Representations of Algebraic Groups
Lecture of 22 Nov. 2016

Throughout: $\mathfrak{g}$ is a semisimple Lie algebra
$B : \mathfrak{g} \times \mathfrak{g} \longrightarrow \mathbb{C}$ Killing form, non-degenerate

Have seen: $X \in \mathfrak{g}$ has Jordan decomposition $X = X_s + X_n$ such that $[X_s, X_n] = 0$ and in any repr. $\rho : \mathfrak{g} \longrightarrow \mathfrak{gl}(V)$ we have: $\rho(X_s)$ is semisimple, $\rho(X_n)$ is nilpotent.

Running example: $\mathfrak{g} = \mathfrak{sl}_n$, $n \geq 2$.

<u>Definition</u> A <u>toral</u> Lie subalg. $\mathfrak{h} \subset \mathfrak{g}$ is a Lie subalg. consisting of semisimple elements. A <u>maximal toral</u> Lie subalg. is a toral Lie subalg that is not contained in a strictly bigger toral Lie subalg.

<u>Example</u> G alg. gp with $\mathfrak{g} = \mathrm{Lie}(G)$ semisimple; $T \subset G$ a torus; then $\mathrm{Lie}(T) \subset \mathfrak{g}$ is toral.

<u>Example</u> $\mathfrak{g} = \mathfrak{sl}_n \supset \mathfrak{h} = \{ \mathrm{diag}(a_1, \ldots, a_n) \mid \sum a_i = 0 \}$ is toral. We will see that this $\mathfrak{h}$ is maximal toral.

<u>Lemma 1</u> $\mathfrak{h} \subset \mathfrak{g}$ is toral $\Rightarrow$ $\mathfrak{h}$ is commutative.

<u>Proof</u> Take $X \in \mathfrak{h}$ and let $\varphi = \mathrm{ad}_{\mathfrak{h}}(X) : \mathfrak{h} \to \mathfrak{h}$ be the restriction of $\mathrm{ad}(X)$ to $\mathfrak{h}$. Since X is semisimple, φ is diagonalizable. Let $Y \in \mathfrak{h}$ be an eigenvector, say $\varphi(Y) = \lambda \cdot Y$. Next consider $\psi = \mathrm{ad}_{\mathfrak{h}}(Y) : \mathfrak{h} \to \mathfrak{h}$. Again this is diagonalizable; choose a basis $e_1, \ldots, e_d$ for $\mathfrak{h}$ such that $\psi(e_i) = \mu_i \cdot e_i$ with $\mu_i \in \mathbb{C}$. On the one hand, $\psi(X) = [Y, X] = -[X, Y] = -\varphi(Y) = -\lambda \cdot Y$, and therefore $\psi \circ \psi(X) = \psi(-\lambda \cdot Y) = [Y, -\lambda \cdot Y] = 0$. On the other hand, if $X = \sum c_i e_i$ then

$4 \circ \psi(X) = \sum \mu_i^2 c_i \cdot e_i$. This gives that $\mu_i^2 c_i = 0$ for all i. Hence also $\mu_i c_i = 0$ for all i, and this gives that $\psi(X) = 0$. So X and Y commute and $\lambda = 0$. As λ was an arbitrary eigenvalue of φ it follows that $\varphi = 0$, i.e., X commutes with every element of $\mathfrak{h}$. $\square$

<u>Lemma 2</u> If V is a vector space $/ \mathbb{C}$, and $A_1, \dots, A_n \in \mathrm{End}(V)$ are semisimple endomorphisms that mutually commute ($[A_i, A_j] = 0$) then there exists a basis for V on which all A_i simultaneously are in diagonal form.

Let $\mathfrak{h} \subset \mathfrak{g}$ be toral, $\mathfrak{h}^* =$ dual vector space. Lemma 2 gives us a decomp.

$$\mathfrak{g} = \bigoplus_{\alpha \in \mathfrak{h}^*} \mathfrak{g}_\alpha \quad ; \quad \mathfrak{g}_\alpha = \{ Y \in \mathfrak{g} \mid [X,Y] = \alpha(X) \cdot Y \text{ for all } X \in \mathfrak{h} \} .$$

By Lemma 1 we have $\mathfrak{h} \subset \mathfrak{g}_0$.

