Representations of Algebraic Groups

Lecture 8 – November 1, 2016

If $\rho : \mathfrak{g} \to \mathfrak{gl}(V)$ is a representation of a Lie algebra $\mathfrak{g}$, define a bilinear form $B_\rho : \mathfrak{g} \times \mathfrak{g} \to \mathbb{C}$ by $B_\rho(X,Y) = \operatorname{trace}(\rho_X \circ \rho_Y)$.
If $\mathfrak{g} \subset \mathfrak{gl}(V)$ then we also write B_V instead of B_ρ.

Special case: for $\rho = \operatorname{ad} : \mathfrak{g} \to \mathfrak{gl}(\mathfrak{g})$ we simply write $B : \mathfrak{g} \times \mathfrak{g} \to \mathbb{C}$ for the associated form; it is called the <u>Killing form</u> of $\mathfrak{g}$. Concretely:
$B(X,Y) = \operatorname{trace}(\operatorname{ad}(X) \circ \operatorname{ad}(Y))$.

<u>Lemma</u> With notation as above, B_ρ is symmetric, and for all $X, Y, Z \in \mathfrak{g}$ we have

$$B_\rho([X,Y], Z) = B_\rho(X, [Y,Z]).$$

Proof For any two $\varphi, \psi \in \operatorname{End}(V)$ we have $\operatorname{trace}(\varphi \circ \psi) = \operatorname{trace}(\psi \circ \varphi)$; this gives the symmetry of B_ρ. Further:

$$\begin{aligned} B_\rho([X,Y],Z) &= \operatorname{trace}(\rho_X\rho_Y\rho_Z - \rho_Y\rho_X\rho_Z) \\ &= \operatorname{trace}(\rho_X\rho_Y\rho_Z) - \operatorname{trace}(\rho_Y\rho_X\rho_Z) \\ &= \operatorname{trace}(\rho_X\rho_Y\rho_Z) - \operatorname{trace}(\rho_X\rho_Z\rho_Y) \\ &= \operatorname{trace}(\rho_X\rho_Y\rho_Z - \rho_X\rho_Z\rho_Y) = B_\rho(X,[Y,Z]). \end{aligned}$$ $\boxtimes$

<u>Corollary</u>: If $\mathfrak{h} \subset \mathfrak{g}$ is an ideal then also

$$\mathfrak{h}^\perp := \{ X \in \mathfrak{g} \mid B(X,Y) = 0 \text{ for all } Y \in \mathfrak{g} \}$$

is an ideal of $\mathfrak{g}$.

Proof: Easy exercise, using the Lemma.

<u>Example</u> Let $\mathfrak{g} = \mathfrak{sl}_2$. The Killing form is given, with respect to the basis $\{Y, H, X\}$, by the matrix $\begin{pmatrix} & & 4 \\ & 8 & \\ 4 & & \end{pmatrix}$. Note that this is a non-degenerate form.

Example Let $\mathfrak{g}$ be a nilpotent Lie algebra. As we have seen: there is a basis $e_1, \dots, e_d$ for $\mathfrak{g}$ under which we get $ad(\mathfrak{g}) \subseteq \mathfrak{n}_d(\mathbb{C})$ (= strictly upper triangular matrices). It follows that $B = 0$.

Theorem Let $\mathfrak{g} \subset \mathfrak{gl}(V)$ be a Lie subalgebra for which $B_V = 0$. Then $\mathfrak{g}$ is solvable.

For the proof we refer to Fulton - Harris, Appendix C.

