Representations of Algebraic Groups
Lecture 4, Sept. 27, 2016

Let $\rho: G \to GL(V)$ be a repr. Suppose we have a collection $\{W_\alpha\}_{\alpha \in A}$ of G-stable subspaces of V. Then their span $\sum_{\alpha\in A} W_\alpha$ is again a G-stable subspace.

We say that $\sum W_\alpha$ is the direct sum of the W_α if every $y \in \sum W_\alpha$ can be written uniquely as $y = \sum y_\alpha$ with $y_\alpha \in W_\alpha$. In this case we write the sum as $\bigoplus_{\alpha\in A} W_\alpha$.

In order for $\sum W_\alpha$ to be a direct sum we must have:

for any subset $B \subseteq A$ and $\alpha \in A \setminus B$: $W_\alpha \cap \left(\sum_{\beta\in B} W_\beta\right) = 0$.

Definition A repr. $\rho: G \to GL(V)$ is irreducible if $V \neq 0$ and V has no G-stable subspaces other than (0) and V itself.

A repr. is semisimple, or: completely reducible, if it is $\cong$ to a direct sum of irreducible repr.

Lemma A: Let $\rho: G \to GL(V)$ be a representation,
$W \subseteq V$ a G-stable subspace
$\{Y_\alpha\}_{\alpha\in A}$ a collection of irreducible G-stable subspaces
such that $V = W + \sum_{\alpha\in A} Y_\alpha$. Then there is a subset $B \subseteq A$ such that V is the direct sum of W and the subspaces Y_β with $\beta \in B$. In other words: $V = W \oplus \bigoplus_{\beta\in B} Y_\beta$.

Proof: Let $B \subseteq A$ be a subset for which the sum $\underbrace{W + \sum_{\beta\in B} Y_\beta}_{\text{is direct}}$, and which is maximal with respect to this property. All we need to show is that $W + \sum_{\beta \in B} Y_\beta = V$. For any $\alpha \in A$ the sum $W + (\sum_{\beta \in B} Y_\beta) + Y_\alpha$ is not direct, and it follows that $U := Y_\alpha \cap (W + \sum_{\beta \in B} Y_\beta)$ is not (0). As U is a G-stable subspace of Y_α and Y_α is irreducible, $U = Y_\alpha$; this shows that $Y_\alpha \subseteq W + \sum_{\beta \in B} Y_\beta$. As this holds for all α, it follows that $V = W + \sum_{\alpha \in A} Y_\alpha \subseteq W + \sum_{\beta \in B} Y_\beta$. $\square$

Proposition Let $\rho : G \to GL(V)$ be a repr. Then the following are equiv:

(1) V is semisimple

(2) V is spanned by its irreducible submodules

(3) for every G-stable subspac $W \subset V$ there exists a G-stable $W' \subset V$ such that $V = W \oplus W'$.

Proof (1) $\Rightarrow$ (2): obvious; (2) $\Rightarrow$ (1): Apply the lemma with $W = (0)$.

(2) $\Rightarrow$ (3): Apply the lemma and take $W' = \bigoplus_{\beta \in B} Y_\beta$.

(3) $\Rightarrow$ (2): Let W be the span of all irreducible submodules in V. If $W \subsetneq V$ then (3) gives a G-stable $W' \subset V$ with $V = W \oplus W'$, and $W' \neq (0)$. Let $W'' \subset W'$ be any minimal G-stable subspace $\neq (0)$. Then W'' is irreducible as a representation of G, hence $W'' \subset W$, contradicting the fact that V is the direct sum of W and W'. Hence $W = V$, i.e., (2) holds. $\boxtimes$

Example Let $G = \left\{ \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \in GL_2(\mathbb{C}) \right\}$ and consider the tautological representation $\rho : G \to GL_2(\mathbb{C})$. This representation is <u>not</u> semisimple: the line $W = \mathbb{C} \cdot \binom{1}{0}$ is a subrepr., and it is the <u>only</u> 1-dim'l subrepr. of $V = \mathbb{C}^2$.