<u>Observation</u> For $\alpha, \beta \in \mathfrak{h}^*$ we have $[\mathfrak{g}_\alpha, \mathfrak{g}_\beta] \subseteq \mathfrak{g}_{\alpha+\beta}$.

<u>Immediate consequences</u> :

(1) $X \in \mathfrak{g}_\alpha$ with $\alpha \neq 0 \Rightarrow X$ is nilpotent

(2) If $\alpha + \beta \neq 0$ then $\mathfrak{g}_\alpha \perp \mathfrak{g}_\beta$ w.r.t. B

(3) $B|_{\mathfrak{g}_0}$ is non-degenerate.

<u>Proposition 1</u> If $\mathfrak{h} \subset \mathfrak{g}$ is maximal toral then $\mathfrak{g}_0 = \mathfrak{h}$.

For the proof of this we refer to the book of Humphreys, Section 8.2.

From now on we assume $\mathfrak{h} \subset \mathfrak{g}$ is maximal toral. Define

$$R := \{ \alpha \in \mathfrak{h}^* \mid \mathfrak{g}_\alpha \neq 0 \} \quad \underline{\text{roots}} \text{ of } \mathfrak{g} \text{ (w.r.t. } \mathfrak{h})$$

Remark We will work with a fixed maximal toral $\mathfrak{h}$. It can be shown that for the results we obtain it doesn't matter which $\mathfrak{h}$ we take.

Example $\mathfrak{g} = \mathfrak{sl}_n \supset \mathfrak{h} = \{\text{diagonal matrices}\}$. Define $L_i \in \mathfrak{h}^*$ by $L_i(\operatorname{diag}(a_1, \ldots, a_n)) = a_i$. Then $\mathfrak{h}^* = \mathbb{C}\cdot L_1 + \cdots + \mathbb{C}\cdot L_n / \mathbb{C}\cdot(L_1 + \cdots + L_n)$.
Let E_{ij} be the (i,j)-th elementary matrix. For $i \neq j$ we find $\mathbb{C}\cdot E_{ij} = \mathfrak{g}_\alpha$ with $\alpha = L_i - L_j \in \mathfrak{h}^*$, and $\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\alpha \in R} \mathfrak{g}_\alpha$, where $R = \{\text{all } L_i - L_j \text{ with } i \neq j\}$. So necessarily $\mathfrak{g}_0 = \mathfrak{h}$, which implies that $\mathfrak{h}$ is maximal toral.

Corollary $B|_{\mathfrak{h}} : \mathfrak{h} \times \mathfrak{h} \longrightarrow \mathbb{C}$ is non-degenerate.

This gives us an isomorphism $\mathfrak{h} \xrightarrow{\sim} \mathfrak{h}^*$ by $X \longmapsto B(X,-)$. In other words: given $\lambda \in \mathfrak{h}^*$, there is a unique $X \in \mathfrak{h}$ with $\lambda = B(X,-)$.

We now start deriving some properties of the set of roots:

Proposition 2 If $\alpha \in R$ then $-\alpha \in R$, and the elements of R span $\mathfrak{h}^*$ as a $\mathbb{C}$-vector space.

Proof Let $0 \neq X \in \mathfrak{g}_\alpha$. There exists an element $Y \in \mathfrak{g}$ with $B(X,Y) \neq 0$. Also $\mathfrak{g} = \mathfrak{g}_0 \oplus \left(\bigoplus_{\beta \in R} \mathfrak{g}_\beta \right)$; write $Y = Y_0 + \sum_{\beta \in R} Y_\beta$.
Then $B(X,Y) = B(X, Y_{-\alpha})$ because $\mathfrak{g}_\alpha \perp \mathfrak{g}_\beta$ if $\alpha + \beta \neq 0$. So $Y_{-\alpha} \neq 0$, and therefore $-\alpha \in R$.