Corollary 1 (Cartan's criterion) Let $\mathfrak{g}$ be a Lie algebra, $\mathcal{D}\mathfrak{g} = [\mathfrak{g}, \mathfrak{g}]$. Then

$$\mathfrak{g} \text{ is solvable} \iff B(\mathfrak{g}, \mathcal{D}\mathfrak{g}) = 0$$

$$(\text{i.e., } B(X,Y) = 0 \text{ for all } X \in \mathfrak{g}, Y \in \mathcal{D}\mathfrak{g})$$

Proof "$\Rightarrow$" We can choose a basis for $\mathfrak{g}$ under which $ad(\mathfrak{g}) \subseteq \mathfrak{b}_d(\mathbb{C})$. If $Y \in \mathcal{D}\mathfrak{g}$ then $ad(Y) \in \mathfrak{n}_d(\mathbb{C})$; hence $ad(X) \circ ad(Y) \in \mathfrak{n}_d(\mathbb{C})$ and $B(X,Y) = \text{trace}(ad(X) \circ ad(Y)) = 0$

"$\Leftarrow$" Suppose $B(\mathfrak{g}, \mathcal{D}\mathfrak{g}) = 0$. The theorem, applied to $ad(\mathcal{D}\mathfrak{g}) \subset \mathfrak{gl}(\mathfrak{g})$, gives that $ad(\mathcal{D}\mathfrak{g})$ is solvable. Hence $\mathcal{D}\mathfrak{g}$ is solvable. From the definition of solvability it is clear that this implies that $\mathfrak{g}$ is solvable $\boxtimes$

Corollary 2 Let $\mathfrak{g}$ be a Lie algebra. Then

$$\mathfrak{g} \text{ is semisimple} \iff \text{its Killing form } B \text{ is non-degenerate}.$$

Proof "$\Rightarrow$" Suppose $\mathfrak{g}$ is semisimple. Saying that B is non-degen. means that its kernel $\mathfrak{g}^\perp = \{X \in \mathfrak{g} \mid B(X,Y) = 0 \text{ for all } Y \in \mathfrak{g}\}$ is zero. As we have seen, $\mathfrak{g}^\perp$ is an ideal of $\mathfrak{g}$. By Cartan's criterion, $\mathfrak{g}^\perp$ is solvable. As $\mathfrak{g}$ is semisimple it follows that $\mathfrak{g}^\perp = 0$.

"$\Leftarrow$" Suppose $\mathfrak{g}^\perp = 0$. To show that $\mathfrak{g}$ is semisimple, we must show that $\mathfrak{g}$ has no non-zero abelian ideals. If $\mathfrak{a} \subset \mathfrak{g}$ is an abelian ideal then for all $X \in \mathfrak{a}$ and $Y \in \mathfrak{g}$ we have:

$\mathrm{ad}(X) \circ \mathrm{ad}(Y)$ maps $\mathfrak{g}$ into $\mathfrak{a}$ because $\mathfrak{a}$ is an ideal

——— " ——— $\mathfrak{a}$ to 0 because $\mathfrak{a}$ is abelian.

Hence $\mathfrak{a} \subset \mathfrak{g}^\perp$. As $\mathfrak{g}^\perp = 0$, it follows that $\mathfrak{a} = 0$. $\boxtimes$

<u>Definition</u> A Lie algebra $\mathfrak{g} \neq 0$ is <u>simple</u> if $\mathfrak{g}$ is not abelian and $\mathfrak{g}$ has no ideals other than (0) and $\mathfrak{g}$.

<u>Note</u>:
- (0) is not simple, by definition; but (0) <u>is</u> semisimple
- if $\mathfrak{g}$ is simple then $\mathfrak{g}$ is also semisimple

<u>Corollary 3</u> Let $\mathfrak{g}$ be a semisimple Lie algebra. Then there exist ideals $\mathfrak{s}_1, \dots, \mathfrak{s}_r$ of $\mathfrak{g}$ that are themselves simple as Lie algebras, such that $\mathfrak{g} = \mathfrak{s}_1 \times \dots \times \mathfrak{s}_r$.

