Representations of Algebraic Groups

Lecture 3, Sept. 20, 2016

As before : V vector space / $\mathbb{C}$ ($\dim < \infty$)

$\mathcal{O}(GL(V))$ = $\mathbb{C}$-algebra of algebraic functions $f: GL(V) \to \mathbb{C}$

Choice of basis $e: \mathbb{C}^n \xrightarrow{\sim} V$ induces an isomorphism

$$\mathcal{O}(GL(V)) \cong \mathbb{C}[x_{ij}, \tfrac{1}{\det}] \qquad (1 \le i,j \le n)$$

<u>Definition</u> Let $G \subset GL(V)$ be an algebraic group. Then we define :

$$I_G := \{ f \in \mathcal{O}(GL(V)) \mid f(g) = 0 \text{ for all } g \in G \}.$$

This is an ideal of $\mathcal{O}(GL(V))$. Further we define

$$\mathcal{O}(G) := \mathcal{O}(GL(V))/I_G .$$

<u>Note</u> : $\mathcal{O}(G) \hookrightarrow$ $\mathbb{C}$-algebra of <u>all</u> functions $G \to \mathbb{C}$

The functions $f: G \to \mathbb{C}$ that lie in $\mathcal{O}(G)$ are called the algebraic functions on G.

<u>Facts</u> : (1) $G = \mathcal{Z}(I_G) \overset{\text{def}}{=} \{ g \in GL(V) \mid f(g) = 0 \text{ for all } f \in I_G \}$

The inclusion "$\subseteq$" is trivial. On the other hand, because $G \subset GL(V)$ is an algebraic subgroup, there exists a subset $S \subset \mathcal{O}(GL(V))$ such that $G = \mathcal{Z}(S)$. Then clearly $S \subseteq I_G$, and it follows that $\mathcal{Z}(I_G) \subseteq \mathcal{Z}(S) = G$.

(2) The ideal $I_G \subset \mathcal{O}(GL(V))$ is finitely generated : $\exists\, f_1, \dots, f_r \in \mathcal{O}(GL(V))$ s.t. $I_G = (f_1, \dots, f_r)$, and then $G = \mathcal{Z}(f_1, \dots, f_r)$ $= \{ g \in GL(V) \mid f_1(g) = \dots = f_r(g) = 0 \}$.

<u>Definition</u> Let $G_1 \subset GL(V_1)$ and $G_2 \subset GL(V_2)$ be algebraic groups. A homomorphism $\rho : G_1 \longrightarrow G_2$ is said to be <u>algebraic</u> if for every $f \in \mathcal{O}(G_2)$ the function

$$\rho^*(f) \overset{\text{def}}{=} f \circ \rho : G_1 \longrightarrow \mathbb{C}$$

is an algebraic function on G_1.

<u>Example</u> Let $\mathbb{G}_m := GL_1(\mathbb{C})$. Note: $\mathbb{G}_m = \mathbb{C}^*$; the special notation indicates that we wish to view $\mathbb{G}_m$ as an algebraic group. Then $\det : GL(V) \longrightarrow \mathbb{G}_m$ is an algebraic homomorphism. To see this:

- choose a basis for V $\rightsquigarrow$ $\mathcal{O}(GL(V)) = \mathbb{C}[x_{ij}, \frac{1}{\det}]$
- $\mathcal{O}(\mathbb{G}_m) = \mathbb{C}[y, \frac{1}{y}]$

<u>Now</u>: $\det^*(y) = \det$; $\det^*(\frac{1}{y}) = \frac{1}{\det}$, and from this it is clear that $\det^*(f)$ is an elt. of $\mathcal{O}(GL(V))$ for any $f \in \mathbb{C}[y, y^{-1}]$.