Proposition Let $\rho: G \to GL(V)$ and $\sigma: G \to GL(W)$ be repr., $h: V \to W$ a homom of repr.

(i) $\mathrm{Ker}(h) \subset V$ and $\mathrm{Im}(h) \subset W$ are G-stable.
(ii) If $h \neq 0$ and V is irreducible then h is $\hookrightarrow$.
(iii) If $h \neq 0$ and W is irreducible then h is $\twoheadrightarrow$.

Proof: left as an (easy) exercise.

Corollary (Schur's lemma) If $\rho: G \to GL(V)$ is irreducible then $\mathrm{End}_G(V) = \mathbb{C} \cdot \mathrm{id}_V$.

(Here $\mathrm{End}_G(V) = \mathrm{End}(\rho) = \{G\text{-equivariant lin. maps } h: V \to V\}$.)

Proof Clear: $\mathbb{C} \cdot \mathrm{id}_V \subseteq \mathrm{End}_G(V)$. If $0 \neq h \in \mathrm{End}_G(V)$ then h is a bijection. Hence $\mathrm{End}_G(V)$ is a division ring. Now take any $0 \neq h \in \mathrm{End}_G(V)$ and let λ be an eigenvalue of h. Then $h - \lambda \cdot \mathrm{id}_V$ is a non-invertible element of $\mathrm{End}_G(V)$; hence $h = \lambda \cdot \mathrm{id}_V$. $\boxtimes$

Definition An algebraic group T is called a torus if $T \cong \mathbb{G}_m^r$ for some $r \geq 0$. In this case r is called the rank of T.

(Caution: do not confuse this with the notion of a complex torus that you may have seen in other courses.)

Example Let

$$T = \{\text{diagonal matrices in } SL_n(\mathbb{C})\} = \left\{ \begin{pmatrix} a_1 & & 0 \\ & \ddots & \\ 0 & & a_n \end{pmatrix} \;\middle|\; \prod a_i = 1 \right\}.$$

This is a torus of rank $n-1$: $\mathbb{G}_m^{n-1} \xrightarrow{\sim} T$ by

$$(a_1, \ldots, a_{n-1}) \longmapsto \mathrm{diag}(a_1, a_2, \ldots, a_{n-1}, (a_1 \cdots a_{n-1})^{-1}).$$

If T is a torus of rank >0 then the isomorphism $T \cong \mathbb{G}_m^r$ is not canonical. Eg., in the previous example with $n=3$ we could just as well have used $\mathbb{G}_m^2 \xrightarrow{\sim} T$ given by

$$(a_1, a_2) \longmapsto \begin{pmatrix} a_1^{-1} a_2^{2} & & 0 \\ & a_1^{2} a_2^{-3} & \\ 0 & & a_1^{-1} a_2 \end{pmatrix}$$

However, as we shall see there is a good way to "repair" this.

<u>Lemma</u> $\mathrm{End}(\mathbb{G}_m) \cong \mathbb{Z}$

$$(z \mapsto z^n) \longleftarrow n$$

Note: $\mathrm{End}(\mathbb{G}_m) = \{ \text{homom of alg. groups } \mathbb{G}_m \to \mathbb{G}_m \}$

<u>Proof</u> (Cf. Exercises of last week): any endomorphism of $\mathbb{C}[x, x^{-1}]$ as a $\mathbb{C}$-algebra is given by $x \mapsto c \cdot x^n$ for some $c \in \mathbb{C}^*$ and $n \in \mathbb{Z}$, because the elements $c \cdot x^n$ are the only units in $\mathbb{C}[x, x^{-1}]$. Now remark that $z \mapsto c \cdot z^n$ gives an endomorphism of $\mathbb{G}_m$ only if $c = 1$. $\boxtimes$

<u>Definition</u> If T is a torus, let

$X_*(T) = \mathrm{Hom}(\mathbb{G}_m, T)$ cocharacter group

$X^*(T) = \mathrm{Hom}(T, \mathbb{G}_m)$ character group.