<u>Proof</u> If $\mathfrak{h} \subset \mathfrak{g}$ is an ideal then so is $\mathfrak{h}^\perp$, and $\mathfrak{h} \cap \mathfrak{h}^\perp$ is solvable by Cartan's criterion. Hence $\mathfrak{h} \cap \mathfrak{h}^\perp = 0$ and $\mathfrak{g} = \mathfrak{h} \times \mathfrak{h}^\perp$. As $\dim(\mathfrak{g}) < \infty$, the Corollary readily follows. $\boxtimes$

<u>Corollary 4</u> Let $\mathfrak{g}$ be a semisimple Lie algebra. Then:

(i) $\mathfrak{g} = \mathcal{D}\mathfrak{g}$

(ii) $Z(\mathfrak{g}) = (0)$ and hence $\mathrm{ad} : \mathfrak{g} \hookrightarrow \mathfrak{gl}(\mathfrak{g})$

(iii) If $\rho : \mathfrak{g} \to \mathfrak{gl}(V)$ is any representation then $\rho(\mathfrak{g}) \subseteq \mathfrak{sl}(V)$.

(iv) Any 1-dimensional repr. of $\mathfrak{g}$ is trivial.

<u>Proof</u> (i) follows from Corollary 3; (ii) follows from the fact that $Z(\mathfrak{g})$ is a solvable ideal; (iii) follows from (i) since $\mathcal{D}(\mathfrak{gl}(V)) \subseteq \mathfrak{sl}(V)$; (iv) follows from (iii) since $\mathfrak{sl}(V) = (0)$ if $\dim(V) = 1$. $\boxtimes$

Example Let $\mathfrak{g} = \mathfrak{sl}_2(\mathbb{C})$. Then $\dim(\mathfrak{g}) = 3$ and the Killing form B is a non-degen symmetric bilinear form. Hence $\mathfrak{so}(\mathfrak{g}, B) \cong \mathfrak{so}_3(\mathbb{C})$. It follows from the relation $B([X,Y],Z) = B(X,[Y,Z])$ that the image of $\mathrm{ad}: \mathfrak{sl}_2 \to \mathfrak{gl}(\mathfrak{g})$ lies in $\mathfrak{so}(\mathfrak{g}, B) = \{$linear maps $\varphi: \mathfrak{g} \to \mathfrak{g} \mid B(\varphi X, Y) + B(X, \varphi Y) = 0$ for all $X, Y\}$. As ad is injective and $\dim(\mathfrak{so}_3) = 3$ it follows that $\mathfrak{sl}_2 \xrightarrow{\sim} \mathfrak{so}_3$.

Remark Let $\mathfrak{g}$ be a simple Lie algebra. We cannot have $\dim(\mathfrak{g}) = 1$, for then $\mathfrak{g}$ would be abelian. We also cannot have $\dim(\mathfrak{g}) = 2$, for if $\{X, Y\}$ is a basis then $\mathcal{D}\mathfrak{g}$ is generated by $[X,Y]$, contradicting $\mathfrak{g} = \mathcal{D}\mathfrak{g}$. So $\dim(\mathfrak{g}) \geq 3$. Further, if $\dim(\mathfrak{g}) = 3$ then the same argument as above shows that $\mathfrak{g} \cong \mathfrak{so}_3 \cong \mathfrak{sl}_2$.

THEOREM (Hermann Weyl) Let $\mathfrak{g}$ be a semisimple Lie algebra. Then any representation $\rho: \mathfrak{g} \longrightarrow \mathfrak{gl}(V)$ is semisimple.

Outline of the proof:

(1) One first proves the following very special case of the theorem:
Let $\rho: \mathfrak{g} \to \mathfrak{gl}(V)$ be a repr., $W \subset V$ a subspace of codimension 1 (so: $\dim(W) = \dim(V) - 1$), and suppose that $X(v) \in W$ for all $X \in \mathfrak{g}$ and $v \in V$. Then there exist a 1-dimensional $\mathfrak{g}$-submodule $L \subset V$ such that $V = W \oplus L$.
(Note: $\mathfrak{g}$ necessarily acts as zero on L, i.e., $X(\lambda) = 0$ for $X \in \mathfrak{g}$ and $\lambda \in L$.)

(2) Assume (1) is proven. Let $\sigma: \mathfrak{g} \to \mathfrak{gl}(M)$ be a repr of $\mathfrak{g}$ and $N \subset M$ a $\mathfrak{g}$-submodule. Our goal is to show that there exists a $\mathfrak{g}$-submodule $N' \subset M$ such that $M = N \oplus N'$.

