Combinatorial Game Theory

An Introduction Guided by the Game of Go

ANTONIE BLOM
4624629

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Supervisor:
Wieb Bosma

Second reader:
Michael Müger

Radboud University

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Introduction

The field of (partizan) Combinatorial Game Theory was developed during the 1960s and 70s, in a joint effort between Elwyn R. Berlekamp, John H. Conway and Richard K. Guy, and has been connected with the game of Go since its very early days. Conway got the idea that certain types of games could be represented as “sums” of simpler games after observing how Go games tended to “break up” into several smaller games in various regions of the board [Rob15, p. 178]. Despite Go having inspired the field’s early development, the first comprehensive reference work on the application of Combinatorial Game Theory to Go only only came out in 1994, namely *Mathematical Go* by Berlekamp and Wolfe [BW94], based in part on David Wolfe’s PhD thesis.

The goal of this Bachelor’s thesis has been to discover the methods and results used within the field of Combinatorial Game Theory, with the idea of understanding its application to Go as the guiding principle. These findings are laid out here, in the form of an introductory level overview of the field, written to be understood by undergraduate students in mathematics. The focus on application to Go means I’ve opted to give matters relating to temperature a more in-depth treatment, while disregarding impartial games entirely.

By the end, we’ll have established a method of evaluating certain frequently occurring Go endgame positions by means of chilling, thereby demonstrating the power of Combinatorial Game Theory in action.

I would like to thank my supervisor Wieb Bosma for trusting me to find my own way through the sources on Combinatorial Game Theory, allowing me to gradually figure out the ultimate direction of the thesis. Our weekly exchange of ideas ensured steady progress, and kept everything focused.

1 Games

On a conceptual level, a combinatorial game is a competition between two players called *Left* and *Right* who alternate moves. We shall use the *normal play* convention, where the objective of a game is to get the last move. That is to say, when a player is called upon to make a move and they don't have any options, they lose the game.

Definition 1.1. A *(partizan) game* G is a pair of sets L and R each consisting of other games. The elements of L are called the *Left options*, and the elements of R are called the *Right options*. These represent the moves available to the Left player and Right player respectively.

Here the word partizan reflects that the options available to Left may differ from those available to Right.[1]

For a game with Left options $a, b, c, \dots$ and Right options $p, q, r, \dots$ we write $\{a, b, c, \dots \mid p, q, r, \dots\}$. If we instead wish to refer to some unspecified game G we write $G = \{G^L \mid G^R\}$, using G^L and G^R as symbols for typical representatives of Left and Right options respectively.

Notice firstly that the definition is recursive, and secondly that we do not attempt to construct a *set* of games. Rather, the definition describes a set-theoretical class of games. Indeed there is no set of all games.

[1] Conway, who had originally dubbed these games *unimpartial*, said about the term:

> That's a terrible word, "unimpartial." But the negative of "impartial" can't be "partial" for a mathematician, because "partial" means there's only part of it present. ...[I was] discussing it with someone, I think it may have been Richard Guy, and we came up with "partizan"—and it's partizan as opposed to partisan.

The spelling "partisan" reminded him too much of *Napisan*, a diaper cleaning product. [Rob15, p. 178]

The games that appear as options of a game G, or as options of options, etc. are known as the *positions* or *followers* of G. Games with finitely many positions are called *short*, those with infinitely many positions *long*. In this thesis we mostly restrict ourselves to the study of short games.

1.1 Simple examples

Definition 1.2. The simplest game we can construct from the definition has $L = R = \varnothing$. We call this game *zero* and write $0 \coloneqq \{|\}$. Neither player can make any legal moves. Whoever starts the game loses.

This leads to an important remark about Definition 1.1, namely that the mathematical objects it describes do not tell us whose turn it is! Board games can perhaps provide a justification for this property, as in such games we cannot typically tell whose turn it is based on the board position alone. Indeed, we could examine a board position by analysing what either player's chances might be if they were to make the next move. This crucial distinction between the act of *playing* a game and the game itself (its *state*, if you will) is precisely what makes CGT so interesting and powerful. Consider the finite perfect information games for instance, where one of the players is guaranteed a winning strategy. Disclosing which player goes first would in some sense give the entire game away! The first player either wins or loses. The player-agnostic analysis of CGT on the other hand can provide us with valuable and nuanced insights about the incentives, advantages and disadvantages of both players.

Using the zero game we can construct other games. For instance, let's play the game $G \coloneqq \{0|\}$. If Left starts, they must move from G to 0, after which Right has no legal moves, so Left wins. If Right starts, they have no legal moves, and Left also wins. We see that Left always wins, no matter who starts. Conversely, the game $\{|0\}$ is always won by Right.

The final game we can construct using 0 alone is important enough to get its own symbol and name.

Definition 1.3. $* \coloneqq \{0|0\}$

It's easy to see that whoever gets to move first in $*$, wins.

1.2 Who wins?

The four games we've analysed so far represent four different types of games, characterised by which player has a winning after a given player starts. We use the following notation.

$G > 0$ – Left wins no matter who starts (e.g. $G = \{0|\}$).

$G < 0$ – Right wins no matter who starts (e.g. $G = \{|0\}$).

$G \parallel 0$ – the first player to move wins (e.g. $G = *$).

$G = 0$ – the second player to move wins.

All games fall into one of these outcome classes.[2]

Later on, these four classes will provide a natural foundation for the reimplementation of the symbols $<$, $>$, $\parallel$ and $=$ as binary relations on games.

The notation $G = 0$ for the last category is perhaps a bit suspect. After all, the game $\{*|*\}$ is a win for the second player but looks quite distinct from 0. How does writing $\{*|*\} = 0$ make any sense? The answer lies in the fact that in CGT, equality is a defined relation. There is a distinction between the *form* of a game and the *value* of a game. Two games can be equal in value yet have different forms. So far we've only defined what equality of value means with respect to zero. We

[2]The casual nature of this remark is perhaps somewhat unjustified, seeing as it's not completely trivial to prove.

shall henceforth use the symbol $\equiv$ to refer to games that are *identical*, that is, equal in form.

The following notation is used to refer to combinations of the four main categories.

$G \geq 0$ –	i.e. $G > 0$ or $G = 0$;	if Right starts then Left wins.
$G \leq 0$ –	i.e. $G < 0$ or $G = 0$;	if Left starts then Right wins.
$G \mathrel{\vert\!\rhd} 0$ –	i.e. $G > 0$ or $G \parallel 0$;	if Left starts then Left wins.
$G \mathrel{\lhd\!\vert} 0$ –	i.e. $G < 0$ or $G \parallel 0$;	if Right starts then Right wins.

