Representations of algebraic groups
Lecture 6, Oct 11, 2016

Today: Representations of SL_2 and $\mathfrak{sl}_2$.

<u>Definition</u> A representation of a Lie algebra $\mathfrak{g}$ on a vector space V is a homomorphism of Lie algebras $\rho: \mathfrak{g} \to \mathfrak{gl}(V)$ $(= \mathrm{End}(V))$.
Concretely this means that for every $X \in \mathfrak{g}$ we have an endomorphism ρ_X of V and $\rho_{[X,Y]} = [\rho_X, \rho_Y]$ for all $X, Y \in \mathfrak{g}$. $(\rho_X = \rho(X))$

<u>Example</u> A representation $r: G \to GL(V)$ gives rise to a representation $\rho = T r: \mathfrak{g} \to \mathfrak{gl}(V)$.

There are some basic notions and constructions that are very similar to what we discussed in the context of repr. of groups:

- $W \subseteq V$ is stable under $\mathfrak{g}$, or a $\mathfrak{g}$-submodule, if $\rho_X(w) \in W$ for all $w \in W$. In this case ρ induces representations $\rho|_W : \mathfrak{g} \to \mathfrak{gl}(W)$ and $\bar{\rho}: \mathfrak{g} \to \mathfrak{gl}(\bar{V})$, where $\bar{V} = V/W$.
- $\rho^\vee : \mathfrak{g} \to \mathfrak{gl}(V^\vee)$ is given by $\rho^\vee(X) = \rho(-X)^\vee$.
- If $\rho_1 : \mathfrak{g} \to \mathfrak{gl}(V_1)$ and $\rho_2 : \mathfrak{g} \to \mathfrak{gl}(V_2)$ are repr. then we get $(\rho_1 \oplus \rho_2): \mathfrak{g} \to \mathfrak{gl}(V_1 \oplus V_2)$ and $\rho_1 \otimes \rho_2 : \mathfrak{g} \to \mathfrak{gl}(V_1 \otimes V_2)$.

  Caution: $(\rho_1 \otimes \rho_2)(X) = \rho_1(X) \otimes \mathrm{id} + \mathrm{id} \otimes \rho_2(X)$
  (for group representations we had $(\rho_1 \otimes \rho_2)(g) = \rho_1(g) \otimes \rho_2(g)$)

  Also we get induced repr. $\mathrm{Sym}^i(\rho): \mathfrak{g} \to \mathfrak{gl}(\mathrm{Sym}^i(V))$
  $\Lambda^i(\rho): \mathfrak{g} \to \mathfrak{gl}(\Lambda^i(V))$

  Make sure you understand how these are given!

As before: a repr. ρ is irreducible if ...

semisimple if ... .

Consider $\mathfrak{sl}_2$ = matrices $A \in M_2(\mathbb{C})$ with trace$(A) = 0$

$= \mathbb{C}\cdot Y \oplus \mathbb{C}\cdot H \oplus \mathbb{C}\cdot X$

where $Y = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$ $\quad H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ $\quad X = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$

Commutation relations:

- $[A, A] = 0$ for every $A \in \mathfrak{sl}_2$
- $[X, H] = 2X$, $\quad [Y, H] = -2Y$, $\quad [X, Y] = H$.

Memorize these relations! As we will see, they are crucial for the calculations that follow.

Consider any representation $\rho : \mathfrak{sl}_2 \to \mathfrak{gl}(V)$. We will simply write Y, H and X for ρ_Y, ρ_H and ρ_X — this will not lead to confusion.

<u>Lemma</u> If $v \in V$ is an eigenvector of H with $H(v) = \alpha \cdot v$ then $X(v)$ is an eigenvector of H with eigenvalue $\alpha + 2$

$Y(v)$ ——— " ——— $\alpha - 2$.

<u>Proof</u> $[X, H] = 2X$ means that $H \circ X = X \circ (H + 2)$; similarly for Y.

We will (for today) admit one fact without proof:

<u>Fact</u>: The endomorphism $H \in \mathrm{End}(V)$ is semisimple (= diagonalisable).

We will see in the next weeks where this comes from. We could do all proofs without using this fact, but this would only lead to more involved calculations that are not very enlightening.

The semisimplicity of H means that we have an eigenspace decomposition

$$V = \bigoplus_{\alpha \in \mathbb{C}} V_\alpha$$

with $H|_{V_\alpha}$ = multiplication by α.

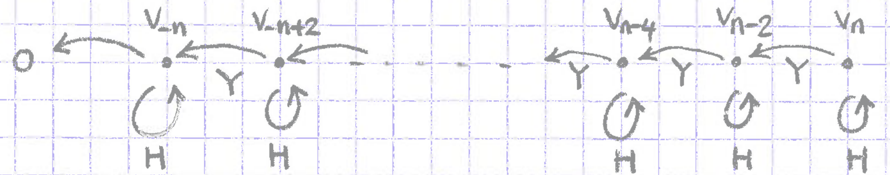
Of course, $V_\alpha = 0$ for almost all α.