If $\langle R \rangle \subsetneq \mathfrak{h}^*$, there exists a $0 \neq H \in \mathfrak{h}$ with $\alpha(H) = 0$ for all $\alpha \in R$. This gives $[H, X] = \alpha(H) \cdot X = 0$ for all $\alpha \in R$ and $X \in \mathfrak{g}_\alpha$.
Also $[H, X] = 0$ for all $X \in \mathfrak{h}$, because $\mathfrak{h}$ is abelian. (Lemma 1)
So $H \in Z(\mathfrak{g})$. But $\mathfrak{g}$ is semisimple $\Rightarrow Z(\mathfrak{g}) = 0$, $\lightning$. $\square$

Theorem Let $\alpha \in R$.

(i) $\dim(\mathfrak{g}_\alpha) = 1$

(ii) The subspace $\mathfrak{h}_\alpha = [\mathfrak{g}_\alpha, \mathfrak{g}_{-\alpha}] \subset \mathfrak{h}$ is 1-dimensional and contains a unique element H_α with $\alpha(H_\alpha) = 2$.

(iii) If $0 \neq X_\alpha \in \mathfrak{g}_\alpha$, there exists a unique element $Y_\alpha \in \mathfrak{g}_{-\alpha}$ such that $\mathfrak{s}_\alpha := \langle Y_\alpha, H_\alpha, X_\alpha \rangle \subset \mathfrak{g}$ is a Lie subalgebra that is isomorphic to $\mathfrak{sl}_2$ under $\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \mapsto Y_\alpha$, $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \longmapsto H_\alpha$, $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \longmapsto X_\alpha$.

Proof Step 1 We have $0 \neq \alpha \in \mathfrak{h}^*$ so there is a unique $0 \neq t_\alpha \in \mathfrak{h}$ with $\alpha(H) = B(t_\alpha, H)$ for all $H \in \mathfrak{h}$. If $X \in \mathfrak{g}_\alpha$ and $Y \in \mathfrak{g}_{-\alpha}$ then $[X,Y] \in \mathfrak{g}_0 = \mathfrak{h}$, so for $H \in \mathfrak{h}$ we have

$$B(H, [X,Y]) = B([H,X],Y) = B(\alpha(H)\cdot X, Y) = \alpha(H)\cdot B(X,Y).$$

It follows that $[X,Y] = B(X,Y)\cdot t_\alpha$ and hence $[\mathfrak{g}_\alpha, \mathfrak{g}_{-\alpha}] = \mathbb{C}\cdot t_\alpha$.

Step 2 Given $0 \neq X \in \mathfrak{g}_\alpha$, choose $Y \in \mathfrak{g}_{-\alpha}$ with $B(X,Y) = 1$, and hence $[X,Y] = t_\alpha$. Claim: $\alpha(t_\alpha) \neq 0$.

To see this, suppose $\alpha(t_\alpha) = 0$. Consider $\mathfrak{n} = \mathbb{C}\cdot Y + \mathbb{C}\cdot t_\alpha + \mathbb{C}\cdot X$, which is a nilpotent Lie subalgebra of $\mathfrak{g}$ with $[\mathfrak{n},\mathfrak{n}] = \mathbb{C}\cdot t_\alpha$ and $[t_\alpha, \mathfrak{n}] = 0$. As $t_\alpha \in [\mathfrak{n},\mathfrak{n}]$ this implies (e.g., using Lie's thm on solvable Lie algebras) that for any representation $\rho: \mathfrak{n} \to \mathfrak{gl}(V)$ the endomorphism $\rho(t_\alpha)$ is nilpotent. In particular, $\mathrm{ad}_\mathfrak{g}(t_\alpha)$ is nilpotent. But also $\mathrm{ad}_\mathfrak{g}(t_\alpha)$ is semisimple. So $\mathrm{ad}_\mathfrak{g}(t_\alpha) = 0$, hence $t_\alpha = 0$, contradiction.

Step 3 By rescaling Y we may now assume that $H_\alpha := [X,Y]$ satisfies $\alpha(H_\alpha) = 2$. (More precisely: $H_\alpha = \frac{2}{\alpha(t_\alpha)}\cdot t_\alpha$.) Then $[X,Y] = H$, $[H,X] = \alpha(H)\cdot X = 2X$ and $[H,Y] = -\alpha(H)\cdot Y = -2Y$, and we see that $\mathfrak{s}_\alpha \cong \mathfrak{sl}_2$.

