Representations of algebraic groups

Lecture 5, Oct. 4, 2016

<u>Today</u>: the Lie algebra of an algebraic group

Idea: G an algebraic group $\rightsquigarrow$ $\mathfrak{g} = \mathrm{Lie}(G)$

As a vector space: $\mathfrak{g}$ = tangent space of G at the origin $1 = e \in G$.

Group structure on G $\rightsquigarrow$ Lie bracket $[\,,\,]$ on $\mathfrak{g}$; this makes $\mathfrak{g}$ into a new kind of algebraic structure, called a Lie algebra.

The construction $(G, \text{gp. str.}) \rightsquigarrow (\mathfrak{g}, [\,,\,])$ can be viewed as a "linearization": we attempt to approximate information about G by linear algebra data.

<u>Situation</u> G an algebraic group, $g \in G$.

We want to define the tangent space $T_g G$ of G at g. Three algebraic ways to define this:

(a) Consider
$$ev_g : \mathcal{O}(G) \longrightarrow \mathbb{C}, \quad f \longmapsto f(g)$$
and let $m(g) = \mathrm{Ker}(ev_g)$.

Then $m(g)$ is a maximal ideal, $\mathcal{O}(G)/m(g) \cong \mathbb{C}$. Inclusion of ideals $m(g) \supset m(g)^2$ = ideal of $\mathcal{O}(G)$ generated by all $f_1 \cdot f_2$ with $f_1, f_2 \in m(g)$.

<u>Note</u>: $m(g)/m(g)^2$ is a vector space over $\mathbb{C}$.

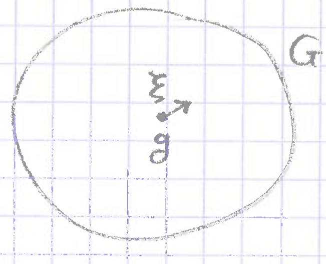
<u>Def</u> $T_g G := \left(m(g)/m(g)^2\right)^\vee$ dual $\mathbb{C}$-vector space

(b) Consider $\mathbb{C}[\varepsilon]$ = ring of dual numbers, $\varepsilon^2 = 0$. Then

$$T_g G = \left\{ \mathbb{C}\text{-algebra homomorphisms } \tau : \mathcal{O}(G) \longrightarrow \mathbb{C}[\varepsilon] \text{ such that } \tau \bmod \varepsilon = ev_g \right\}$$

(c) $T_g G := \{ \mathbb{C}$-linear maps $D : \mathcal{O}(G) \to \mathbb{C}$ with the property that $D(f_1 \cdot f_2) = f_1(g) \cdot D(f_2) + f_2(g) D(f_1)$ for all $f_1, f_2 \in \mathcal{O}(G)$ $\}$.

Intuition : If we think of a tangent vector $\xi \in T_g G$ as an "infinitesimal vector" then the corresponding $D: \mathcal{O}(G) \to \mathbb{C}$ sends a function f to its derivative in the direction ξ.



Explanation of why the three given definitions describe the same space :

- A homomorphism τ as in (b) is of the form $\tau(f) = f(g) + D(f)\cdot \varepsilon$ for some $\mathbb{C}$-linear map $D: \mathcal{O}(G) \to \mathbb{C}$. Saying that $\tau(f_1 f_2) = \tau(f_1)\cdot\tau(f_2)$ then means precisely that D satisfies $D(f_1 f_2) = f_1(g)\cdot D(f_2) + f_2(g)\cdot D(f_1)$. This gives the link between (b) and (c).

- Given D as in (c), it restricts to a $\mathbb{C}$-linear map $m(g) \to \mathbb{C}$ that vanishes on $m(g)^2 \subset m(g)$ and therefore factors via a $\mathbb{C}$-linear map $m(g)/m(g)^2 \to \mathbb{C}$. So D gives an element of $(m(g)/m(g)^2)^\vee$.