<u>Proposition</u> (i) If T has rank r then $X^*(T)$ and $X_*(T)$ are free abelian groups of rank r (i.e., $\cong \mathbb{Z}^r$).

(ii) The pairing

$$\langle \, , \rangle : X_*(T) \times X^*(T) \longrightarrow \mathrm{End}(\mathbb{G}_m) = \mathbb{Z}$$

$$(\lambda, \chi) \longmapsto \chi \circ \lambda$$

is a perfect pairing; this means that

$$X^*(T) \xrightarrow{\sim} \mathrm{Hom}(X_*(T), \mathbb{Z}) \quad \text{by } \chi \mapsto \langle -, \chi \rangle$$

$$X_*(T) \xrightarrow{\sim} \mathrm{Hom}(X^*(T), \mathbb{Z}) \quad \text{by } \lambda \mapsto \langle \lambda, - \rangle.$$

Proof Choose an isomorphism $T \xrightarrow{\sim} \mathbb{G}_m^r$; this induces isomorphisms $X^*(T) \cong \mathrm{Hom}(\mathbb{G}_m^r, \mathbb{G}_m) \cong \mathbb{Z}^r$ and $X_*(T) \cong \mathrm{Hom}(\mathbb{G}_m, \mathbb{G}_m^r) \cong \mathbb{Z}^r$. (Use the lemma.) Taking these as identifications, $\langle -,- \rangle$ becomes the standard pairing $\mathbb{Z}^r \times \mathbb{Z}^r \longrightarrow \mathbb{Z}$ given by $\langle (k_1, \dots, k_r), (l_1, \dots, l_r) \rangle = \sum k_i l_i$. $\boxtimes$

Remark If $f : T_1 \to T_2$ is an algebraic homomorphism between tori, we get induced homomorphisms of groups

$$X_*(f) : X_*(T_1) \longrightarrow X_*(T_2)$$

$$X^*(f) : X^*(T_2) \longrightarrow X^*(T_1) \qquad \text{note the order.}$$

Remark If $\chi_1, \dots, \chi_r$ is a $\mathbb{Z}$-basis for $X^*(T)$, the homom. $T \longrightarrow \mathbb{G}_m^r$ given by $t \longmapsto (\chi_1(t), \dots, \chi_r(t))$ is an $\cong$. Conversely, if $\varphi : T \xrightarrow{\sim} \mathbb{G}_m^r$ is an isomorphism, the characters $\chi_i = pr_i \circ \varphi$ (with $pr_i : \mathbb{G}_m^r \longrightarrow \mathbb{G}_m$ the projections) form a basis of $X^*(T)$.

(Formally we could even write $T = \mathbb{G}_m^{X^*(T)} \dots$)

Theorem Let $\rho : T \longrightarrow GL(V)$ be an algebraic representation. For $\chi \in X^*(T)$, define

$$V_\chi = \{ v \in V \mid \rho_t(v) = \chi(t) \cdot v \text{ for all } t \in T \}.$$

Then

$$V = \bigoplus_{\chi \in X^*(T)} V_\chi .$$

Proof Step 1: the special case $T = \mathbb{G}_m$. In this case, we identify $X^*(T) = \mathbb{Z}$ and for $n \in \mathbb{Z}$ we let

$$V_n = \{ v \in V \mid \rho_t(v) = t^n \cdot v \text{ for all } t \in \mathbb{G}_m = \mathbb{C}^* \}.$$

The claim is that $V = \bigoplus_{n\in\mathbb{Z}} V_n$.

