Repr. theory, 8 Nov.

$\mathfrak{sl}_3$; dim = 8 ; trivial centre

Direct calculation : Killing form is non-degen $\Rightarrow$ $\mathfrak{sl}_3$ is semisimple

Alternative : Exercise 6 of October 18

We try to make the step from $\mathfrak{sl}_2$ to $\mathfrak{sl}_3$. Many things we will see generalize to arbitrary semisimple Lie algebras.

$$\mathfrak{sl}_2 = \mathbb{C}\cdot Y + \mathbb{C}\cdot H + \mathbb{C}\cdot X$$

Y, X : nilpotent, H : semisimple

If $\rho : \mathfrak{sl}_2 \to \mathfrak{gl}(V)$ is a repr., we start by decomposing V into eigenspaces wrt. H.

Making the step to $\mathfrak{sl}_3$: We no longer work with a single ss. element H, but with a subspace $\mathfrak{h} \subset \mathfrak{sl}_3$ of such elements.

Namely, we take : $\mathfrak{h} = \left\{ \begin{pmatrix} a_1 & & \\ & a_2 & \\ & & a_3 \end{pmatrix} \;\middle|\; \sum a_i = 0 \right\}$

Note : $\mathfrak{h}$ is a 2-dimensional commutative Lie subalg. of $\mathfrak{sl}_3$, consisting of semisimple elements.

Define $\mathfrak{h}^* = \mathrm{Hom}_{\mathbb{C}}(\mathfrak{h}, \mathbb{C})$, dual vector space ;

$L_i \in \mathfrak{h}^*$ defined by $L_i \begin{pmatrix} a_1 & & \\ & a_2 & \\ & & a_3 \end{pmatrix} = a_i$

Then $\mathfrak{h}^* = \mathbb{C}\cdot L_1 + \mathbb{C}\cdot L_2 + \mathbb{C}\cdot L_3 \,/\, \mathbb{C}\cdot(L_1 + L_2 + L_3)$.

<u>Definition</u> If $\rho : \mathfrak{sl}_3 \to \mathfrak{gl}(V)$ is a repr., $\alpha \in \mathfrak{h}^*$, let

$$V_\alpha := \{ v \in V \mid H(v) = \alpha(H)\cdot v \text{ for all } H \in \mathfrak{h} \}$$

If $V_\alpha \neq 0$ then we call α a <u>weight</u> of V (wrt $\mathfrak{h}$).

Proposition For any repr. $\rho : \mathfrak{sl}_3 \to \mathfrak{gl}(V)$ we have

$$V = \bigoplus_{\alpha \in \mathfrak{h}^*} V_\alpha .$$

[Proof: Introduce elts $H_{i,j}$ for $1 \leq i < j \leq 3$. Then $\mathfrak{h} = \mathbb{C} \cdot H_{1,2} + \mathbb{C} \cdot H_{2,3}$; simultaneous diagonalization of commuting semisimple endomorphisms.]

Define $\Lambda_W \subset \mathfrak{h}^*$ as the lattice spanned by L_1, L_2, L_3; so in fact $\Lambda_W = \mathbb{Z} \cdot L_1 \oplus \mathbb{Z} \cdot L_2 \subset \mathfrak{h}^*$. We call Λ_W the <u>weight lattice</u>; this name will be justified.

<u>Copies of $\mathfrak{sl}_2$ in $\mathfrak{sl}_3$</u>: Let E_{ij} = elementary matrix with 1 on position (i,j) and 0 everywhere else.

Then: $\langle E_{2,1}, H_{1,2}, E_{1,2} \rangle \subset \mathfrak{sl}_3$ is a Lie subalg that is $\cong \mathfrak{sl}_2$.