These generally turn out to be way more convenient in proofs.[3] Indeed, to prove $G = 0$ we can show that Right wins if Left starts ($G \geq 0$) and Left wins if Right starts ($G \leq 0$), and to prove $G > 0$ we show that Left wins if Left starts ($G \mathrel{\vert\!\rhd} 0$) and Left wins if Right starts ($G \geq 0$), and so on. Proofs of this kind are known as *strategic proofs*, and are a combinatorial game theorist's bread and butter.

Theorem 1.4 (Simplicity theorem, preliminary version)**.** *Let G be a game such that $G^L \mathrel{\lhd\!\vert} 0$ for all Left options G^L and $G^R \mathrel{\vert\!\rhd} 0$ for all Right options G^R. Then $G = 0$.*

Proof. We give a strategic proof. If Left starts and has no legal moves, then Right wins. If Left moves to some position G^L, then Right must have a winning response, since $G^L \mathrel{\lhd\!\vert} 0$. By analogy, if Right starts then Left wins. ■

1.3 Sums of games

We can interweave G and H into a new game by playing them side-by-side. The player must make a move in either G or H, leaving the

[3] In fact, it's easier to formulate rigorous definitions for $\leq$ and $\geq$ and express $>$, $<$, $\parallel$ and $=$ in those terms than to do the opposite. Namely, let $G \geq 0$ if there is no G^R with $G^R \leq 0$, and let $G \leq 0$ if there is no G^L with $G^L \geq 0$.

other game unchanged. The other player may then respond to the move directly, or make a move in the other game. This combination of G and H is called the (direct) sum of a G and H.

Definition 1.5. The direct sum of two games G and H is written $G+H$, given by

$$G + H \coloneqq \{G^L + H, G + H^L \mid G^R + H, G + H^R\}$$

Keep in mind that G^L, G^R, H^L and H^R do not refer to specific elements. Instead, these symbols are understood to cycle through all of their options. The expression $G^L + H$, for instance, generally adds a multitude of different Left options to $G + H$, or sometimes none at all.

Let's calculate some sums, it's only a matter of filling in the definition after all. Because the zero game has no Left or Right options, we quickly find that $0+0 \equiv \{|\} \equiv 0$. We can use this to see that $\{0|\}+0 \equiv \{0+0|\} \equiv \{0|\}$. Indeed, adding zero to things doesn't seem to do much. Adding things to zero seems equally unproductive: $0 + \{0|\} \equiv \{0 + 0|\} \equiv \{0|\}$.

Let's try some non-zero summands, like $\{0|\}+\{0|\} \equiv \{0+\{0|\}, \{0|\}+0 \mid \} \equiv \{\{0|\}|\}$. At last, a new game! Where among the four categories does this game fall? We can find out by just playing the game. If Left starts, they must move to $\{0|\}$, after which a win for Left is guaranteed (remember $\{0|\} > 0$). If Right starts, they lose instantly. Thus $\{\{0|\}|\} > 0$, a guaranteed win for Left. Similarly, $\{|0\} + \{|0\} \equiv \{|\{|0\}\} < 0$.

Another interesting sum is $* + *$. We could of course calculate the result directly, as we did above, but we can also use the fact that addition is nothing more than playing two games at once. Let's say Left starts the game $*+*$ by moving to $0+*$. Right can only respond by moving to $0+0$, after which Left has no legal moves. Had Left started by moving to $*+0$, they would've ended up in the same predicament. If Right starts the game, the situation for them looks equally dire. The first player to move loses, so $* + * = 0$.

1.4 The negative of a game

Addition leads naturally to the construction of a game's additive inverse—its negative.

Definition 1.6. $-G := \{-G^R \mid -G^L\}$

Negating a game corresponds to having the players swap seats, or—for a board game like checkers, chess or go—exchanging the colours of the pieces. So it's seen quite quickly that if $G > 0$ then $-G < 0$; if $G < 0$ then $-G > 0$; if $G \parallel 0$ then $-G \parallel 0$; if $G = 0$ then $-G = 0$ and that negating a game twice does not affect said game.

A sum of the form $G + (-H)$ is called a *difference game*, and may also be written as $G - H$.

Proposition 1.7. *Let G be a game, then $G - G = 0$.*

Proof. We're playing the games G and $-G$ side-by-side. If the first player moves G to one of their options, H say. The second player, presented with $H + (-G)$ can simply move $-G$ to $-H$. Had the first player moved in $-G$ instead, the second player would have mimicked their move in G. By keeping this up, the second player is ensured never to run out of moves. ∎

1.5 Comparing games

We're now equipped to define $<$, $>$, $\parallel$ and $=$ as binary operations.

Definition 1.8. For games G and H

(i) $G > H$ if $G - H > 0$,

(ii) $G < H$ if $G - H < 0$,

(iii) $G \parallel H$ if $G - H \parallel 0$ (we say G is *confused with* H),

(iv) $G = H$ if $G - H = 0$.

Noticing that $G - H \equiv G$ when $H \equiv 0$, we observe that these more general definitions are perfectly compatible with the earlier definitions of > 0, < 0, $\parallel 0$ and $= 0$. It should be clear how $\geq$, $\leq$, $\rhd\!\!\!|$ and $\lhd\!\!|$ are defined analogously. Beware that $\lhd\!\!|$ and $|\!\!\rhd$ are not transitive!

Proposition 1.9. *Let G be any game. Then $G^L \lhd\!\!| \; G$ for all G^L and $G \lhd\!\!| \; G^R$ for all G^R.*

Proof. Right can move the game $G^L - G$ to 0 by moving $-G$ to $-G^L$, winning the game, whereby $G^L \lhd\!\!| \; G$. We have $G \lhd\!\!| \; G^R$ by symmetry. ∎

Now that we've laid the groundwork for CGT, we should confirm that it is solid. That is, whether the operations and relations thus far defined behave as we would expect. The following propositions ensure us of just that. We shall not, however, prove these statements, as this would in my estimation add more verbosity than insight. To do so requires knowledge of *induction* on games, which will be introduced in section 2.1.