To see this, write $\mathcal{O}(\mathbb{G}_m) = \mathbb{C}[x,x^{-1}] = \bigoplus_{n\in\mathbb{Z}} \mathbb{C}\cdot x^n$. Also, choose a basis $\mathbb{C}^d \xrightarrow{\sim} V$, which gives an identification $GL(V) \xrightarrow{\sim} GL_d(\mathbb{C})$, and write $\mathcal{O}(GL_d(\mathbb{C})) = \mathbb{C}[y_{ij}, \frac{1}{\det}]_{1\le i,j\le d}$. Let $\rho^*: \mathbb{C}[y_{ij}, \frac{1}{\det}] \longrightarrow \mathbb{C}[x,x^{-1}]$ be the homomorphism induced by ρ, and write

$$\rho^*(y_{ij}) = \sum_{n\in\mathbb{Z}} \gamma_{ij}^{(n)} \cdot x^n \qquad \text{with} \quad \gamma_{ij}^{(n)} \in \mathbb{C}$$

$$(\gamma_{ij}^{(n)} = 0 \text{ for almost all } n).$$

Let $\Gamma^{(n)}$ be the $d\times d$ matrix with coeff. $\gamma_{ij}^{(n)}$; so $\Gamma^{(n)} \in M_d(\mathbb{C}) = \mathrm{End}(\mathbb{C}^d)$. Now $\rho: \mathbb{G}_m \longrightarrow GL_d(\mathbb{C})$ is given by

$$\rho(\alpha) = \Big(\sum_{n\in\mathbb{Z}} \gamma_{ij}^{(n)}\cdot\alpha^n\Big)_{i,j} = \sum_{n\in\mathbb{Z}} \Gamma^{(n)}\cdot\alpha^n.$$

Then $\rho(\alpha\beta) = \rho(\alpha)\circ\rho(\beta)$ gives:

$$\Big(\sum_{p\in\mathbb{Z}} \Gamma^{(p)}\cdot\alpha^p\Big)\circ\Big(\sum_{q\in\mathbb{Z}} \Gamma^{(q)}\cdot\beta^q\Big) = \sum_{n\in\mathbb{Z}} \Gamma^{(n)}\cdot\alpha^n\beta^n$$

for all $\alpha,\beta \in \mathbb{C}^*$. This implies:

$$\Gamma^{(p)}\circ\Gamma^{(q)} = \begin{cases} 0 & \text{if } p\neq q \\ \Gamma^{(n)} & \text{if } p=q=n. \end{cases}$$

Moreover, $\rho(1) = \mathbb{1}$ gives that $\sum_{n\in\mathbb{Z}} \Gamma^{(n)} = \mathbb{1}$. So the $\Gamma^{(n)}$ (which are almost all 0) form an orthogonal collection of idempotents in the ring $\mathrm{End}(\mathbb{C}^d)$. Taking $V_n = \mathrm{Im}(\Gamma^{(n)})$ we get a decomposition $V = \mathbb{C}^d = \bigoplus_{n\in\mathbb{Z}} V_n$ such that $\Gamma^{(n)}$ = projection onto V_n, and we find:

$$\rho(\alpha) = \text{multiplication by } \alpha^n \text{ on } V_n.$$

Step 2: The general case.

Use induction on $r = \mathrm{rank}(T)$. If $r > 1$, fix an isomorphism $T \xrightarrow{\sim} \mathbb{G}_m^r$, let $T' = \mathbb{G}_m^{r-1} \times \{1\}$ and $T'' = \{1\} \times \mathbb{G}_m \cong \mathbb{G}_m$, so that $T = T' \times T''$. On character groups: $X^*(T) = X^*(T') \oplus X^*(T'')$ with $X^*(T'') = \mathbb{Z}$. By induction:

$$V = \bigoplus_{\psi \in X^*(T')} V_\psi \quad \text{where}$$

$V_\psi = \{ v \in V \mid \rho_{t'}(v) = \psi(t') \cdot v \text{ for all } t' \in T' \}$. <u>Note</u>: the action of T'' on V commutes with the action of T' and therefore maps each V_ψ into itself. So for each ψ we get a further decomposition $V_\psi = \bigoplus_{n \in \mathbb{Z}} V_{\psi, n}$. In total this gives the desired decomposition

$$V = \bigoplus_{\psi \in X^*(T')} \bigoplus_{n \in \mathbb{Z}} V_{\psi, n} = \bigoplus_{\chi \in X^*(T)} V_\chi . \qquad \boxtimes$$