In fact, we have

$$i_{12} : \mathfrak{sl}_2 \hookrightarrow \mathfrak{sl}_3 \quad \text{by} \quad \begin{cases} Y \longmapsto E_{2,1} \\ H \longmapsto H_{1,2} \\ X \longmapsto E_{1,2} \end{cases}$$

In a similar way:

$$i_{13} : \mathfrak{sl}_2 \hookrightarrow \mathfrak{sl}_3 \quad \text{by} \quad \begin{cases} Y \longmapsto E_{3,1} \\ H \longmapsto H_{1,3} \\ X \longmapsto E_{1,3} \end{cases}$$

$$i_{2,3} : \mathfrak{sl}_2 \hookrightarrow \mathfrak{sl}_3 \quad \text{by} \quad \begin{cases} Y \longmapsto E_{3,2} \\ H \longmapsto H_{2,3} \\ X \longmapsto E_{2,3} \end{cases}$$

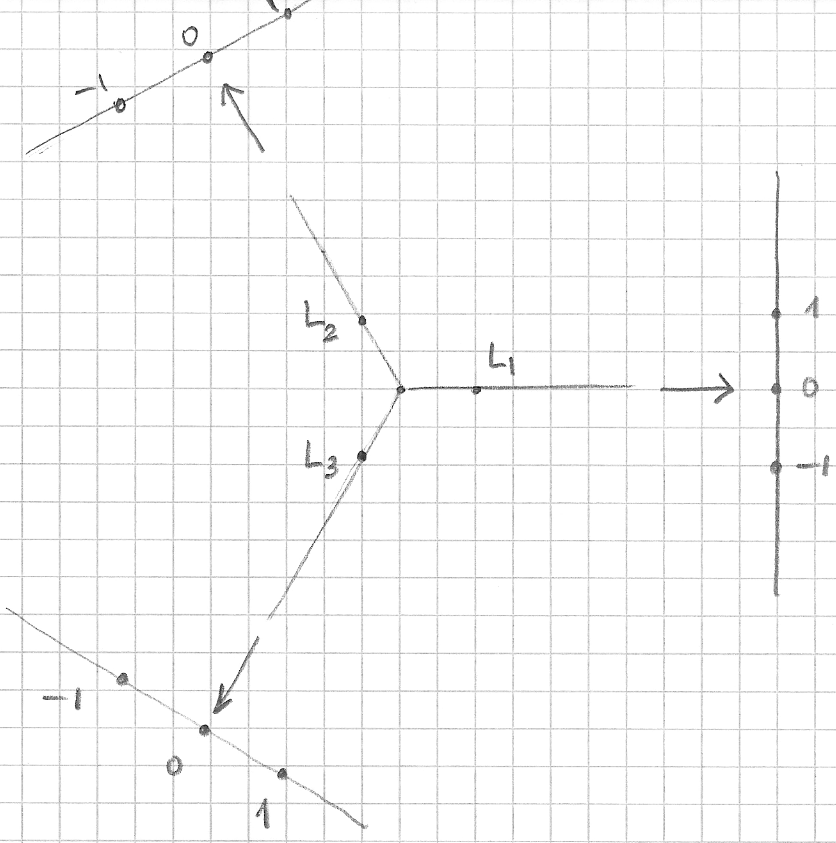
(Of course, there are many more copies of $\mathfrak{sl}_2$ inside $\mathfrak{sl}_3$.)

For each pair (p,q) with $1 \leq p < q \leq 3$ the map i_{pq} induces a map

$$\begin{tikzcd} \mathbb{C} \cdot H \arrow[r, "i_{pq}"] \arrow[d, phantom, "\cap"] & \mathfrak{h} \arrow[d, phantom, "\cap"] \\ \mathfrak{sl}_2 & \mathfrak{sl}_3 \end{tikzcd}$$

and this gives a linear map $i^*_{pq} : \mathfrak{h}^* \longrightarrow (\mathbb{C} \cdot H)^* \cong \mathbb{C}$.

given by
$f \longmapsto f(H)$



Let $\rho : \mathfrak{sl}_3 \to \mathfrak{gl}(V)$ be a repr. Then each $\rho \circ i_{pq} : \mathfrak{sl}_2 \to \mathfrak{gl}(V)$ is a repr. of $\mathfrak{sl}_2$. If $\alpha \in \mathfrak{h}^*$ occurs as a weight then $i_{pq}^*(\alpha)$ is an eigenvalue of $H \in \mathfrak{sl}_2$ acting on V.

Corollary 1 If $\alpha \in \mathfrak{h}^*$ occurs as a weight in V then $\alpha \in \Lambda_W$.

Remark: In particular, all weights lie in the real span of L_1, L_2, L_3; so we can draw real pictures inside $\Lambda_W \otimes \mathbb{R} \subset \mathfrak{h}^*$.