<u>Example of a non-algebraic homomorphism</u>: Take

$$G = \left\{ \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} \in GL_2(\mathbb{C}) \;\middle|\; b \in \mathbb{C} \right\}.$$

<u>Note</u>: $G \cong \mathbb{C}$ with addition; we call this algebraic group $\mathbb{G}_a$.
Then

$$\exp : \mathbb{G}_a \longrightarrow \mathbb{G}_m, \quad \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} \longmapsto e^b$$

is a homomorphism that is <u>not</u> algebraic. Indeed, $\mathcal{O}(\mathbb{G}_a) \cong \mathbb{C}[x]$ with $x : G \longrightarrow \mathbb{C}$, $\begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} \longmapsto b$ the function

and then $\exp^*(y) = e^x \notin \mathbb{C}[x]$.

By construction, an algebraic homomorphism $\rho : G_1 \to G_2$ induces a homomorphism of $\mathbb{C}$-algebras

$$\rho^* : \mathcal{O}(G_2) \longrightarrow \mathcal{O}(G_1) .$$
$$f \longmapsto \rho^*(f) = f \circ \rho$$

Important : The homomorphism ρ is fully determined by ρ^*. To see this, suppose $G_1 \subset GL(V_1)$, $G_2 \subset GL(V_2)$, and choose bases for V_1 and V_2.

This gives

$$\mathcal{O}(GL(V_1)) \cong \mathbb{C}\left[x_{ij}, \tfrac{1}{\det}\right] \quad 1 \le i,j \le n$$
$$\mathcal{O}(GL(V_2)) \cong \mathbb{C}\left[y_{pq}, \tfrac{1}{\det}\right] \quad 1 \le p,q \le m$$

(⚠ "det" has 2 different meanings here)

Write
$$I_1 = I_{G_1} \subset \mathbb{C}\left[x_{ij}, \tfrac{1}{\det}\right]$$
$$I_2 = I_{G_2} \subset \mathbb{C}\left[y_{pq}, \tfrac{1}{\det}\right]$$

so that
$$\mathcal{O}(G_1) \cong \mathbb{C}\left[x_{ij}, \tfrac{1}{\det}\right]/I_1 ;$$
$$\mathcal{O}(G_2) \cong \mathbb{C}\left[y_{pq}, \tfrac{1}{\det}\right]/I_2 .$$

Let $\rho^* : \mathcal{O}(G_2) \to \mathcal{O}(G_1)$ be given by $y_{pq} \longmapsto h_{pq} \bmod I_1$ for certain polynomials $h_{pq} \in \mathbb{C}\left[x_{ij}, \tfrac{1}{\det}\right]$.

This just means that $\rho : G_1 \longrightarrow G_2 \subset GL_m(\mathbb{C})$ sends $g \in G_1$ to the matrix $\left(h_{pq}(g)\right) \in GL_m(\mathbb{C})$.

Note : • This matrix is independent of the chosen representatives h_{pq} for the $\rho^*(y_{pq})$.

• This matrix defines a point of G_2 because for any $f \in I_2$ we have
$$f\left(h_{pq}(g)\right) \overset{!}{=} \rho^*(f \bmod I_2)$$
$$= \rho^*(\bar{0}) = 0 ,$$

<u>Example</u> Suppose $\dim(V) = 2$ and e_1, e_2 is a basis of V.
Let $W = \mathrm{Sym}^2(V)$; then $\dim(W) = 3$ and

$$e_1^2 \quad , \quad e_1 e_2 \quad , \quad e_2^2$$

is a basis of W.

