v) = \varphi(v,w)$ for all $v, w \in V$

anti-symmetric $\varphi(w,v) = -\varphi(v,w)$ ——— " ——— .

<u>Lemma</u>. (i) Suppose $\varphi : V \times V \to \mathbb{C}$ is a non-degenerate symmetric bilinear form. Then there exists a basis $e_1, \dots, e_n$ for V with respect to which φ is given by the matrix

$$B = \begin{pmatrix} 0 & & 1 \\ & \iddots & \\ 1 & & 0 \end{pmatrix} \qquad \text{(anti-diagonal matrix)}.$$

(ii) Suppose φ is a non-degenerate anti-symmetric bilinear form. (We call this a symplectic form.) Then $\dim(V)$ is even, say $\dim V = 2m$, and there exists a basis $e_1, \dots, e_m, e_{m+1}, \dots, e_{2m}$ for V such that φ is given on this basis by the matrix

$$B = \begin{array}{c} \left(\begin{array}{ccc|ccc} & & & 0 & & 1 \\ & 0 & & & \iddots & \\ & & & 1 & & 0 \\ \hline 0 & & -1 & & & \\ & \iddots & & & 0 & \\ -1 & & 0 & & & \end{array}\right) \end{array} \begin{array}{l} \\ m \\ \\ \\ m \\ \\ \end{array}$$

Sketch of the proof of (ii) : We argue by induction on $\dim(V) = n$.

$n=0$: void.

$n=1$: does not occur, for if $V = \mathbb{C}\cdot e$ then anti-symmetry implies $\varphi(e,e) = 0$ but the requirement that φ is non-degenerate forces $\varphi(e,e) \neq 0$.

$n=2$: Choose any $0 \neq e_1 \in V$. Note : $\varphi(e_1,e_1) = 0$ because of anti-symmetry. As φ is non-degenerate, there exists a vector e_2 with $\varphi(e_1,e_2) \neq 0$, and because $\varphi(e_1,\lambda e_2) = \lambda\cdot\varphi(e_1,e_2)$ we may assume $\varphi(e_1,e_2) = 1$ and hence $\varphi(e_2,e_1) = -1$. Also $\varphi(e_2,e_2) = 0$. Further it is clear that e_2 is linearly independent from e_1, and hence e_1, e_2 is a basis of V on which φ is given by the matrix $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$.

Induction step : Assume (ii) is correct for all (V',φ') of dimension $< n = \dim(V)$.

Choose $0 \neq e_1 \in V$. As above, find $e_n \in V$ with $\varphi(e_1,e_n) = 1$. Let $W = \mathbb{C}\cdot e_1 + \mathbb{C}\cdot e_n$. Then $\dim(W) = 2$, and $W^\perp \subset V$ has $\dim(W^\perp) = n-2$. Further : $W \cap W^\perp = (0)$ and so $V = W \oplus W^\perp$.
The form $\varphi|_{W^\perp}$ is again symplectic, so by induction $\dim(W^\perp)$ is even and we can find a basis $e_2, \dots, e_{n-1}$ for $W^\perp$ such that $\varphi|_{W^\perp}$ is given by the matrix

$$\begin{array}{cc} \left(\begin{array}{ccc|ccc} & & & 0 & & 1 \\ & 0 & & & \iddots & \\ & & & 1 & & 0 \\ \hline 0 & & -1 & & & \\ & \iddots & & & 0 & \\ -1 & & 0 & & & \end{array}\right) & \begin{array}{c} \frac{n-2}{2} \\ \\ \\ \frac{n-2}{2} \end{array} \end{array}$$

on this basis. Then the basis $e_1, \dots, e_n$ does the job for φ.

__Definition.__ (i) For $\varphi : V \times V \to \mathbb{C}$ a non-degenerate symmetric bilinear form, let

$$O(V,\varphi) = \left\{ g \in GL(V) \;\middle|\; \begin{array}{l} \varphi(gv,gw) = \varphi(v,w) \\ \text{for all } v,w \in V \end{array} \right\};$$

This is called the __orthogonal group__ of φ.

(ii) For $\varphi : V \times V \to \mathbb{C}$ symplectic, let

$$Sp(V,\varphi) = \left\{ g \in GL(V) \;\middle|\; \begin{array}{l} \varphi(gv,gw) = \varphi(v,w) \\ \text{for all } v,w \in V \end{array} \right\};$$

This is called the __symplectic group__ of (or associated with) φ.

For $V = \mathbb{C}^n$ with φ given by $\begin{pmatrix} 0 & & 1 \\ & \iddots & \\ 1 & & 0 \end{pmatrix}$ we simply write O_n.

For $V = \mathbb{C}^{2m}$ with φ given by $\begin{pmatrix} & & & & & 1 \\ & 0 & & & \iddots & \\ & & & 1 & & \\ & & -1 & & & \\ & \iddots & & & 0 & \\ -1 & & & & & \end{pmatrix}$ we simply write Sp_{2m}.

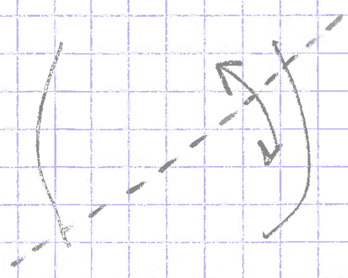
__Theorem__: $O(V,\varphi)$ (orthogonal case)
$Sp(V,\varphi)$ (symplectic case)

are algebraic subgroups of $GL(V)$.

Proof for $O(V,\varphi)$: If $g \in GL_n(\mathbb{C})$, let $g^\dagger$ denote the matrix given by

$$(g^\dagger)_{i,j} = g_{n+1-j,\,n+1-i}$$

$g \mapsto g^\dagger$ is reflection in the anti-diagonal



If we let $J = \begin{pmatrix} 0 & & 1 \\ & \diagup & \\ 1 & & 0 \end{pmatrix}$ then $J = J^{-1}$ and $g^{\dagger} = J \cdot {}^t g \cdot J$.

Now choose a basis for V such that φ is given by the matrix J. Then if $g \in GL_n(\mathbb{C})$ the form $(v, w) \longmapsto \varphi(gv, gw)$ is given by the matrix ${}^t g \cdot J \cdot g$. So:

$$
\begin{aligned}
g \in O(V, \varphi) \quad &\Longleftrightarrow \quad {}^t g \cdot J \cdot g = J \\
&\Longleftrightarrow \quad J \cdot {}^t g \cdot J \cdot g = \mathbb{1} = \begin{pmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{pmatrix} \\
&\Longleftrightarrow \quad g^{\dagger} \cdot g = \mathbb{1}
\end{aligned}
$$

It is clear that this last identity is a system of n^2 polynomial equations in the coefficients of g.

Symplectic case: easy variant on this proof.