Proposition 1.10. *For all games G, H and K we have*

(i) $G = G$ *(put another way: if $G \equiv H$ then $G = H$),*

(ii) If $G = H$ then $H = G$,

(iii) If $G = H$ and $H = K$ then $G = K$.

Whereby $=$ defines an equivalence relation.

Proposition 1.11. *For all games G, H and K we have*

(i) $(G + H) + K \equiv G + (H + K)$,

(ii) $G + H \equiv H + G$,

(iii) If $H = 0$ then $H + G = G + H = G$.

Corollary 1.12. *If $G = G'$ and $H = H'$, then $G + H = G' + H'$. That is to say: addition is well-defined modulo $=$.*

Lemma 1.13. *For games $G \geq 0$ and $H \geq 0$ we have $G + H \geq 0$.*

Proposition 1.14. *For all games G, H and K we have*

(i) $G \geq G$,

(ii) $G = H$ *if and only if* $G \geq H$ *and* $H \geq G$,

(iii) *If* $G \geq H$ *and* $H \geq K$ *then* $G \geq K$.

Whereby $\geq$ defines a partial order.

Proposition 1.15. *If H and K are games such that $H \geq K$, then $G + H \geq G + K$ for any game G.*

The class of all (not necessarily short!) games modulo $=$ is known as **Pg** (partizan games), or also **Ug** (unimpartial games) in older literature. Per the above results, the class **Pg** equipped with the well-defined binary operation $+$, forms a partially ordered abelian group with negation as inversion, neutral element 0 and partial order $\geq$.

2 Numbers

Perhaps the most widely known application of combinatorial game theory has been John Conway's construction of the *surreal numbers*, popularised by Knuth [Knu76]. Although the class of surreal numbers is vast, our focus on short games means we will only require a (comparatively) small portion of them in our analyses.

2.1 Discovering numbers, day by day

Definition 2.1. A *(surreal) number* $x = \{x^L \mid x^R\}$ is a game whose options are all surreal numbers, with $x^L < x^R$ for all x^L and x^R.

Let's construct the class of surreal numbers from the ground up, using the recursive definition step by step, or—using Conway's more poetic language—day by day. On the zeroth day we discover the number $0 \equiv \{|\}$.

On the first day we encounter the candidates $\{0|\}$, $\{|0\}$ and $* \equiv \{0|0\}$. Of these, the first two fit the definition of a surreal number, and we dub them 1 and -1 respectively. The game $*$ is not a number, since $0 \not< 0$.

We've already seen how these three first numbers compare to one another, namely as $-1 < 0 < 1$.

On the second day, we use the numbers created so far to generate

- $\{1|\}, \{1,0|\}, \{1,-1|\}, \{1,0,-1|\}$,
- $\{0|1\}, \{-1,0|1\}$,
- $\{|1\}, \{-1|1\}, \{-1|\}$,

- $\{-1|0\}\,\{-1|0,1\}$,
- $\{|-1\}, \{|-1,0\}, \{|-1,1\}, \{|-1,0,1\}$.

You may suspect the numbers grouped together are somehow related, and indeed these all turn out to be equal in value. Verifying this means playing an awful lot of difference games, and while I do encourage you to verify at least one equality as a helpful exercise, I hope you agree that our time can be spent more effectively. Don't worry, by the way, this was the last such exhaustive list!

Theorem 1.4 tells us that $0 = \{|1\} = \{-1|1\} = \{-1|\}$, so these are merely different forms of a number we've seen before. Let's give the others some suggestive names, say

- $2 := \{1|\}$,
- $\frac{1}{2} := \{0|1\}$,
- $-\frac{1}{2} := \{-1|0\}$,
- $-2 := \{|-1\}$.

When we were familiarising ourselves with the definition of $+$, we saw that $\{0|\} + \{0|\} \equiv \{\{0|\}|\}$. Using the new names, this simply says $1 + 1 \equiv 2$.

The ordering of the numbers born thus far is $-2 < -1 < -\frac{1}{2} < 0 < \frac{1}{2} < 1 < 2$. Again, verifying one of these (e.g. $\frac{1}{2} > 0$) can be a helpful exercise.

The recursive definition of numbers (and games, for that matter) allows us to prove theorems about them by *induction*. That is, if a theorem can be shown to hold for a game G by assuming that it holds for all G^L and G^R, then the theorem holds universally [Con76, p. 64]. We can use this to prove a proposition that may already be intuitive after studying the ordering of numbers on day two.

Proposition 2.2. *Let x be a number. Then $x^L < x < x^R$ for all x^L and x^R.*

Proof. We prove $x^L < x$ by induction. Suppose that for all x^L we have $x^{LL} < x^L$. If Left starts the game $x - x^L$ they can win trivially by moving

x to x^L. If Right starts, then a move from x to any x^R is no good, since $x^R > x^L$ (by definition). So Right must move $-x^L$ to some $-x^{LL}$, but since $x^L > x^{LL}$ (by assumption), Left can win by moving x to x^L. ■

Notice the remarkable parallel between Conway's surreal numbers and the construction of the real numbers according to Dedekind! Which of the two one sees as more fundamental is a matter of perspective.

Corollary 2.3. *Between any two numbers $x < y$ lies the distinct number $\{x|y\}$. Furthermore, if X is a set of numbers, then the number $l = \{X|\}$ is strictly greater than all numbers in X, and the number $r = \{|X\}$ strictly smaller.*

Many of the numbers constructed on the second day were different forms of numbers we'd already encountered, and this holds true for all subsequent days. The following result allows us to more easily recognise when a number is equivalent to another, or even whether a given game is actually a number in disguise.[1]

Theorem 2.4 (Simplicity theorem [Con76, Thm. 11])**.** *Let G be a game. Suppose that some number x satisfies $G^L \lhd\!\!| \; x \lhd\!\!| \; G^R$ for all G^L and G^R, but that no option of x satisfies the same condition. Then $G = x$.*

Proof. Consider the games $(G - x)^L$. These have one of the following forms.

- $G^L - x$. Then $G^L - x \lhd\!\!| \; 0$ by supposition.
- $G - x^R$. For any G^L we have $G^L \lhd\!\!| \; x < x^R$, whereby $G^L \lhd\!\!| \; x^R$. Then x^R may not satisfy $x^R \lhd\!\!| \; G^R$ for all G^R, guaranteeing the existence of some Right option G^{R_0} with $G^{R_0} \leq x^R$. If Right then moves $G - x^R$ to $G^{R_0} - x^R \leq 0$, they win the game. Thus $G - x^R \lhd\!\!| \; 0$.