Final step Fix $\alpha \in R$. We show that $\dim(\mathfrak{g}_{-\alpha}) = 1$. If $\dim(\mathfrak{g}_{-\alpha}) > 1$ then there is a $0 \neq Z \in \mathfrak{g}_{-\alpha}$ with $B(X_\alpha, Z) = 0$. By Step 1: $[X_\alpha, Z] = 0$. So Z is a highest weight vector for the action of $\mathfrak{s}_\alpha \cong \mathfrak{sl}_2$ on $\mathfrak{g}$. But $[H_\alpha, Z] = -2Z$, which is not possible for a highest weight vector. $\boxtimes$

Example $\mathfrak{g} = \mathfrak{sl}_n$, $R \ni \alpha = L_i - L_j$ $(i \neq j)$, then we can take

$$Y_\alpha = E_{ji}, \qquad H_\alpha = H_{ij} := E_{ii} - E_{jj}, \qquad X_\alpha = E_{ij}.$$

Remark For $\alpha \in R$ the element $H_\alpha \in \mathfrak{h}$ does not depend on choices. We have $H_{-\alpha} = -H_\alpha$. The element $X \in \mathfrak{g}_\alpha$ may be rescaled by a factor $c \neq 0$; the corresponding Y_α is then rescaled by c^{-1}.

Corollary For $X, Y \in \mathfrak{h}$ we have

$$B(X,Y) = \sum_{\gamma \in R} \gamma(X) \cdot \gamma(Y). \qquad (*)$$

Proof Write $R = \{\alpha_1, \dots, \alpha_N\}$ and choose a generator e_i for $\mathfrak{g}_{\alpha_i}$. Let $e_{N+1}, \dots, e_M$ be a $\mathbb{C}$-basis for $\mathfrak{h}$. Then $e_1, \dots, e_M$ is a basis for $\mathfrak{g}$, and for $X \in \mathfrak{h}$ the matrix of $\mathrm{ad}_{\mathfrak{g}}(X)$ on this basis is

$$\mathrm{diag}(\alpha_1(X), \dots, \alpha_N(X), 0, \dots, 0). \qquad \boxtimes$$

Consider finite dimensional repr. $\rho : \mathfrak{g} \longrightarrow \mathfrak{gl}(V)$. By Lemmas 1 and 2:

$$V = \bigoplus_{\lambda \in \mathfrak{h}^*} V_\lambda \quad \text{with } V_\lambda = \{ v \in V \mid H(v) = \lambda(H) \cdot v \text{ for all } H \in \mathfrak{h} \}.$$

Also: $\mathfrak{g}_\alpha(V_\lambda) \subseteq V_{\alpha+\lambda}$ for all $\alpha \in \{0\} \cup R$ and $\lambda \in \mathfrak{h}^*$.

If $V_\lambda \neq 0$ then we call λ a weight of V.

Fix $\alpha \in R$, and let $\mathfrak{s}_\alpha = \langle Y_\alpha, H_\alpha, X_\alpha \rangle \subset \mathfrak{g}$ be as before. We have $\mathbb{C} \cdot H_\alpha \subset \mathfrak{h}$; this induces a projection $pr_\alpha : \mathfrak{h}^* \to \mathbb{C}$ by $\lambda \mapsto \lambda(H_\alpha)$.

If λ is a weight of V then $\bigoplus_{i \in \mathbb{Z}} V_{\lambda + i\alpha}$ is an $\mathfrak{s}_\alpha$-submodule of V. It follows that the multiset of all $\lambda + i\alpha(H_\alpha) = \lambda(H_\alpha) + 2i$, counted with multiplicities $\dim(V_{\lambda+i\alpha})$ is the multiset of weights of an $\mathfrak{sl}_2$-representation:

- $\lambda(H_\alpha) \in \mathbb{Z}$
- the multiset is symmetrical under $k \leftrightarrow -k$
- if $\lambda + i\alpha$ and $\lambda + j\alpha$ are weights and $0 \leq \lambda(H_\alpha) + 2i \leq \lambda(H_\alpha) + 2j$ then $\dim(V_{\lambda+i\alpha}) \geq \dim(V_{\lambda+j\alpha})$.