- Conversely, if we have $\delta: m(g)/m(g)^2 \to \mathbb{C}$, define a map $D: \mathcal{O}(G) \to \mathbb{C}$ by $D(f) = \delta(\overline{f - f(g)})$. (Note that $f - f(g) \in m(g)$.) One easily checks that $D(f_1 f_2) = f_1(g)\, D(f_2) + f_2(g)\cdot D(f_1)$. This explains how to link (a) and (c).

Induced maps on tangent spaces :

Let $\rho: G_1 \to G_2$ be an algebraic map (not necessarily a homom.) ; $g_1 \in G_1$ and $g_2 := \rho(g_1) \in G_2$. Then ρ induces a $\mathbb{C}$-linear map

$$T\rho : T_{g_1} G_1 \longrightarrow T_{g_2} G_2 .$$

Again we have three descriptions :

(a) $\rho^* : \mathcal{O}(G_2) \to \mathcal{O}(G_1)$ maps $m(g_2)$ into $m(g_1)$. Hence we get an induced linear map $m(g_2)/m(g_2)^2 \longrightarrow m(g_1)/m(g_1)^2$ and $T\rho$ is by definition the dual of this map.

(b) $T\rho$ is the map $\tau \mapsto \tau \circ \rho^*$.

(c) $T\rho$ is the map $D \longmapsto D \circ \rho^*$.

Algebraic groups are homogeneous: For $g \in G$, define $L_g : G \to G$ by $x \longmapsto g \cdot x$. (L_g = left multiplication by g.) Then $L_e = id$ and $L_h \circ L_g = L_{hg}$. It follows that each L_g is bijective, with inverse $L_{g^{-1}}$. Therefore the induced map on tangent spaces

$$TL_g : T_eG \xrightarrow{\sim} T_gG \quad \text{is an isomorphism.}$$

Via these isomorphisms we can reduce the study of T_gG to the case $g = e$.

Example $G = GL_n(\mathbb{C})$ with $\mathcal{O}(G) = \mathbb{C}[x_{ij}, \frac{1}{\det}]$.

Claim: $T_eG \cong M_n(\mathbb{C})$ the vector space of all $n \times n$ matrices

Informal explanation: T_eG is the space of all invertible matrices in $GL_n(\mathbb{C}[\varepsilon])$ of the form $\mathbb{1} + \varepsilon \cdot M$. But $\mathbb{1} + \varepsilon \cdot M$ is invertible for every $n \times n$ matrix M, since $(\mathbb{1} + \varepsilon \cdot M)(\mathbb{1} - \varepsilon \cdot M) = \mathbb{1}$.

Using (b) as definition: $ev_{\mathbb{1}} : \mathcal{O}(G) \to \mathbb{C}$ is given by $ev_{\mathbb{1}}(x_{ij}) = \delta_{ij}$ (Kronecker δ), and then $ev_{\mathbb{1}}(\det) = 1$. If $\tau : \mathcal{O}(G) \to \mathbb{C}[\varepsilon]$ is a homomorphism with $\tau \bmod \varepsilon = ev_{\mathbb{1}}$ then we obtain an $n \times n$ matrix $M = (M_{ij})$ by the rule that $\tau(x_{ij}) = \delta_{ij} + M_{ij}\,\varepsilon$. Conversely, if M is an $n \times n$ matrix, we may define $\tau : \mathcal{O}(G) \to \mathbb{C}[\varepsilon]$ by this rule. Note that $\tau(\det) = 1 + a \cdot \varepsilon$ for some $a \in \mathbb{C}$; hence $\tau(\det) \in \mathbb{C}[\varepsilon]^*$ and we can set $\tau(\det^{-1}) = (1 + a\varepsilon)^{-1} = 1 - a\varepsilon$.

More intrinsic version of the same :

If $G = GL(V)$ for some vector space V then $T_e G = \mathrm{End}(V)$.

(To justify this, one needs to check the following : the choice of a basis $\mathbb{C}^n \xrightarrow{\sim} V$ induces isomorphisms $GL(V) \xrightarrow{\sim} GL_n(\mathbb{C})$ and $\mathrm{End}(V) \xrightarrow{\sim} M_n(\mathbb{C})$. The induced isomorphism

$$T_e GL(V) \xrightarrow{\sim} T_e GL_n(\mathbb{C}) = M_n(\mathbb{C}) \xrightarrow{\sim} \mathrm{End}(V)$$

is independent of the chosen basis.)