A linear transformation $g \in GL(V)$ induces a linear transformation $\mathrm{Sym}^2(g)$ of W. This gives a homomorphism $\rho : GL(V) \longrightarrow GL(W)$, and we claim this ρ is algebraic. Concretely:

$$\rho : \begin{pmatrix} a & b \\ c & d \end{pmatrix} \longrightarrow \begin{pmatrix} a^2 & ab & b^2 \\ 2ac & ad+bc & 2bd \\ c^2 & cd & d^2 \end{pmatrix}$$

This means that

$$\rho^* : \mathbb{C}\left[y_{pq}, \frac{1}{\det} \right]_{1 \leq p,q \leq 3} \longrightarrow \mathbb{C}\left[x_{ij}, \frac{1}{\det} \right]_{1 \leq i,j \leq 2}$$

is given by

$$\left.\begin{aligned} y_{11} &\longmapsto x_{11}^2 \\ y_{12} &\longmapsto x_{11} x_{12} \\ y_{13} &\longmapsto x_{12}^2 \\ y_{21} &\longmapsto 2 x_{11} x_{21} \end{aligned}\right\}$$

this just tells you how the coefficients of the matrix $\rho(g)$ are expressed as functions of the coefficients of g.

<u>etc</u>

Basic notions of representation theory

<u>Definition</u> A representation of a group G is a homomorphism $\rho: G \longrightarrow GL(V)$.

If ρ is as given, we call it a representation of G on the vector space V.
In this case, G acts on V: we have $G \times V \xrightarrow{R} V$

$$(g, v) \longmapsto \rho(g)(v) \quad \text{or}: \; g \cdot v$$

In practice we often write ρ_g for $\rho(g)$. Conversely, if we have an action R of G on V s.t. the maps $\rho_g : V \longrightarrow V$ given by $\rho_g(v) = R(g,v)$ are linear transformations of V, then the ρ_g give a repr. of G on V.

One sometimes refers to V as a "G-module"; this comes from the fact that the representation gives V the structure of a module over the group ring $\mathbb{C}[G]$.

<u>Definition</u> If $\rho_1 : G \longrightarrow GL(V_1)$
$\rho_2 : G \longrightarrow GL(V_2)$ are repr. of G then a linear map $h : V_1 \longrightarrow V_2$ is said to be <u>G-equivariant</u>

or: <u>a homomorphism of representations</u>

if $h(\rho_{1,g}(v)) = \rho_{2,g}(h(v))$ for all $g \in G$ and $v \in V$.

(Written differently: $h(g \cdot v) = g \cdot h(v)$ for all ...)

For those familiar with category theory: with this notion of a (homo)morphism we get a category $\mathrm{Rep}(G)$. <u>Notation</u>: $\mathrm{Hom}_G(V_1, V_2) = \{ h \; \ldots \}$

We say ρ_1 and ρ_2 are <u>isomorphic</u> as repr. of G if there exists a bijective G-equivariant $h : V_1 \xrightarrow{\sim} V_2$. Note that h^{-1} in this case is again G-equivariant.

Constructing new representations out of given ones :

- Direct sums : If $\rho_1 : G \to GL(V_1)$ are repr we get
$\rho_2 : G \to GL(V_2)$

$\rho_1 \oplus \rho_2 : G \to GL(V_1 \oplus V_2)$ by $(\rho_1 \oplus \rho_2)_g (v_1 \oplus v_2)$
$= \rho_{1,g}(v_1) \oplus \rho_{2,g}(v_2)$.

Alternative phrasing :

$GL(V_1) \times GL(V_2) \hookrightarrow GL(V_1 \oplus V_2)$

In terms of matrices : $(A_1, A_2) \longmapsto \left(\begin{array}{c|c} A_1 & 0 \\ \hline 0 & A_2 \end{array}\right)$

and now $\rho_1 \oplus \rho_2$ is obtained by composing this with :

$G \longrightarrow GL(V_1) \times GL(V_2)$; $g \longmapsto (\rho_1(g), \rho_2(g))$

- Tensor products :

$\rho_1 \otimes \rho_2 : G \to GL(V_1 \otimes V_2)$ by $g \longmapsto \rho_{1,g} \otimes \rho_{2,g}$

So on pure tensors : $(\rho_1 \otimes \rho_2)(g) : v_1 \otimes v_2 \longmapsto \rho_{1,g}(v_1) \otimes \rho_{2,g}(v_2)$.