[1]The fact that there are games that do not satisfy the conditions of Definition 2.1, yet are still equal to a number *in value* (e.g. $\{*|*\} = 0$), can be a bit iffy. We permit ourselves to say that such games *are* numbers, and conversely only say that a game *isn't* a number if it is not equal in value to *any* number. However, when speaking of "a number" as in "let x be a number", we do require adherence to Definition 2.1.

Similarly it can be shown that all $(G-x)^R \vartriangleright 0$. Applying Theorem 1.4 gives $G-x=0$. ■

This result regularly allows us take shortcuts, for example when calculating sums. We initially use the definition of direct sums to find $\frac{1}{2}+\frac{1}{2} = \{0|1\}+\{0|1\} = \{0+\frac{1}{2}\frac{1}{2}+0 \mid 1+\frac{1}{2}, \frac{1}{2}+1\} = \{\frac{1}{2} \mid 1+\frac{1}{2}\}$. Since $\frac{1}{2} > 0$ we have $1+\frac{1}{2} > 1$. Then Theorem 2.4 with $x=1$ gives $\{\frac{1}{2} \mid 1+\frac{1}{2}\} = 1$. Great!

I do not see a better way to conclude the construction of numbers than by citing Conway verbatim:

> [If] L and R are sets of numbers chosen from those we already have, then since we suspect these numbers are totally ordered, in any expression $x = \{x^L \mid x^R\}$ we need only consider the greatest x^L (if any) and the least x^R (ditto). This gives us for the next "day" only the numbers
>
> $$0 < \{0|\tfrac{1}{2}\} < \tfrac{1}{2} < 1 < \{1|2\} < 2 < \{2|\}$$
>
> and their negatives. What are the proper names for these numbers? We suspect that $\{2|\} = 3$, and indeed we can verify that
>
> $$1+1+1 = \{0+1+1, 1+0+1, 1+1+0 \mid\} = \{2|\}$$
>
> The equation $\{1|2\}$ is almost as easy to guess and verify. So we shall make $1\frac{1}{2}$ a permanent name for this number.
>
> The two likely guesses for $\{0|\frac{1}{2}\}$ are $\frac{1}{3}$ and $\frac{1}{4}$. If anything, the first might seem the better guess, since otherwise it's hard to see what $\frac{1}{3}$ will be. But in fact it turns out that $\{0|\frac{1}{2}\}$ is $\frac{1}{4}$—at least we can verify that twice this number is $\frac{1}{2}$. In a similar way, $\{\frac{1}{2}|1\}$ turns out to be $\frac{3}{4}$ rather than $\frac{2}{3}$.
>
> It is now easy to guess the pattern for the numbers which take only finitely much work to define. ...[Considering

> these numbers by day of birth, they] seem to form a tree [as shown in figure 2.1]. Each node of the tree has two "children", namely the first later numbers born just to the left and right of it. We guess that on the n'th day the extreme numbers to be born are n and $-n$, and that each other number is the arithmetic mean of the numbers to the left and right of it. Happily, of course, this turns out to be the case. Supposing all of this, we know all numbers born on finite days. [Con76, pp. 10–12]

The surreal numbers that are born within a finite number of days are the dyadic rationals, i.e. rational numbers whose denominator is a power of two. All short games that are numbers are therefore dyadic rationals. Beyond the dyadic rationals we can build up the reals as Dedekind cuts in the dyadic rationals, and even construct infinite and infinitesimal numbers. For instance

$$\omega := \{0, 1, 2 \ldots \mid \}$$

is a surreal number greater than any real number, and

$$\tfrac{1}{\omega} := \{0 \mid 1, \tfrac{1}{2}, \tfrac{1}{4} \ldots\}$$

is a positive surreal number smaller than any positive real number.

The classes $\mathbb{D}$ of dyadic rationals and **No** of all surreal numbers, form totally ordered groups with group operation $+$ and order $<$. A proof of this (for the class **No**) is given by theorems 5 and 6 in Conway [Con76]. In fact, these classes can be equipped with a multiplication operation turning them into a totally ordered field. Beyond the fact that such a multiplication operation exists (and that its behaviour on the dyadic rationals and real numbers behaves as we would expect—of course), we need not know much about it for our purposes.

2.2 Interplay between games and numbers

We can use our knowledge of numbers to play games more effectively.

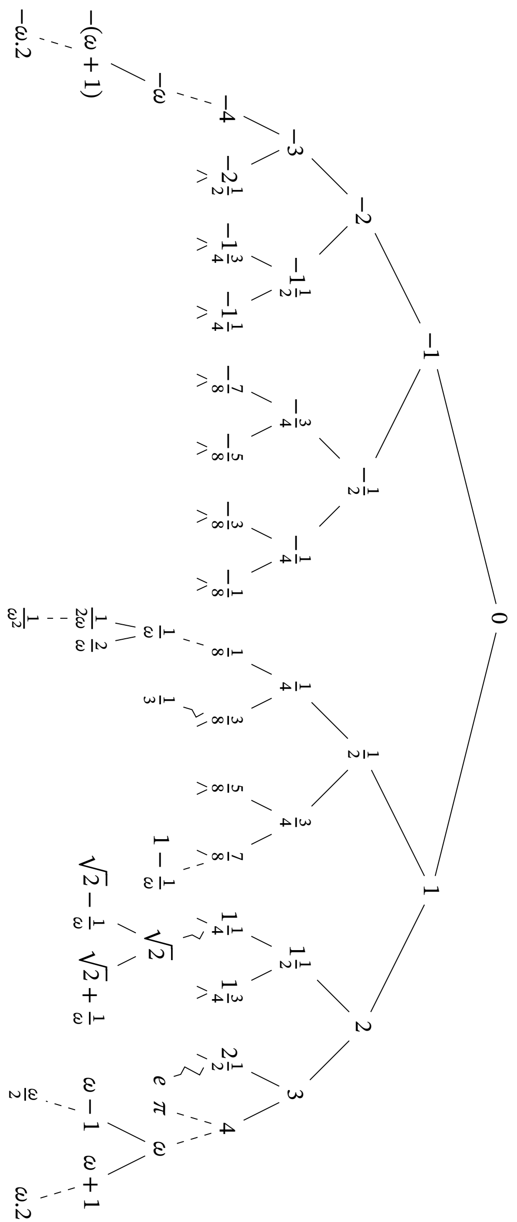


Figure 2.1: The first few surreal numbers. Adapted from Conway [Con76, fig. 0].