Consider the representation $\mathrm{Hom}(M, N) = M^\vee \otimes N$ of $\mathfrak{g}$. Concretely: if $X \in \mathfrak{g}$ and $\varphi \in \mathrm{Hom}(M,N)$ then $X(\varphi) \in \mathrm{Hom}(M,N)$ is given by $X(\varphi)(m) = X(\varphi(m)) - \varphi(X(m))$. Define:

$$V = \{ \varphi \in \mathrm{Hom}(M,N) \mid \varphi|_N : N \to N \text{ is a scalar multiple of } \mathrm{id}_N \}$$
$$\cup$$
$$W = \{ \varphi \in \mathrm{Hom}(M,N) \mid \varphi|_N = 0 \}$$

Note: $\dim(W) = \dim(V) - 1$.

If $X \in \mathfrak{g}$ and $\varphi \in V$ then $X(\varphi) \in W$. Indeed, if $\varphi \in V$ then $\varphi|_N = c \cdot \mathrm{id}_N$ for some $c \in \mathbb{C}$, and then
$X(\varphi)(n) = X(\varphi(n)) - \varphi(X(n)) = X(c \cdot n) - c \cdot X(n) = 0$, for all $n \in N$.

In particular, V and W are $\mathfrak{g}$-submodules of $\mathrm{Hom}(M,N)$. By (1) it follows that there is a $\mathfrak{g}$-stable line $L \subset V$ such that $V = W \oplus L$.

Let $\lambda \in L$ be the unique element for which $\lambda|_N = \mathrm{id}_N$. We know that $X(\lambda) = 0$ for all $X \in \mathfrak{g}$; this means that $\lambda(X(m)) = X(\lambda(m))$ for all $m \in M$. In other words: $\lambda : M \to N$ is a homomorphism of repr. of $\mathfrak{g}$. It follows that $N' := \mathrm{Ker}(\lambda)$ is a $\mathfrak{g}$-submodule of M. Finally, $\lambda|_N = \mathrm{id}_N$ implies that $M = N \oplus N'$, and we are done $\boxtimes$

Application: Jordan decomposition in semisimple Lie algebras

Example: Take $\mathfrak{g} = \mathbb{C}$ with $[\,,\,] = 0$. (Not semisimple.) Consider:

$\rho_1 : \mathfrak{g} \to \mathfrak{gl}_1(\mathbb{C})$ given by $t \mapsto (t)$

$\rho_2 : \mathfrak{g} \to \mathfrak{gl}_2(\mathbb{C})$ $\quad t \mapsto \begin{pmatrix} 0 & t \\ 0 & 0 \end{pmatrix}$

$\rho_3 : \mathfrak{g} \to \mathfrak{gl}_2(\mathbb{C})$ $\quad t \mapsto \begin{pmatrix} t & t \\ 0 & 0 \end{pmatrix}$

Then:

- $\rho_1(t)$ is semisimple for all $t \in \mathbb{C}$
- $\rho_2(t)$ is nilpotent for all $t \in \mathbb{C}$
- $\rho_3(t)$ is neither nilpotent nor semisimple for $t \in \mathbb{C} \setminus \{0\}$;

  moreover: $\rho_3(t)_s$ and $\rho_3(t)_n$ are not in the image of ρ_3.

For semisimple Lie algebras the situation is much nicer!

<u>Lemma 1</u>: Let $\varphi \in \mathfrak{gl}(V) = \mathrm{End}(V)$ with Jordan decomposition $\varphi = \varphi_s + \varphi_n$. Then the Jordan decomposition of $\mathrm{ad}_{\mathfrak{gl}(V)}(\varphi) : \mathfrak{gl}(V) \longrightarrow \mathfrak{gl}(V)$, $\psi \longmapsto [\varphi, \psi]$ is given by $\mathrm{ad}_{\mathfrak{gl}(V)}(\varphi) = \mathrm{ad}_{\mathfrak{gl}(V)}(\varphi_s) + \mathrm{ad}_{\mathfrak{gl}(V)}(\varphi_n)$.