From now on, if G is an algebraic group, we write

$$\mathfrak{g} = \mathrm{Lie}(G) := T_e G .$$

The group G acts on itself by inner automorphisms : for $g \in G$, define $\mathrm{Inn}(g) : G \longrightarrow G$ by $x \longmapsto g x g^{-1}$. Of course, $e \longmapsto e$, so we get an induced map on tangent spaces

$$\mathrm{Ad}(g) = T_e \mathrm{Inn}(g) : \mathfrak{g} \longrightarrow \mathfrak{g} .$$

Clear : • $\mathrm{Inn}(e) = \mathrm{id}_G$ so $\mathrm{Ad}(e) = \mathrm{id}_{\mathfrak{g}}$.

• $\mathrm{Inn}(hg) = \mathrm{Inn}(h) \circ \mathrm{Inn}(g)$ so $\mathrm{Ad}(hg) = \mathrm{Ad}(h) \circ \mathrm{Ad}(g)$.

It follows that each $\mathrm{Ad}(g)$ is invertible, with $\mathrm{Ad}(g)^{-1} = \mathrm{Ad}(g^{-1})$, and that

$$\mathrm{Ad} : G \longrightarrow GL(\mathfrak{g})$$

is a homomorphism. Define

$$\mathrm{ad} : \mathfrak{g} \longrightarrow \mathrm{End}(\mathfrak{g})$$

to be the induced map on tangent spaces, and for $X, Y \in \mathfrak{g}$ define

$$[X, Y] := \mathrm{ad}(X)(Y) .$$

The map $[\ ,\ ] : \mathfrak{g} \times \mathfrak{g} \longrightarrow \mathfrak{g}$ is called the Lie-bracket on $\mathfrak{g}$.

<u>Example</u> $G = GL(V)$; $\mathfrak{g} = \mathrm{End}(V)$

For $A \in GL(V)$ the map $\mathrm{Ad}(A) : \mathfrak{g} \longrightarrow \mathfrak{g}$ is given by
$$Y \longmapsto A \cdot Y \cdot A^{-1} .$$

So $\mathrm{Ad} : G \longrightarrow GL(\mathfrak{g})$ is given by $A \longmapsto \mathrm{Inn}(A)$, where we now view $\mathrm{Inn}(A)$ as a map on all of $\mathrm{End}(V)$. Then we find that

$$\mathrm{ad} : \mathfrak{g} \longmapsto \mathrm{End}(\mathfrak{g}) \text{ is given by } X \mapsto (Y \mapsto X \cdot Y - Y \cdot X).$$

Indeed : working with matrices with coefficients in $\mathbb{C}[\varepsilon]$ we find :

- if $A = \mathbb{1} + \varepsilon \cdot X$ then $A^{-1} = \mathbb{1} - \varepsilon X$
- hence $\mathrm{Inn}(A) : \mathrm{End}(V) \longrightarrow \mathrm{End}(V)$ is the map given by

$$\begin{aligned} Y \longmapsto & (\mathbb{1} + \varepsilon \cdot X) \cdot Y \cdot (\mathbb{1} - \varepsilon \cdot X) \\ = & \mathbb{1} + \varepsilon \cdot XY - \varepsilon \cdot YX \qquad (\varepsilon^2 = 0 \,!) \\ = & \mathbb{1} + \varepsilon \cdot (XY - YX) \end{aligned}$$

Conclusion : the Lie bracket on $\mathfrak{g} = \mathrm{End}(V)$ is given by $[X,Y] = X \cdot Y - Y \cdot X$.