Of course, every character $\chi \in X^*(T) = \mathrm{Hom}(T, GL_1(\mathbb{C}))$ can be viewed as a 1-dim'l repr. of T, automatically irreducible. If $d(\chi) = \dim(V_\chi)$ then the choice of a basis for V_χ gives an isomorphism of representations $V_\chi \cong \chi^{\oplus d(\chi)}$. Hence:

<u>Corollary</u> Every (algebraic) repr. of a torus is semisimple.

The theorem nicely illustrates the concept of an <u>isotypic decomposition</u>:

If G is any alg. group, define

$$\mathrm{Irrep}(G) = \left\{ \begin{array}{l} \cong \text{ classes of irreducible (finite dim'l} \\ \text{algebraic) repr. of } G \end{array} \right\}$$

E.g.: $\mathrm{Irrep}(T) \cong X^*(T)$.

In practice we choose a representative for each class in $\mathrm{Irrep}(G)$; so if we write $\chi \in \mathrm{Irrep}(G)$ we think of χ as being an actual repr. of G.

If $\rho : G \longrightarrow GL(V)$ is a semisimple repr. then $V \cong \oplus$ of irreducible repr.; but such a decomposition is in general not unique or canonical. Example: if $G = \{1\}$ then such a decomposition is the same as writing V as a direct sum of lines.

For $\chi \in \mathrm{Irrep}(G)$, define

$$V_\chi = \text{Span of all } \underline{\text{irreducible}} \text{ sub-repr. } W \subseteq V \text{ with } W \cong \chi .$$

We call V_χ the <u>χ-isotypic component</u> of V.

<u>Proposition</u> Let $\rho : G \longrightarrow GL(V)$ be a semisimple repr. Then

$$V = \bigoplus_{\chi \in \mathrm{Irrep}(G)} V_\chi$$

and for each χ : $V_\chi \cong$ direct sum of copies of χ.
(Of course $V_\chi = 0$ for almost all χ.)

<u>Proof</u> Let A be the collection of all irreducible submodules of V, and for $\alpha \in A$ denote the corresponding irr. submod. by $W_\alpha \subseteq V$. For $\chi \in \mathrm{Irrep}(G)$, let $A(\chi) \subset A$ be the subset of all α for which $W_\alpha \cong \chi$. We have $A = \coprod_{\chi \in \mathrm{Irrep}(G)} A(\chi)$.

Because V is semisimple, there is a subset $B \subseteq A$ such that $V = \bigoplus_{\beta \in B} W_\beta$.
Let $B(\chi) = B \cap A(\chi)$, and $W_\chi = \bigoplus_{\beta \in B(\chi)} W_\beta$.

If $\alpha \in A(\chi)$, consider the compositions

$$\pi_\beta : W_\alpha \hookrightarrow V = \bigoplus_{\beta \in B} W_\beta \xrightarrow{pr_\beta} W_\beta$$

with pr_β the projection onto the component W_β.

Then $\pi_\beta = 0$ for $\beta \notin B(\chi)$; hence $W_\alpha \subseteq W_\chi = \bigoplus_{\beta \in B(\chi)} W_\beta$. It follows that $V_\chi = W_\chi$, which is a direct sum of copies of χ . Finally, it is clear that $V = \bigoplus_\chi W_\chi = \bigoplus_\chi V_\chi$. $\boxtimes$

Remarks (1) In the proof we use : if $\psi \neq \chi \in \mathrm{Irrep}(G)$ then $\mathrm{Hom}(\chi, \psi) = 0$

(2) The isotypic decomposition is canonical : if $h : V \longrightarrow W$ is a G-equivariant homomorphism of semisimple repr. then for any $\chi \in \mathrm{Irrep}(G)$, h restricts to a map $h_\chi : V_\chi \longrightarrow W_\chi$.