<u>Corollary</u> For all $\alpha, \beta \in R$ we have $\alpha(H_\beta) \in \mathbb{Z}$.

<u>Proposition</u> Let $\alpha \in R$. Then the only multiples of α in R are $\pm\alpha$.

<u>Proof</u> Suppose $c\alpha \in R$. Then $2c = c\alpha(H_\alpha) \in \mathbb{Z}$, so $c \in \frac{1}{2}\mathbb{Z}$.

We have $\dim(\mathfrak{g}_\beta) = 1$ for all $\beta \in R$. It follows that the $\mathfrak{s}_\alpha$-submodule $V \subset \mathfrak{g}$ spanned by $\mathfrak{g}_{c\alpha}$ is irreducible and contains all $\mathfrak{g}_{k\alpha}$ for $k \in \{-c, -c+1, \dots, c\} \setminus \{0\}$. (w.l.o.g., $c > 0$) On the other hand, $\mathfrak{s}_\alpha = \mathfrak{g}_{-\alpha} + \mathbb{C} \cdot H_\alpha + \mathfrak{g}_\alpha$ is itself an $\mathfrak{s}_\alpha$-submodule of $\mathfrak{g}$. If $c \in \mathbb{Z}_{>0}$ it follows that $c = 1$. In particular, $2\alpha \notin R$. Hence also $\frac{1}{2}\alpha \notin R$. Coming back to the submodule V, it follows that $c \in \frac{1}{2} + \mathbb{Z}$ is impossible. $\boxtimes$

Next we look at a root $\beta \in R$ with $\beta \neq \pm\alpha$.

<u>Proposition</u> Let $\beta \in R \setminus \{\pm\alpha\}$. Then the roots of the form $\beta + i\alpha$ $(i \in \mathbb{Z})$ form a progression $\beta - p\cdot\alpha, \beta + (1-p)\cdot\alpha, \dots, \beta, \dots, \beta + (q-1)\alpha, \beta + q\alpha$ for some $p, q \in \mathbb{Z}_{\geq 0}$, and $\beta(H_\alpha) = p - q$.

<u>Proof</u> We know that $\bigoplus_{i\in\mathbb{Z}} \mathfrak{g}_{\beta+i\alpha}$ is an $\mathfrak{s}_\alpha$-submodule of $\mathfrak{g}$ with $\dim(\mathfrak{g}_{\beta+i\alpha}) \leq 1$ for all i. This gives the first assertion. Further, there is an integer n with $\beta - p\alpha(H_\alpha) = \beta(H_\alpha) - 2p = -n$ and $\beta + q\alpha(H_\alpha) = \beta(H_\alpha) + 2q = n$; this gives $\beta(H_\alpha) = p - q$. $\boxtimes$

As we have seen, $\mathfrak{h} \xrightarrow{\sim} \mathfrak{h}^*$ by $X \longmapsto B(X, -)$. For $\lambda \in \mathfrak{h}^*$, let $t_\lambda \in \mathfrak{h}$ be the unique element with $\lambda = B(t_\lambda, -)$. We can now transfer the Killing form on $\mathfrak{h}$ to a symmetric bilinear form

$$(\, , \,) : \mathfrak{h}^* \times \mathfrak{h}^* \longrightarrow \mathbb{C}$$

by $(\lambda, \mu) = B(t_\lambda, t_\mu) = \lambda(t_\mu)$.

Let $\alpha, \beta \in R$. We have:

- $B(H_\alpha, H_\beta) \in \mathbb{Z}$ and $B(H_\alpha, H_\alpha) > 0$: use $(*)$ plus the previous Corollary
- $B(t_\alpha, H_\beta) = \alpha(H_\beta) \in \mathbb{Z}$
- $H_\alpha = \frac{2}{\alpha(t_\alpha)} \cdot t_\alpha$.