Theorem 2.5 (Number avoidance theorem)**.** *Suppose x is a number and G is not. When playing the game $G + x$, we need only consider moves in G.*

Formally put: if $G + x \mathrel{|\!\!\rhd} 0$, then there is some G^L with $G^L + x \geq 0$.

Proof. We proceed by induction on x.

Left starts $G + x$ and has a winning move. If this winning move is of the form $G^L + x \geq 0$, then we're done. Otherwise, it is of the form $G + x^L \geq 0$. Since G isn't a number it is certainly $\neq -x^L$, so $G + x^L > 0$. By induction there is then some G^L with $G^L + x > G^L + x^L \geq 0$. ∎

An *ordinal number* is a number that can be expressed in a form without Right options. The ordinals born within a finite number of days are simply the natural numbers.

Theorem 2.6 (The Archimedean principle [Con76, Thm. 55])**.** *For any game G (not necessarily short!) there exists an ordinal number α such that $-\alpha < G < \alpha$.*

Proof. We prove by induction on G. Let α_L be the numbers such that $-\alpha_L < G^L < \alpha_L$, and α_R similar for G^R. We may assume these numbers have no Right options. We construct numbers l, r and α such that we have $\alpha > l > \alpha_L$ and $\alpha > r > \alpha_R$ for all α_L and α_R. By Corollary 2.3 we can use $l = \{\alpha_L|\}$, $r = \{\alpha_R|\}$, $\alpha = \{l, r|\}$.

Consider the game $G+\alpha$. Left can start by moving α to r. Right's only replies (if any) are moves from G to G^R, but $G^R + r > G^R + \alpha_R > 0$, so Left wins. If Right starts, they lose for the same reason. Thus $-\alpha < G$. The argument for $G < \alpha$ is analogous. ∎

For short games G, it suffices to take $\alpha =$ the total number of positions of G, plus one.

The Archimedean principle can be used to derive a "number recognition" mechanism in the spirit of Theorem 2.4.

Proposition 2.7. *If a game has no Left options or no Right options, then it is a number.*

Proof. Let G and α as in Theorem 2.6, and suppose G is not a number. Then by the number avoidance theorem, $G < \alpha$ guarantees the existence of a Right option of G, and $-\alpha < G$ guarantees the existence of a Left option of G. ■

The following is often helpful. Its proof relies on a slightly stronger version of the number avoidance theorem (see [ANW07, Thm. 6.18]), and is omitted here.

Theorem 2.8 (Number translation theorem)**.** *Let G be a game that isn't a number. For any number x, we have $G + x = \{G^L + x \mid G^R + x\}$.*

3 Go

The purpose of this chapter is to lay the groundwork for analysing Go, as well as to see the theory from the preceding chapters in action.[1] In lieu of adopting any standardised rule set used in Go (Japanese, Chinese, AGA, Tromp-Taylor, etc.), which don't necessarily lend themselves naturally to CGT-type analysis, we shall only establish a few very basic principles.[2] These will turn out to capture enough of the game's essence that we'll be able to correctly analyse many frequently occurring endgame positions.

Principle 1. A player may move by placing a stone on the board, in ways permitted by the "regular" rules of Go. The basics of placing a stone and capturing your opponent's stones are unaltered. We refer to the players Left and Right as *Black* and *White*.

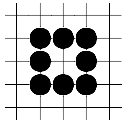
Principle 2. We do not allow a game to loop. Positions mustn't ever repeat (the *ko* rule, *positional superko* specifically). Passing and single-stone suicides are forbidden.

Instead of looking at entire board positions, we shall usually analyse only portions of the goban, and use the convention that the stones framing our diagrams are safe—i.e. unconditionally alive, immortal.

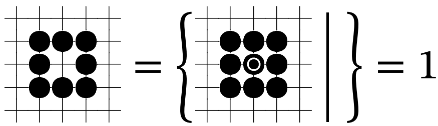
[1] In some sense this chapter could be seen as the bridge between Conway [Con76] and Berlekamp and Wolfe [BW94], providing a justification for the methods used in the latter source on a more fundamental level.

[2] Ways to model these different rule sets are given in Berlekamp and Wolfe [BW94, appendix B]. Although rather elegant in a way, these solutions still end up feeling a bit contrived, for they are certainly more complex than the way the rules of Go are usually expressed in natural language. I cannot help but wonder whether they can actually provide insight, or whether it's all just an exercise in formalisation for formalisation's sake.

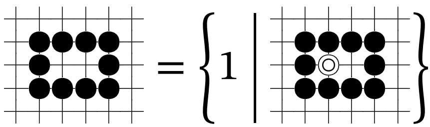
Let's start simple and use our principles to derive the value of the following rather basic position.

$G =$ 

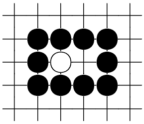
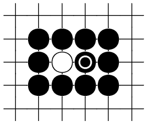
It's clear that Black has exactly one move and White has none, since suicide and passing are forbidden. But after Black's move neither player has any moves left, so the final position is just the zero game. Then G is the game $\{0 \mid \}$, which you may remember also goes by the name "1".



Already we start hoping that the theory of numbers discussed in chapter 2 might correspond naturally to a Go player's concept of territory. Let's work out a similar position with two empty intersections. In such a position, both players seem to have two moves. By symmetry we need only consider one of each. Black can move to obtain the same one-intersection position which we earlier saw to be the number 1, and White can make only the rather silly looking move of simply giving up a stone. But perhaps we shouldn't be so unkind, we forbade them to pass after all.



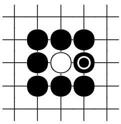
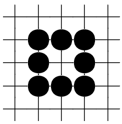
Now we need to work out the position after White ◎. White has no moves because the board may not repeat, and Black can move to capture the White stone.

 $= \Bigg\{$  $\Bigg| \; \Bigg\}$

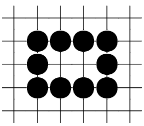
Now, because any position where neither player can move is the zero game irrespective of how many stones of either colour are in the area, we cannot naturally retrieve any kind of area scoring from our basic principles. We must therefore give Black some kind of compensation for capturing a stone.

Principle 3. Captured stones of the opponent's colour become a player's *prisoners*. A player may "pass" by giving up one of their prisoners, returning the stone to the pot.