- Subspaces and quotients : Let $\rho : G \to GL(V)$ be a repr.

Def : A lin. subspace $W \subset V$ is a subrepresentation
or : is stable under G

if $\rho_g(W) \subseteq W$ for all $g \in G$. (Equiv : $\rho_g(W) = W$.)

For $W \subset V$ a G-stable subspace, we get induced repr.

$\rho|_W : G \to GL(W)$

$\overline{\rho} : G \to GL(\overline{V})$ where $\overline{V} = V/W$

$\left(\overline{\rho}_g (v \bmod W) = \rho_g(v) \bmod W \right)$

- Dual (or contragredient) representation : $\rho : G \longrightarrow GL(V)$ induces $\check{\rho} : G \longrightarrow GL(V^\vee)$ by $(\rho^\vee)_g = ((\rho_g)^\vee)^{-1} = ((\rho_g)^{-1})^\vee = (\rho_{g^{-1}})^\vee$

Note : If we choose a basis $e_1, \dots, e_n$ for V and as basis for $V^\vee$ we take the dual basis $e_1^\vee, \dots, e_n^\vee$ then :

$$A = \text{matrix of } \rho_g$$

$$\Updownarrow$$

$$^tA \quad \text{matrix of } \rho_g^\vee$$

So : $\rho_g^\vee : V^\vee \longrightarrow V^\vee$ is given by the matrix $^tA^{-1}$.

In this way we indeed get a homom. $G \longrightarrow GL(V^\vee)$:

$$(\rho_{hg})^\vee = (\rho_g)^\vee \circ (\rho_h)^\vee \qquad \text{(note the order)}$$

$$\rightsquigarrow \quad (\rho^\vee)_{hg} = (\rho^\vee)_h \circ (\rho^\vee)_g \qquad \underline{\text{OK}} .$$

- Symmetric & exterior powers :

$$T(V) = \bigoplus_{n \geq 0} V^{\otimes n} \qquad \text{(non-comm.) ring}$$

Ideal $I \subset T(V)$ generated by all $v \otimes w - w \otimes v$; then $I = \bigoplus_{n \geq 0} I_n$ with $I_n = I \cap V^{\otimes n}$, and by def. :

$$\mathrm{Sym}^n(V) := V^{\otimes n} / I_n$$

Let $\rho : G \to GL(V)$ be a repr. We get an induced action of G on $T(V)$, and this induced action preserves the grading and the ring structure. Also : $\rho_g \circlearrowright T(V)$ sends I into itself.

Therefore $I_n \subset V^{\otimes n}$ is stable under the action of G and we get an induced representation

$$\mathrm{Sym}^n(\rho) : G \longrightarrow \mathrm{Sym}^n(V)$$

In exactly the same way: $J \subset T(V)$ the 2-sided ideal generated by all $v \otimes v$; then $J = \bigoplus_{n \geq 0} J_n$ and $\Lambda^n V = V^{\otimes n}/J_n$. Then the G-action on $T(V)$ sends J into itself and we get an induced repr.

$$\Lambda^n \rho : G \longrightarrow \Lambda^n V .$$

Let $\rho : G \to GL(V)$ be a repr. Suppose we have a collection $\{W_\alpha\}_{\alpha \in A}$ of G-stable subspaces. Then their span $\sum W_\alpha \subset V$ is again a G-stable subspace.

We say that $\sum W_\alpha$ is the <u>direct sum</u> of the subspaces W_α if each $y \in \sum W_\alpha$ can be written <u>uniquely</u> as $y = \sum_{\alpha \in A} w_\alpha$ with $w_\alpha \in W_\alpha$ and $w_\alpha = 0$ for almost all α. In this case we write the sum as $\bigoplus_\alpha W_\alpha$.
In order for $\sum W_\alpha$ to be a direct sum we must have:

$\forall$ subset $B \subseteq A$,

$$\forall \alpha \notin B : \quad W_\alpha \cap \Big(\sum_{\beta \in B} W_\beta \Big) = 0 .$$

<u>Definition</u> A repr. $\rho : G \to GL(V)$ is <u>irreducible</u> if $V \neq 0$ and V has no subspaces $0 \subsetneq W \subsetneq V$ that are stable under the action of G.