<u>Proposition</u> Let $\mathfrak{g} \subset \mathfrak{gl}(V)$ be a semisimple Lie subalgebra. Let $X \in \mathfrak{g}$, and let $X = s + n$ be its Jordan decomposition as an endomorphism of V. (s = semisimple part, n = nilpotent part) Then also $s, n \in \mathfrak{g}$.

Sketch of the proof:

(1) Let $\mathfrak{n} = \{ A \in \mathfrak{gl}(V) \mid [A, \mathfrak{g}] \subseteq \mathfrak{g} \}$; this is called the normalizer of $\mathfrak{g}$ in $\mathfrak{gl}(V)$. Clear: $\mathfrak{g} \subseteq \mathfrak{n}$.

If $W \subseteq V$ is a $\mathfrak{g}$-submodule, define

$$\mathfrak{h}_W := \{ A \in \mathfrak{gl}(V) \mid A(W) \subseteq W \text{ and } \mathrm{trace}(A|_W) = 0 \}.$$

Note: $\mathfrak{g} \subseteq \mathfrak{h}_W$.

<u>Claim</u>:
$$\mathfrak{g} = \mathfrak{n} \cap \Big(\bigcap_{\substack{W \subseteq V \\ \mathfrak{g}\text{-submodule}}} \mathfrak{h}_W \Big)$$

Proof of the claim: Let $\mathfrak{g}'$ be the RHS, which is a Lie subalg. of $\mathfrak{gl}(V)$ with $\mathfrak{g} \subseteq \mathfrak{g}'$. By Weyl's thm, there exists a $\mathfrak{g}$-submodule $U \subseteq \mathfrak{g}'$ such that $\mathfrak{g}' = \mathfrak{g} \oplus U$ as repr. of $\mathfrak{g}$. Our goal is to show that $U = 0$.

We know: V is a $\oplus$ of irreducible $\mathfrak{g}$-modules, say $V = W_1 \oplus \cdots \oplus W_k$.
If $Y \in \mathfrak{U}$ then in particular $Y \in \mathfrak{h}_{W_i}$ for all i; so $Y(W_i) \subseteq W_i$.
On the other hand, $\mathfrak{g}' \subseteq \mathfrak{n}$, so $\mathfrak{g}$ is an ideal of $\mathfrak{g}'$, which implies that $[\mathfrak{g}, Y] = 0$. This means that the action of Y on W_i commutes with the action of $\mathfrak{g}$. By Schur's lemma, $Y = c \cdot \mathrm{id}_{W_i}$ for some constant c. But also $Y \in \mathfrak{h}_{W_i}$ so $\operatorname{trace}(Y|_{W_i}) = 0$, and we conclude that $c = 0$. Hence $Y|_{W_i} = 0$ for all i; so $Y = 0$; so $\mathfrak{U} = 0$. This proves the claim.

(2) Lemma 2: Let M be a vector space, $\varphi \in \operatorname{End}(M)$ with Jordan decomposition $\varphi = \varphi_s + \varphi_n$. Suppose $K \subseteq L \subseteq M$ are subspaces with $\varphi(L) \subseteq K$. Then also $\varphi_s(L) \subseteq K$ and $\varphi_n(L) \subseteq K$.

(3) By (1), it suffices to show that $s, n \in \mathfrak{n}$, and also $s, n \in \mathfrak{h}_W$ for every $\mathfrak{g}$-submodule $W \subseteq V$.

$s, n \in \mathfrak{n}$: $\operatorname{ad}_{\mathfrak{gl}(V)}(X) : \mathfrak{gl}(V) \longrightarrow \mathfrak{gl}(V)$ maps $\mathfrak{g} \subset \mathfrak{gl}(V)$ into itself. Now apply Lemmas 1 and 2.

$s, n \in \mathfrak{h}_W$: By Lemma 2 we have $s(W) \subseteq W$ and $n(W) \subseteq W$. Further, $n|_W$ is nilpotent so $\operatorname{trace}(n|_W) = 0$ and also $\operatorname{trace}(s|_W) = \operatorname{trace}(X|_W) = 0$. Hence $s, n \in \mathfrak{h}_W$.