This qualified passing right corresponds exactly to our earlier definition of the integers as games. Namely, if Black has $n > 0$ prisoners, that number corresponds to the game $\{n - 1 \mid \}$, wherein Black's allowed to move ("pass") by decreasing their number (of prisoners), and White has no moves. Prisoners taken by White are negative integers. These numbers are added to the go position.

 $=$  $+ 1 = 2$

We now see our two-intersection example to be

 $= \{1 \mid 3\} = 2$

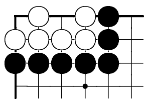
Note that this localised approach to Go requires that a game has indeed "broken up" into fully independent subgames. This means we're forced to disregard positions where there's a ko on the board, a rather limiting factor indeed!

Theorem 3.1 (Principle of settled territory). *Let G be an isolated area of the Go board that is surrounded by unconditionally alive Black groups, and in which White cannot live or make a ko. Suppose G contains n empty intersections and m White stones. Then $G = n + 2m$.*

Proof. Consider $G-(n+2m)$, Black to move. Black *must* make a move in G, having no options in $-(n+2m)$. How many moves can Black make in succession, if White never responds? They can fill in the n empty intersections, after which all m White stones are captured. Then they can fill in those m new intersections, and pass another m times. Black can make $n + 2m$ moves in a row. So if White just keeps increasing $-(n + 2m)$ by 1 each move, Black is the first to run out of options.

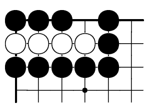
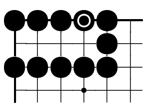
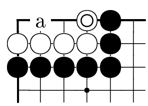
But if White's to move first, their strategy can't be simply to ignore G and increase $-(n + 2m)$, since Black can move $n + 2m$ times in a row. What's more, since White cannot make life, Black always has a response to any White move in G. And even if White manages to capture a certain number of Black stones p, Black just gets to fill p additional intersections. So Black can always move last. ■

The fact that—in general—we disallow passing has its drawbacks, and means our approach is practically of no use with regard to life and death problems. For example, consider the following position.

$x =$ 

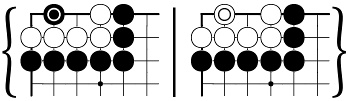
We'd obviously like to have $x = -2$, But since White can't pass if it's their turn, they must fill in one of their eyes, so that $x = \{\ |\ 14\} = 0$.

It is interesting to note however, that in some contexts the exact value of x doesn't really matter. Take for instance

$G =$  $= \Big\{$  $+ 4 \ \Big|$ 

$- 3 \Big\}$

Where we find



$$= \{12 \mid x\}$$

Whereby $G = \{8 \mid \{9 \mid x - 3\}\} = 9$ (theorems 2.8 and 2.4), irrespective of whether $x = -2$ or $x = 0$.

So while the simple and principled Go rule set we've presented so far lends itself readily to CGT analysis, it can't cope with ko, seki and life & death situations very well. It's helpful to formally establish which types of Go positions we shall generally disregard.

Definition 3.2. A Go position is *elementary* if, when completely played out in any environment, every point on the board is either occupied by a live stone, or becomes territory for a player.

4 Simplifying games

The main purpose of this chapter is to arrive at a canonical way of representing short games, which allow us to more easily spot equality between them. Laying out these results is almost obligatory for any work on applied combinatorial game theory. I essentially follow Conway [Con76] in my presentation, making explicit some of the proofs left there as an exercise to the reader.

4.1 Dominated and reversible options

The terms dominated and reversible option are more daunting than the concepts they describe. Plainly put: if two Left options G^{L_0} and G^{L_1} of G are comparable with one another, then G^{L_1} is said to be *dominated* by G^{L_0} if it is *worse* than G^{L_0} (that is to say $G^{L_1} \leq G^{L_0}$). The Right option G^{R_1} is similarly dominated by G^{R_0} if $G^{R_1} \geq G^{R_0}$.

A Left option is said to be *reversible* if it has a one move refutation. Specifically, the option G^{L_0} is reversible if there is a Right option $G^{L_0 R_0}$ such that $G^{L_0 R_0} \leq G$. The analogous definition for Right options is obvious.

Let's see what makes these concepts powerful before looking at an example.

Lemma 4.1. *Suppose $H \lhd\!\!| \; G$. Let $G' = \{H, G^L \mid G^R\}$, then $G' = G$.*

Proof. It's easy to see that $G' - G \geq 0$, since Left can use a mirroring strategy. If Left starts $G' - G$ by moving to $H - G$, then Right wins, since $H - G \lhd\!\!| \; 0$. Any other Left opening can just be mirrored by Right. Thus $G' - G \leq 0$, whereby $G' = G$. ■

Theorem 4.2 ([Con76, Thm. 68]). *We may do the following without affecting the value of G.*

(i) Delete any dominated option.

(ii) Reverse any reversible option. That is, if Left option G^{L_0} is reversible through $G^{L_0R_0}$, to replace G^{L_0} with the Left options of $G^{L_0R_0}$. For reversible Right options similar.

Proof. We only consider dominated/reversible Left options. Analogous proofs work for Right options. Recall that for any game G we have $G^L \lhd| \, G \lhd| \, G^R$ for all options G^L, G^R (Proposition 1.9).

(i) Suppose $G^{L_1} \leq G^{L_0}$ and let G' be as G but with G^{L_1} deleted. By construction $G = \{G'^L, G^{L_1} \mid G'^R\}$ and since $G^{L_1} \lhd| \, G$, Lemma 4.1 gives $G' = G$.

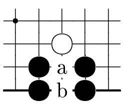
(ii) Let G' be as G but with G^{L_0} reversed. Consider $G' - G$.

If Left opens by moving G' to $G^{L_0R_0L}$, then since $G^{L_0R_0L} \lhd| \, G^{L_0R_0}$ and $G^{L_0R_0} \leq G$ (by reversibility), we have $G^{L_0R_0L} \lhd| \, G$, making this a bad move for Left.

If Right opens by moving $-G$ to $-G^{L_0}$, then Left moves from $-G^{L_0}$ to $-G^{L_0R_0}$. Thereafter Left can mirror in G' any Right move in $-G^{L_0R_0}$, so Right must move G' to some G^R, leaving the position $G^R - G^{L_0R_0} \geq G^R - G \, |\rhd \, 0$, which is bad for Right.