A repr. ρ is <u>semisimple</u> (or: completely reducible) if it is $\cong$ to a direct sum of irreducible repr.

Proposition Let $\rho : G \longrightarrow GL(V)$ be a repr. Then the following are equivalent:

(1) V is semisimple

(2) V is spanned by its irreducible submodules

(3) for every G-stable subspace $W \subset V$ there exists a G-stable $W' \subset V$ such that $V = W \oplus W'$.

Proof (1) $\Rightarrow$ (2) : obvious.

(2) $\Rightarrow$ (3) : Let $\{V_\alpha\}_{\alpha \in A}$ be the collection of all irreducible sub-reps of V. By assumption, $V = \sum_{\alpha \in A} V_\alpha$. Given W, let $B \subset A$ be a maximal subset for which the sum $U = W + \sum_{\beta \in B} V_\beta$ is direct. If $\alpha \notin B$ then the sum $U + V_\alpha$ is not direct; hence $U \cap V_\alpha \neq 0$. But V_α is irreducible and $U \cap V_\alpha$ is a G-submodule. Hence $V_\alpha \subset U$. It follows that $V_\alpha \subseteq U$ for all $\alpha \in A$ and hence $U = V$. Now take $W' = \sum_{\beta \in B} V_\beta$.

(3) $\Rightarrow$ (1) : easy.

Example Let $G = \left\{ \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \in GL_2(\mathbb{C}) \right\}$ and consider the tautological repr. $\rho : G \longrightarrow GL(\mathbb{C}^2) = GL_2(\mathbb{C})$. This repr. is not semisimple: the line $W = \mathbb{C} \cdot \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ is a subrep., and it is the unique 1-dim'l subrepr. of $V = \mathbb{C}^2$.

Caution If a repr. V is semisimple then its decomposition as a $\oplus$ of irreducible repr. is not unique, in gen'l. Just think of the case when $G = \{1\}$. We'll see later that there is a somewhat coarser repr. that is canonical.

Proposition. Let $\rho : G \longrightarrow GL(V)$ and $\sigma : G \longrightarrow GL(W)$ be repr.,
$h : V \longrightarrow W$ a homom. of repr.

(i) $\mathrm{Ker}(h) \subset V$ and $\mathrm{Im}(h) \subset W$ are subreps.
(ii) Suppose $h \neq 0$ and V is irreducible. Then h is $\hookrightarrow$
(iii) Suppose $h \neq 0$ and W is irreducible. Then h is $\twoheadrightarrow$.

The easy proof is left as an exercise

Corollary (Schur's lemma) : If $\rho : G \rightarrow GL(V)$ is irreducible then
$\mathrm{End}_G(V) = \mathbb{C} \cdot \mathrm{id}_V$.
$\left(\hookrightarrow = \{ h : V \longrightarrow V \text{ linear} \mid h \text{ is } G\text{-equivariant} \} \right)$

Proof Clear: $\mathbb{C} \cdot \mathrm{id}_V \subset \mathrm{End}_G(V)$. If $0 \neq h \in \mathrm{End}_G(V)$ then
h is a bijection. Hence $\mathrm{End}_G(V)$ is a division ring. Now take any
$0 \neq h \in \mathrm{End}_G(V)$ and let λ be an eigenvalue. Then $h - \lambda \cdot \mathrm{id}_V$ is
a non-invertible element of $\mathrm{End}_G(V)$; hence $h = \lambda \cdot \mathrm{id}_V$ $\boxtimes$