As we have seen, if $0 \neq v \in V_\alpha$ then $X^k(v) \in V_{\alpha+2k}$ for all $k \geqslant 0$; so there is some $k \geqslant 0$ with $X^k(v) \neq 0$ in $V_{\alpha+2k}$ and $X^{k+1}(v) = 0$.
We start by analysing what happens to such vectors under the action of sl_2.

Definition $\quad V_\alpha^{max} = \{ v \in V_\alpha \mid X(v) = 0 \}$.

Suppose $0 \neq v \in V_\alpha^{max}$ for some $\alpha \in \mathbb{C}$. Then $Y^k(v) \in V_{\alpha-2k}$; let n be the largest integer $\geqslant 0$ for which $Y^n(v) \neq 0$. (So $Y^{n+1}(v) = 0$.)

We now define
$$v_n = v \;;$$
$$v_{n-2} = Y(v)\;;$$
$$v_{n-4} = Y^2(v)\;; \qquad \text{in general}: \quad v_{n-2k} = Y^k(v)$$
(We will see the logic behind this numbering scheme!) By construction, $v_n, v_{n-2}, v_{n-4}, \ldots, v_{-n}$ are all $\neq 0$; $v_{-n-2\ell} = 0$ if $\ell > 0$.
Also: $\quad v_{n-2k} \in V_{\alpha-2k}$.



Now we see how X acts; for this we use $H = [XY - YX]$:

$$\alpha \cdot v_n = H(v_n) = (XY - YX)(v_n) \underset{\substack{\uparrow \\ X(v_n)=0}}{=} X(v_{n-2})$$

So: $X(v_{n-2}) = \alpha \cdot v_n$, and hence $YX(v_{n-2}) = \alpha \cdot v_{n-2}$.

Next:

$$(\alpha - 2) \cdot v_{n-2} = H(v_{n-2}) = (XY - YX)(v_{n-2}) = X(v_{n-4}) - \alpha \cdot v_{n-2}$$

So: $X(v_{n-4}) = (2\alpha - 2) \cdot v_{n-4}$

By induction: $\boxed{X(v_{n-2k}) = (k\alpha - k(k-1)) \cdot v_{n-2k+2}}$

Now take $k = n+1$ and note that $v_{-n-2} = 0$.
$v_{-n} \neq 0$

We find (!!) $\boxed{\alpha = n}$

Conclusions:

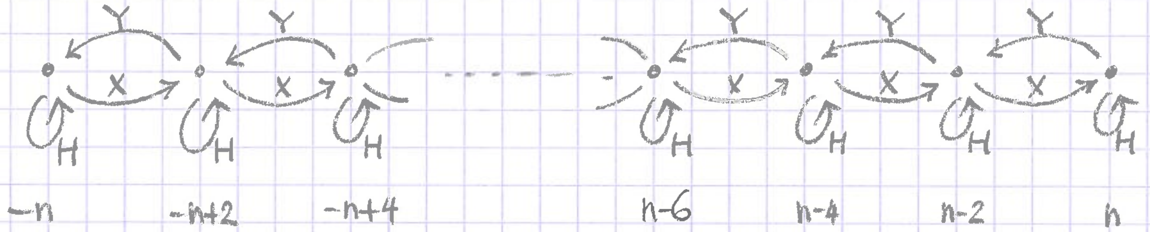
(1) If $V_\alpha \neq 0$ then $\alpha \in \mathbb{Z}$. Indeed, as we have already noted, if $0 \neq u \in V_\alpha$ then $X^k(u) \in V^{max}_{\alpha+2k}$ for some $k \geq 0$, and then the previous calculation shows that $\alpha + 2k \in \mathbb{Z}_{\geq 0}$.

(2) If $0 \neq v \in V_n^{max}$ then

$$\langle v \rangle := \mathbb{C} \cdot v \oplus \mathbb{C} \cdot Yv \oplus \cdots \oplus \mathbb{C} \cdot Y^n v$$

is an $\mathfrak{sl}_2$ - submodule of V, and this repr. is of a particularly simple form: setting $v_{n-2k} := Y^k(v)$ we have:

$$\begin{cases} H(v_{n-2k}) = (n-2k)\cdot v_{n-2k} \\ Y(v_{n-2k}) = v_{n-2k-2} \\ X(v_{n-2k}) = \underbrace{k(n+1-k)}_{\neq 0 \text{ for } k=1,\dots,n}\cdot v_{n-2k+2} \end{cases}$$



In particular we see that $\langle v\rangle$ is irreducible as an $\mathfrak{sl}_2$ − repr : if we set $\langle v\rangle_{n-2k} = \langle v\rangle \cap V_{n-2k}$ then we find that :

- for each $k \in \{0,\dots,n\}$,
  $$\langle v\rangle_{n-2k} = \mathbb{C}\cdot v_{n-2k} \quad \text{is 1-dimensional}$$
  and it generates all of $\langle v\rangle$ as an $\mathfrak{sl}_2$ − module.