Combining these, we find that $\alpha(t_\alpha) \in \mathbb{Q}$, and hence:

$$(\alpha, \beta) = B(t_\alpha, t_\beta) = \frac{\alpha(t_\alpha)\cdot\beta(t_\beta)}{4} \cdot B(H_\alpha, H_\beta) \in \mathbb{Q}.$$

<u>Remark</u>:

$$(\lambda, \mu) = B(t_\lambda, t_\mu) \overset{(*)}{=} \sum_{\alpha\in R} \alpha(t_\lambda)\cdot\alpha(t_\mu) = \sum_{\alpha\in R} (\alpha,\lambda)\cdot(\alpha,\mu). \qquad (**)$$

Proposition Let $E_{\mathbb{Q}} \subset \mathfrak{h}^*$ be the $\mathbb{Q}$-linear span of the roots $\alpha \in R$. Then $\dim_{\mathbb{Q}}(E_{\mathbb{Q}}) = \dim_{\mathbb{C}}(\mathfrak{h}^*)$.

Equivalent :

- The canonical map $\mathbb{C} \otimes_{\mathbb{Q}} E_{\mathbb{Q}} \longrightarrow \mathfrak{h}^*$ is an isomorphism.

or :

- Let $\alpha_1, \dots, \alpha_r \in R$ be a basis for $\mathfrak{h}^*$. If $\beta \in R$ is written as $\beta = \sum_{i=1}^{r} c_i \cdot \alpha_i$ then $c_i \in \mathbb{Q}$ for all $i = 1, \dots, r$.

Proof We use the last formulation. For any i we have :

$$(\beta, \alpha_i) = \sum_{j=1}^{r} c_j \cdot (\alpha_j, \alpha_i)$$

This gives a system of r linear equations in the unknowns c_j, which has a unique solution, since the α_i span $\mathfrak{h}^*$ and the form $(\ ,\ )$ on $\mathfrak{h}^*$ is non-degenerate. As $(\beta, \alpha_i) \in \mathbb{Q}$ and $(\alpha_j, \alpha_i) \in \mathbb{Q}$ for all i, we get $c_j \in \mathbb{Q}$. $\boxtimes$

Note : $(\ ,\ )$ restricts to a $\mathbb{Q}$-valued form on $E_{\mathbb{Q}}$ with $(\lambda, \lambda) > 0$ for all $\lambda \in E_{\mathbb{Q}} \setminus \{0\}$ by $(**)$.

In what follows we will mostly work inside

$$E = \mathbb{R}\text{-linear span of } R = \mathbb{R} \otimes_{\mathbb{Q}} E_{\mathbb{Q}}$$

on which $(\ ,\ )$ is an inner product.

For $\alpha \in R$, let $s_\alpha : E \longrightarrow E$ be the orthogonal reflection in the hyperplane $(\mathbb{R} \cdot \alpha)^{\perp} = \{ y \in E \mid y(H_\alpha) = 0 \}$. Concretely :

$$s_\alpha : x \longmapsto x - \frac{2(\alpha, x)}{(\alpha, \alpha)} \cdot \alpha .$$

(Of course, $s_\alpha^2 = \mathrm{id}$.)

Theorem The reflections s_α map $R \subset E$ into itself.

Proof Let $\beta \in R$. Let $\beta - p\alpha, \dots, \beta + q\alpha$ be the α-string through β, and recall that $\beta(H_\alpha) = p - q$. Now:

$$s_\alpha(\beta) = \beta - \frac{2(\alpha,\beta)}{(\alpha,\alpha)} \cdot \alpha$$

$$= \beta - \frac{2\,\beta(t_\alpha)}{\alpha(t_\alpha)} \cdot \alpha$$

$$= \beta - \frac{2\,\beta(H_\alpha)}{\alpha(H_\alpha)} \cdot \alpha = \beta - (p-q) \cdot \alpha$$

and this is indeed one of the roots in the α-string. $\boxtimes$

Corollary The subgroup $W \subset GL(E)$ generated by the s_α is finite.

Pf By the theorem $W \longrightarrow \mathfrak{S}(R)$, the permutation group of R. As R spans E, this homomorphism is injective. $\boxtimes$

The group W is called the <u>Weyl group</u> of R.

What we have proven means that $R \subset E$ is a <u>root system</u>. The main point of this for us is:

- Root systems can be classified.
- A semisimple Lie algebra is fully determined by the associated root system.