<u>Definition</u> A Lie algebra (over $\mathbb{C}$!) is a $\mathbb{C}$-vector space L together with a $\mathbb{C}$-bilinear map $[\, , \,] : L \times L \longrightarrow L$ such that for all $X, Y, Z \in L$:

- $[Y, X] = -[X, Y]$
- Jacobi identity : $[X,[Y,Z]] + [Y,[Z,X]] + [Z,[X,Y]] = 0$

<u>Example</u> Let A be an associative $\mathbb{C}$-algebra. Then the bracket $[\, , \,] : A \times A \longrightarrow A$ defined by $[X,Y] = X \cdot Y - Y \cdot X$ makes A into a complex Lie algebra. The Jacobi identity is verified by simply writing it out.

If $(L, [\,,\,])$ is a Lie algebra, there is an obvious notion of a Lie-subalgebra: consider subspaces $L' \subset L$ with the property that $[X, Y] \in L'$ for all $X, Y \in L'$. Note that in this case $(L', [\,,\,])$ is again a Lie algebra.

<u>Theorem</u> Let G be an algebraic group. Then $\mathfrak{g} = \mathrm{Lie}(G)$ with the bracket defined by $[X, Y] = \mathrm{ad}(X)(Y)$ is a Lie algebra.

<u>Proof</u> For $G = GL(V)$ this follows from the previous example, as in this case $\mathfrak{g}$ is the Lie algebra given by the algebra $\mathrm{End}(V)$. It therefore suffices to show the following general assertion:

> If G_1 is an algebraic subgroup of G_2 then $\mathfrak{g}_1 \subset \mathfrak{g}_2$ is a Lie subalgebra and the restriction of $[\,,\,]_2 : \mathfrak{g}_2 \times \mathfrak{g}_2 \longrightarrow \mathfrak{g}_2$ to $\mathfrak{g}_1$ is the same as the Lie bracket $[\,,\,]_1 : \mathfrak{g}_1 \times \mathfrak{g}_1 \longrightarrow \mathfrak{g}_2$

To see this, let $H \subset GL(\mathfrak{g}_2)$ be the subgroup of those linear automorphisms that map $\mathfrak{g}_1 \subset \mathfrak{g}_2$ into itself, and note that we have a natural homomorphism $\pi : H \longrightarrow GL(\mathfrak{g}_1)$. For $g_1 \in G_1$ we have $\mathrm{Ad}_{G_1}(g_1) : \mathfrak{g}_1 \to \mathfrak{g}_1$ but also $\mathrm{Ad}_{G_2}(g_1) : \mathfrak{g}_2 \longrightarrow \mathfrak{g}_2$.
It is clear that $\mathrm{Ad}_{G_2}(g_1)$ lies in H and that $\pi(\mathrm{Ad}_{G_2}(g_1)) = \mathrm{Ad}_{G_1}(g_1)$.
So we have a commutative diagram

$$\begin{tikzcd}
G_1 \arrow[r, hook] \arrow[d] \arrow[dd, bend right, "\mathrm{Ad}_{G_1}"'] & G_2 \arrow[d, "\mathrm{Ad}_{G_2}"] \\
H \arrow[r, hook] \arrow[d, "\pi"] & GL(\mathfrak{g}_2) \\
GL(\mathfrak{g}_1) &
\end{tikzcd}$$

Taking derivatives at the origin this induces a commutative diagram

$$\begin{tikzcd}
\mathfrak{g}_1 \arrow[r, hook] \arrow[d] \arrow[dd, bend right=60, "\mathrm{ad}_{G_1}"'] & \mathfrak{g}_2 \arrow[d, "\mathrm{ad}_{G_2}"] \\
T_e H \arrow[r, hook] \arrow[d, "T\pi"] & \mathrm{End}(\mathfrak{g}_2) \\
\mathrm{End}(\mathfrak{g}_1) &
\end{tikzcd}$$

where $T_e H = \{ \alpha \in \mathrm{End}(\mathfrak{g}_2) \mid \alpha(\mathfrak{g}_1) \subseteq \mathfrak{g}_1 \}$ and $T\pi$ sends α to $\alpha|_{\mathfrak{g}_1}$. The claim readily follows from this. $\boxtimes$

The Lie algebra of an algebraic group $G \subset GL_n(\mathbb{C})$.