$\boxtimes$

If $\mathfrak{g}$ is semisimple then $\mathfrak{g} \overset{\operatorname{ad}}{\hookrightarrow} \mathfrak{gl}(\mathfrak{g})$. By the theorem, for $X \in \mathfrak{g}$ there exist $X_s, X_n \in \mathfrak{g}$ such that $X = X_s + X_n$ and $\operatorname{ad}(X)_s = \operatorname{ad}(X_s)$, $\operatorname{ad}(X)_n = \operatorname{ad}(X_n)$. We call this the Jordan decomposition of X in $\mathfrak{g}$.

If $X = X_s$ we say that X is semisimple
$X = X_n$ X is nilpotent.

Theorem Let $\rho : \mathfrak{g} \to \mathfrak{gl}(V)$ be any representation of the semisimple Lie algebra $\mathfrak{g}$. Then for every $X \in \mathfrak{g}$ we have $\rho(X)_s = \rho(X_s)$ and $\rho(X)_n = \rho(X_n)$. In other words: $\rho(X) = \rho(X_s) + \rho(X_n)$ is the Jordan decomposition of the endomorphism $\rho(X)$.

Example Take $\mathfrak{g} = \mathfrak{sl}_2$. With respect to the basis $\{Y, H, X\}$ we have

$$\mathrm{ad}(Y) = \begin{pmatrix} 0 & 2 & 0 \\ & 0 & -1 \\ & & 0 \end{pmatrix} \qquad \mathrm{ad}(H) = \begin{pmatrix} -2 & & \\ & 0 & \\ & & 2 \end{pmatrix} \qquad \mathrm{ad}(X) = \begin{pmatrix} 0 & & \\ 1 & 0 & \\ 0 & -2 & 0 \end{pmatrix}$$

So: X and Y are nilpotent, H is semisimple. By the theorem, for any repr. ρ of $\mathfrak{sl}_2$ the endom ρ_H is semisimple — this is a fact we have used. Also: ρ_X and ρ_Y are nilpotent, which agrees with what we have found.

Sketch of the proof

(1) Reduction to the case where $\rho : \mathfrak{g} \to \mathfrak{gl}(V)$ is injective: Starting from the general case, factor ρ as $\mathfrak{g} \overset{\pi}{\twoheadrightarrow} \mathfrak{g}' \xhookrightarrow[\rho']{} \mathfrak{gl}(V)$, and show that $\pi(X_s) = \pi(X)_s$ and $\pi(X_n) = \pi(X)_n$.

(2) Suppose that $\mathfrak{g} \subset \mathfrak{gl}(V)$. By the proposition, there exist $s, n \in \mathfrak{g}$ such that $X = s + n$ is the Jordan decomposition of X as an endomorphism of V. By Lemma 1, $\mathrm{ad}_{\mathfrak{gl}(V)}(s) : \mathfrak{gl}(V) \to \mathfrak{gl}(V)$ is semisimple and $\mathrm{ad}_{\mathfrak{gl}(V)}(n) : \mathfrak{gl}(V) \to \mathfrak{gl}(V)$ is nilpotent. Now, $\mathrm{ad}_{\mathfrak{g}}(s)$ and $\mathrm{ad}_{\mathfrak{g}}(n)$ are simply the restrictions of $\mathrm{ad}_{\mathfrak{gl}(V)}(s)$ and $\mathrm{ad}_{\mathfrak{gl}(V)}(n)$ to $\mathfrak{g} \subseteq \mathfrak{gl}(V)$. So $\mathrm{ad}_{\mathfrak{g}}(s)$ is semisimple, $\mathrm{ad}_{\mathfrak{g}}(n)$ is nilpotent, and $\mathrm{ad}_{\mathfrak{g}}(X) = \mathrm{ad}_{\mathfrak{g}}(s) + \mathrm{ad}_{\mathfrak{g}}(n)$. By definition of X_s and X_n it follows that $s = X_s$ and $n = X_n$, and we are done $\boxtimes$