If $\alpha \in \Lambda_W$ occurs as weight in a repr. $\rho : \mathfrak{sl}_3 \to \mathfrak{gl}(V)$, write $\mathrm{mult}_V(\alpha) = \dim(V_\alpha)$: the multiplicity of α as a weight.

Some first examples of repr., and the weights that occur:

(1) The trivial repr. $\rho_{triv} : \mathfrak{sl}_3 \to \mathfrak{gl}_1$: only $0 \in \mathfrak{h}^*$ occurs.

(2) The standard repr. $St : \mathfrak{sl}_3 \to \mathfrak{gl}_3$: the weights are L_1, L_2, L_3.

(3) The dual of the standard repr. $St^* : \mathfrak{sl}_3 \to \mathfrak{gl}_3$: the weights are $-L_1, -L_2, -L_3$.

General rule: if $\mathfrak{sl}_3 \longrightarrow \mathfrak{gl}(V)$ and $\mathfrak{sl}_3 \longrightarrow \mathfrak{gl}(W)$ are repr., then $V = \bigoplus_{\alpha \in \Lambda_W} V_\alpha$; $W = \bigoplus_{\beta \in \Lambda_W} W_\beta$; this gives:

$$V \otimes W = \bigoplus_{\alpha, \beta \in \Lambda_W} (V_\alpha \otimes W_\beta)$$

and $V_\alpha \otimes W_\beta$ has weight $\alpha + \beta$.

So the weights $\gamma \in \Lambda_W$ that occur in $V \otimes W$ are all $\alpha + \beta$ with

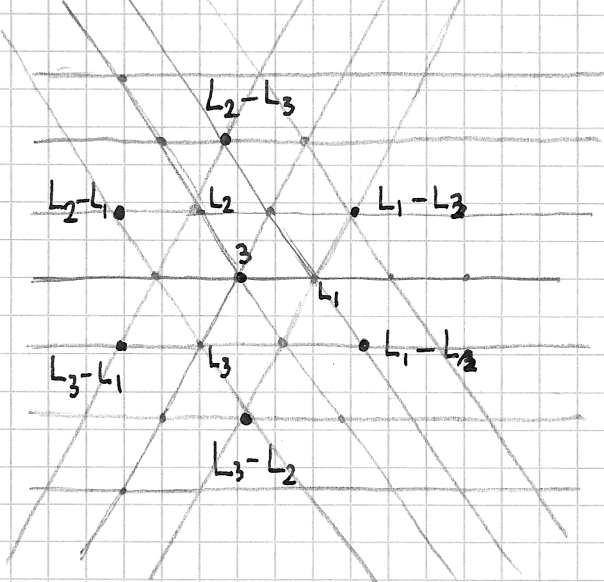
α a weight of V

β a weight of W

and

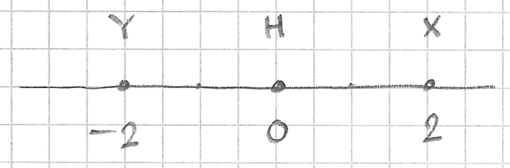
$$\mathrm{mult}_{V\otimes W}(\gamma) = \sum_{\substack{\alpha, \beta \in \Lambda_W \\ \alpha + \beta = \gamma}} \mathrm{mult}_V(\alpha) \cdot \mathrm{mult}_W(\beta).$$

(4) The repr. $St^* \otimes St$. If we write the standard repr. as $\mathfrak{sl}_3 \longrightarrow \mathfrak{gl}(V)$ then $St^* \otimes St$ is the induced Lie algebra action on $\mathrm{End}(V)$. Weights:



(5) The adjoint repr. With notation as in (4), $\mathrm{End}(V) = \mathbb{C} \cdot \mathrm{id}_V \oplus \mathrm{ad}$ as repr. of $\mathfrak{sl}_3$. So the weights are the same, except that now the origin has multiplicity 2. Explicitely: the E_{ij} with $i \neq j$ are the six non-trivial eigenvectors and $\mathfrak{h} \subset \mathfrak{sl}_3$ is the weight 0 subspace.