From now on we write $\mathfrak{gl}_n(\mathbb{C})$ for the Lie algebra of $GL_n(\mathbb{C})$; so: $\mathfrak{gl}_n(\mathbb{C}) = M_n(\mathbb{C})$ with Lie bracket $[A,B] = A\cdot B - B\cdot A$.

Recall: if we describe the tangent space at $\mathbb{1} \in GL_n(\mathbb{C})$ as the space of homomorphisms $\tau : \underset{\mathbb{C}[x_{ij}, \frac{1}{\det}]}{\mathcal{O}(GL_n(\mathbb{C}))} \longrightarrow \mathbb{C}[\varepsilon]$ with $(\tau \bmod \varepsilon) = ev_{\mathbb{1}}$

then the identification with $M_n(\mathbb{C})$ sends τ to the $n\times n$ matrix A that is given by the rule $\tau(x_{ij}) = \delta_{ij} + \varepsilon \cdot A_{ij}$

$= (i,j)$-coefficient of the matrix $\mathbb{1} + \varepsilon A$.

Note (important): for any $f \in \mathbb{C}[x_{ij}, \frac{1}{\det}]$ and any $B \in GL_n(\mathbb{C}[\varepsilon])$ we can talk about $f(B)$, and the homomorphism τ is simply given by $\tau(f) = f(\mathbb{1} + \varepsilon \cdot A)$.

Lemma For $f \in \mathbb{C}[x_{ij}, \frac{1}{\det}]$ we have

$$\tau(f) = f(\mathbb{1} + \varepsilon\cdot A) = f(\mathbb{1}) + \varepsilon \cdot \left(\sum_{i,j=1}^{n} \frac{\partial f}{\partial x_{ij}}(\mathbb{1}) \cdot A_{ij} \right)$$

(This is just a 1^{st} order Taylor expansion!)

Now let $G \subset GL_n(\mathbb{C})$ be an algebraic subgroup, $\mathfrak{g} \subset \mathfrak{gl}_n(\mathbb{C})$ its Lie algebra. We have $\mathcal{O}(G) = \mathcal{O}(GL_n(\mathbb{C}))/I_G$, where $I_G = \{ f \in \mathcal{O}(GL_n(\mathbb{C})) \mid f|_G = 0 \}$. It follows that

$$\begin{aligned}
\mathfrak{g} &= \{ \tau : \mathcal{O}(G) \to \mathbb{C}[\varepsilon] \mid (\tau \bmod \varepsilon) = ev_{\mathbb{1}} \} \\
&= \{ \tau : \mathcal{O}(GL_n(\mathbb{C})) \to \mathbb{C}[\varepsilon] \mid (\tau \bmod \varepsilon) = ev_{\mathbb{1}} \text{ and } \tau(f) = 0 \text{ for all } f \in I_G \} \\
&\cong \{ A \in \mathfrak{gl}_n(\mathbb{C}) \mid f(\mathbb{1} + \varepsilon \cdot A) = 0 \text{ for all } f \in I_G \} \\
&= \{ A \in \mathfrak{gl}_n(\mathbb{C}) \mid \sum_{i,j} \frac{\partial f}{\partial x_{ij}}(\mathbb{1}) \cdot A_{ij} = 0 \text{ for all } f \in I_G \}.
\end{aligned}$$

Example : $SL_n(\mathbb{C}) \subset GL_n(\mathbb{C})$ is given by the single equation "$\det = 1$". For $i, j \in \{1, \dots, n\}$ we have

$$\frac{\partial \det}{\partial x_{ij}}(\mathbb{1}) = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{else} \end{cases}$$

(check this!). Hence: $\mathfrak{sl}_n(\mathbb{C}) \subset \mathfrak{gl}_n(\mathbb{C})$ is the Lie subalgebra given by

$$\mathfrak{sl}_n(\mathbb{C}) = \{ A \in \mathfrak{gl}_n(\mathbb{C}) = M_n(\mathbb{C}) \mid \operatorname{trace}(A) = 0 \}.$$