Example : Let $r: SL_2(\mathbb{C}) \longrightarrow GL(W)$, with $W=\mathbb{C}^2$, be the standard representation ; $\rho: \mathfrak{sl}_2(\mathbb{C}) \longrightarrow \mathfrak{gl}(W)$ the induced Lie algebra representation. Let

$$e_1 = \begin{pmatrix}1\\0\end{pmatrix} \qquad e_{-1} = \begin{pmatrix}0\\1\end{pmatrix} \qquad \in W.$$

We find : $H(e_i) = i\cdot e_i \qquad i = \pm 1$

$$\begin{tikzcd} e_{-1} \arrow[r, bend right, "X"'] & e_1 \arrow[l, bend right, "Y"'] \end{tikzcd}$$

This is the irreducible repr. as above with $n=1$.

Next consider

$$\mathrm{Sym}^2(\rho) : \mathfrak{sl}_2 \longrightarrow \mathfrak{gl}(\mathrm{Sym}^2 W)$$

basis for $\mathrm{Sym}^2(W)$:

$$\begin{tikzcd}
2e_{-1}^2 \arrow[r, bend right, "X"'] \arrow[loop below, "H=-2"] & 2e_{-1}e_1 \arrow[l, bend right, "Y"'] \arrow[r, bend right, "X"'] \arrow[loop below, "H=0"] & e_1^2 \arrow[l, bend right, "Y"'] \arrow[loop below, "H=2"]
\end{tikzcd}$$

$$Y(e_1^2) = 2 \cdot e_{-1} e_1 \qquad X(2e_{-1}^2) = 4 \cdot e_{-1} e_1$$
$$Y(2e_{-1}e_1) = 2 \cdot e_{-1}^2 \qquad X(2e_{-1}e_1) = 2\, e_1^2$$

This is the irreducible representation that we find in the above when $n=2$.

In general :

Theorem (i) If W is the standard 2-dim'l repr. of $\mathfrak{sl}_2$ then $\mathrm{Sym}^n W$ is irreducible of dimension $n+1$.

(ii) If V is any repr. of $\mathfrak{sl}_2$ and $0 \neq v \in V_n^{max}$ for some n then the irreducible repr. $\langle v \rangle$ spanned by v is isomorphic to $\mathrm{Sym}^n(W)$.

Corollary 1 : The representations $\mathrm{Sym}^n(W)$ for $n \geq 0$ give the full list of all irreducible representations of $\mathfrak{sl}_2$.

Corollary 2 : For any $n \geq 0$, $\mathrm{Sym}^n(W)$ is irreducible as a repr. of $SL_2(\mathbb{C})$.

[As we will see these are all irred. repr. of $SL_2(\mathbb{C})$.]

Theorem Every representation of $\mathfrak{sl}_2$ is semisimple.

Proof Let $\rho: \mathfrak{sl}_2(\mathbb{C}) \longrightarrow \mathfrak{gl}(V)$ be a representation. We have $V = \bigoplus_{m \in \mathbb{Z}} V_m$ with $H|_{V_m}$ = multiplication by m.

We want to show that every $0 \neq v \in V_m$ (for some m) is contained in a sum of irreducible subrepresentations of V.

Define $\ell(v) :=$ smallest $\ell \geqslant 0$ such that $X^{\ell+1}(v) = 0$.

We prove the assertion by induction on $\ell(v)$.

$\underline{\ell(v) = 0}$: This precisely means that $v \in V_m^{max}$, and as we have seen, $\langle v \rangle$ is in this case irreducible.

Induction step : Assume that every $0 \neq w \in V$ with $\ell(w) < \ell(v)$ is contained in a sum of irreducible submodules of V.

Let $\ell = \ell(v)$; we may assume $\ell > 0$. By construction : $w := X^\ell(v)$ is $\neq 0$ but $X(w) = 0$; this just means that $w \in V_{m+2\ell}^{max}$, and hence $\langle w \rangle$ is of the type described before. It follows that $X^\ell Y^\ell(w) = c \cdot w$ for some constant $c \neq 0$. Hence, if we set

$$v' = v - \frac{1}{c} \cdot Y^\ell(w)$$

then $v = v' + \frac{1}{c} \cdot Y^\ell(w)$ and $X^\ell(v') = 0$; so $\ell(v') < \ell$.

Clear : $\frac{1}{c} \cdot Y^\ell(w) \in \langle w \rangle$ and $\langle w \rangle$ is irreducible

Assumption $\Rightarrow$ v' is contained in sum of irreps. $\boxtimes$

[We will see later that the analogous theorem is true for repr. of $SL_2(\mathbb{C})$.]