All other opening moves by either player are easily refuted through mirroring. Thus $G' = G$. ∎

Now consider the following Go position.

$G =$  $= \{B_\mathrm{a}, B_\mathrm{b} \mid W_\mathrm{a}, W_\mathrm{b}\}$

Where B_a and B_b represent Black placing a stone at a and b respectively, and W_a, W_b similar for White. It's obvious to any Go player that

B_b and W_b are bad moves, and indeed we can demonstrate this using the idea of dominated and reversible options.

First observe that $B_a = 1$, $B_b = W_a = *$ and $W_b = \{2 \mid 0\}$, so that $G = \{1, * \mid *, \{2 \mid 0\}\}$. It should be clear straight away that $* < 1$, meaning B_b is dominated by B_a and can hence be removed. We could use the same logic to remove W_b, since it doesn't take much effort to verify that $* < \{2 \mid 0\}$. But it's perhaps more immediately apparent that W_b is reversible through 2. Deleting B_b, and reversing W_b by replacing it with the Right options of 2—i.e. nothing—gives $G = \{1 \mid *\}$.

4.2 Canonical form

For any short game, deleting dominated options and reversing reversible options can be done inductively, yielding a game of equal value without dominated options, and without reversible options. This is called its *canonical form* (also *simplest form*), and is unique by the following theorem.

Theorem 4.3 ([Con76, Thm. 79]). *If games G and H have neither dominated nor reversible options, then $G = H$ if and only if the options of G and H are equal.*

Proof. If G and H are equal, consider the difference game $G - H$. Say Right moves to $G^R - H$, then Left must have a reply in either G^R or H. If Left has G^{RL} as a reply, then $G^{RL} \geq H = G$, making G^{RL} reversible, a contradiction. Thus left has some $-H^R$ as a reply, so that for each G^R we have some H^R with $G^R \geq H^R$.

Had we started with a move by Left to some $G - H^{R'}$, we would have similarly found a reply $G^{R'} - H^{R'}$. In general, then, we have for every H^R a G^R such that $H^R \geq G^R$.

Combining these two, we can find $G^R \geq H^R \geq G^{R'}$ for any H^R. Because the game G contains no dominated options, we find $G^R = G^{R'} = H^R$.

The argument for the Left options of G and H is analogous. ∎

5 Temperature

In this chapter we consider only short games and dyadic rational numbers.

5.1 Stops

While the theory of surreal numbers is certainly interesting, the games these numbers represent are, in some sense, quite dull. Not only is there no way either player can affect the outcome of such a game (except in the zero game, in which the only way to win is not to play at all), but any move a player makes actually *worsens* their position. So if while playing a game either player moves to a position that is a number, we can agree to end the game there. Such a number is said to be a *stopping position* of the game.

Definition 5.1. When a short game G is played in isolation, the best stopping positions attainable for Left and Right are given by the game's *Left stop* $\mathbf{LS}(G)$ and *Right stop* $\mathbf{RS}(G)$ respectively. These are defined as follows.

$$\mathbf{LS}(G) \coloneqq \begin{cases} G & \text{if } G \text{ is a number} \\ \max_{G^L} \mathbf{RS}(G^L) & \text{otherwise} \end{cases}$$

$$\mathbf{RS}(G) \coloneqq \begin{cases} G & \text{if } G \text{ is a number} \\ \min_{G^R} \mathbf{LS}(G^R) & \text{otherwise} \end{cases}$$

Because any game without Left options or Right options is a number (Proposition 2.7), all is good and proper.

Proposition 5.2. *Let G be a short game and x a number. Then the following are true.*[1]

(i) Suppose $x \neq \mathbf{LS}(G)$. Then $x < \mathbf{LS}(G) \iff x \lhd\!\!| \; G$.

(ii) Suppose $x \neq \mathbf{RS}(G)$. Then $x > \mathbf{RS}(G) \iff x \;|\!\!\rhd G$.

Proof. We assume that G isn't a number, as otherwise the statement is trivial. The strategy is to prove (i) and (ii) together, inductively. Since the argumentation for (ii) is analogous to (i), we shall only prove the latter explicitly.

Suppose $x < \mathbf{LS}(G)$. Then, by definition of $\mathbf{LS}(G)$ and since G isn't a number, there is a Left option G^L with $x < \mathbf{RS}(G^L)$. By induction, (ii) gives $x < G^L$, meaning Left can win $G - x$ by moving G to G^L. Thus $G - x \;|\!\!\rhd 0$, whereby $x \lhd\!\!| \; G$.

Suppose $x \lhd\!\!| \; G$. The number avoidance theorem gives us a Left option G^L with $x \leq G^L$. Then (ii) gives $x \leq \mathbf{RS}(G^L)$, so certainly $x \leq \mathbf{LS}(G)$. ■

Corollary 5.3. *Let G and H be games.*

(i) If $G = H$ then $\mathbf{LS}(G) = \mathbf{LS}(H)$ and $\mathbf{RS}(G) = \mathbf{RS}(H)$.

(ii) $\mathbf{LS}(G) \geq \mathbf{RS}(G)$.

Proof. Both results can be shown by supposition of the contrary. For (i) suppose that $\mathbf{LS}(G) \neq \mathbf{LS}(H)$, without loss of generality assume $\mathbf{LS}(G) > \mathbf{LS}(H)$. For (ii) suppose $\mathbf{LS}(G) < \mathbf{RS}(G)$. Picking a number in between and invoking Proposition 5.2 quickly leads to a contradiction. ■

Corollary 5.4. *Let G be a game and x a number.*

[1] We could have used this as the defining property for Left and Right stops, which is what Conway does for his definition of Left and Right *sections* [Con76, pp. 98–99]. Not only does this extend the concept to long (infinite) games, but the subtle difference in definition means the equivalences of Proposition 5.2 hold even without the $x \neq \ldots$ suppositions. Alternatively one can use *adorned stops* [ANW07, p. 161].

(i) *If* $x < \mathbf{RS}(G)$ *then* $x < G$.

(ii) *If* $\mathbf{RS}(G) < x < \mathbf{LS}(G)$ *then* $x \parallel G$.

(iii) *If* $x > \mathbf{LS}(G)$ *then* $x > G$.

Proposition 5.5. *Let G be a game and x a number. We have* $\mathbf{LS}(G+x) = \mathbf{LS}(G) + x$ *and* $\mathbf{RS}(G + x) = \mathbf{RS}(G) + x$.