For comparison: for $\mathfrak{sl}_2$, the weight pattern for the adj. repr. is



<u>Definition</u> The <u>non-zero</u> weights in the adjoint repr. are called the <u>roots</u> of $\mathfrak{sl}_3$ (more generally: ...) with respect to $\mathfrak{h}$. If R denotes the set of roots,

$$\mathfrak{sl}_3 = \mathfrak{h} \oplus \bigoplus_{\alpha \in R} (\mathfrak{sl}_3)_\alpha .$$

By $\Lambda_R \subset \mathfrak{h}^*$ we denote the lattice spanned by the roots: the <u>root lattice</u>. It follows from Cor 1 that $\Lambda_R \subset \Lambda_W$, and direct calculation shows that $[\Lambda_W : \Lambda_R] = 3$.

<u>Key observation</u> Let $\rho: \mathfrak{sl}_3 \longrightarrow \mathfrak{gl}(V)$ be any repr. Then for all $\alpha, \beta \in \mathfrak{h}^*$ we have:

$$(\mathfrak{sl}_3)_\alpha \left(V_\beta \right) \subseteq V_{\alpha+\beta} .$$

<u>In other words</u>: if $X \in (\mathfrak{sl}_3)_\alpha$ and $v \in V_\beta$ then $X(v) \in V_{\alpha+\beta}$.

<u>Proof</u> $v \in V_\beta$ means: for all $Z \in \mathfrak{h}$ we have $Z(v) = \beta(Z) \cdot v$;
$X \in (\mathfrak{sl}_3)_\alpha$ means: 〃 $[Z, X] = \alpha(Z) \cdot X$.

This gives:

$$Z(X(v)) = X(Z(v)) + [Z,X](v)$$
$$= \beta(Z) \cdot X(v) + \alpha(Z) \cdot X(v) = (\alpha+\beta)(Z) \cdot X(v)$$

$\boxtimes$

This generalizes what we have seen for $\mathfrak{sl}_2$.

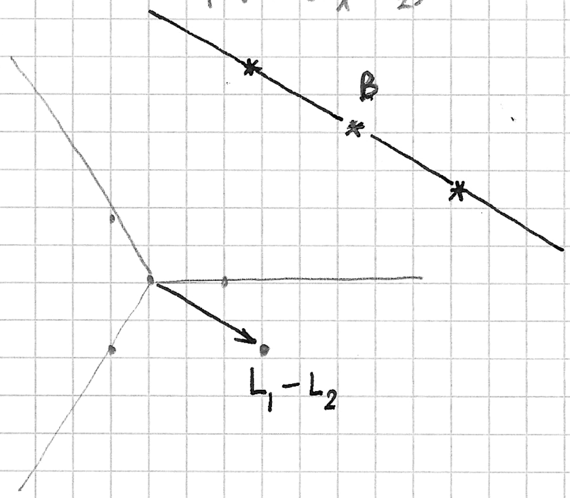
Corollary 2 If ρ is irreducible then all weights of V lie in the same class in Λ_W / Λ_R $(\cong \mathbb{Z}/3\mathbb{Z})$.

Let $\rho: \mathfrak{sl}_3 \to \mathfrak{gl}(V)$ be any repr, $v \in V_\beta$ for some weight β.
Consider the copies of $\mathfrak{sl}_2$ inside $\mathfrak{sl}_3$ obtained as image of $i_{pq}: \mathfrak{sl}_2 \to \mathfrak{sl}_3$.

For example: $\mathrm{Im}(i_{12}) = \mathrm{Span}(E_{2,1}, H_{1,2}, E_{1,2})$

$E_{i,j} \in \mathfrak{sl}_3$ $(i \neq j)$ spans the line corresponding to the root $L_i - L_j$.

So we find: the $\mathfrak{sl}_2$-submodule spanned by v lies in the sum of weight spaces $V_{\beta + m(L_1 - L_2)}$ with $m \in \mathbb{Z}$.



The results about $\mathfrak{sl}_2$ tell us that if we project the set of weights $\beta + m(L_1 - L_2)$ that occur via the map $i_{12}^*: \mathfrak{h}^* \to \mathbb{C}$

$$L_1 - L_2 \mapsto 2$$

then we get a set of values of the form $-n, -n+2, \dots, n-2, n$ for some n.

Corollary 3 For $i \in \{1,2,3\}$, consider the reflection of $\Lambda_W \otimes \mathbb{R}$ in the line spanned by L_i, exchanging the other two L_j. Then the set of weights occurring in a repr. of $\mathfrak{sl}_3$ is mapped into itself (counting the multiplicities!) under this reflection.