Proof. This follows immediately from Theorem 2.8. ■

Proposition 5.6. *Let G be a game that has both Left and Right options. Defining* $L \coloneqq \max_{G^L} \mathbf{RS}(G^L)$ *and* $R \coloneqq \min_{G^R} \mathbf{LS}(G^R)$, *we have*

(i) *If* $L < R$ *then G is a number;*

(ii) *Otherwise, if* $L \geq R$ *then* $\mathbf{LS}(G) = L$ *and* $\mathbf{RS}(G) = R$.

This result is often useful when calculating a game's Left and Right stops, since Definition 5.1 demands we be able to recognise when a game is a number. Take for instance $G = \{0, \{2|\frac{1}{2}\} \mid 1\}$. Naively we might find its Right stop to be 1 and its Left stop the maximum between $\mathbf{RS}(0) = 0$ and $\mathbf{RS}(\{2|\frac{1}{2}\}) = \frac{1}{2}$, namely $\mathbf{LS}(G) = \frac{1}{2}$. But then $\mathbf{LS}(G) = \frac{1}{2} < 1 = \mathbf{RS}(G)$, contradicting Corollary 5.3(ii)! The resolution is given by Corollary 5.6(i), which states G is actually a number in disguise. By putting G in canonical form ($\{2|\frac{1}{2}\}$ is reversible through $\frac{1}{2}$) we find $G = \frac{1}{2}$, so that $\mathbf{LS}(G) = \mathbf{RS}(G) = \frac{1}{2}$.

Proof. Part (i) essentially follows immediately from Corollary 5.3(ii) and Definition 5.1.

Suppose $L \geq R$, and let x be a number $> L$. Then there must be some G^R with $\mathbf{LS}(G^R) < x$, so that $G^R < x$, hence $G \lhd\!| \; x$. Similarly, all G^L satisfy $\mathbf{RS}(G^L) < x$, thus also $G^L \lhd\!| \; x$ whereby $G \leq x$. We conclude that $G < x$. Analogously, $G > x$ for all numbers $x < R$. Now if x is a number between R and L, then there is some G^L with $\mathbf{RS}(G^L) > x$, hence $G^L > x$ and $G \; |\!\rhd x$, and there is some G^R showing that $G \lhd\!| \; x$. Thus $G \parallel x$.

In light of Corollary 5.4, we now know that necessarily $\mathbf{LS}(G) = L$ and $\mathbf{RS}(G) = R$. ■

The discrepancy between a game's Left and Right stop corresponds to what's at stake, and therefore how *eager* both players are to go first. In qualitative terms, we call a game G

cold if it is a number;

tepid if it is not a number, but $\mathbf{LS}(G) = \mathbf{RS}(G)$, e.g. $G = *$;

hot if $\mathbf{LS}(G) > \mathbf{RS}(G)$, e.g. $G = \{5 \mid -2\}$ [ANW07, Def. 6.21].

Continuing this thermal analogy, if $*$ is tepid and $\{1 \mid -1\}$ hot, then surely $\{1000 \mid -1000\}$ is even hotter! This anticipates the definition of *temperature*, which is a way of quantifying the 'heat' of a game.

5.2 Cooling

A simple way to get rid of a game's heat is by imposing a tax on playing, thus decreasing the stakes. This allows us to examine 1) at which "tax rate" a game is no longer hot (the game's temperature), and 2) to which numeric value the game settles down after taxation (the game's mean value). In certain classes of hot games, notably one-point Go endgames, cooling by some amount preserves a game's structure and strategy, while making them easier to analyse.

Definition 5.7. For any game G and number $t \geq 0$, the game G_t represents the game G *cooled* by t degrees. Let $G(t) = \left\{ {G^L}_t - t \mid {G^R}_t + t \right\}$. We set $G_t = G(t)$, unless for some $\tau < t$ we have $\mathbf{LS}(G(\tau)) = \mathbf{RS}(G(\tau))$. In that case we call the lowest[2] such τ the *temperature* $t(G)$ of G, and call $\mathbf{LS}(G(t(G)))$ the *mean value* $m(G)$ of G. We set $G_t = m(G)$ for all $t > t(G)$, and say G *freezes* to $m(G)$.

Observe that a game G is cold or tepid if and only if $t(G) = 0$.

Many important results about cooling and temperature are conventionally derived through *thermography*—that is, by examinging the geometry of the graphs of $\mathbf{LS}(G_t)$ and $\mathbf{RS}(G_t)$. For our purposes, however, thermography will be of little use beyond one very important theorem.

Theorem 5.8. *Let G be a game. There is a number $r = 1/2^i$ so that if we take any $\delta = 1/2^j \leq r$, the following are true whenever t is a non-negative integer multiple of δ.*

(i) Both $\mathbf{LS}(G_t)$ and $\mathbf{RS}(G_t)$ are integer multiples of δ.

(ii) We have either

(a) $\mathbf{LS}(G_{t+x}) = \mathbf{LS}(G_t)$ for all non-negative $x \leq \delta$ or

(b) $\mathbf{LS}(G_{t+x}) = \mathbf{LS}(G_t) - x$ for all non-negative $x \leq \delta$.

(iii) Similarly, either

(c) $\mathbf{RS}(G_{t+x}) = \mathbf{RS}(G_t)$ for all non-negative $x \leq \delta$ or

(d) $\mathbf{RS}(G_{t+x}) = \mathbf{RS}(G_t) + x$ for all non-negative $x \leq \delta$.

Compared to the geometric approach to thermography—for which I'd have to introduce some matters which might lead us astray from the main point—the wording here is deliberately quite abstract.[3] The key takeaway is that $\mathbf{LS}(G_t)$ and $\mathbf{RS}(G_t)$ are (as functions of t) non-increasing and non-decreasing respectively, and have a somewhat "discrete" nature. That is, they are given piecewise on intervals of uniform size, as functions that are constant or have slope ± 1.

Before I give a proof, let's look at two immediate consequences that are often more useful in practice.

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[2]The phrase "the lowest such τ" is a bit sneaky of course, as it contains an assertion—namely that if it is non-empty, the set $\{t \in \mathbb{D}_{\geq 0} : \mathbf{LS}(G(t)) = \mathbf{RS}(G(t))\}$ has a minimal element. As is tradition in texts on combinatorial game theory, we proceed with the definition as if it were consistent.

[3]Compare [Con76, Thm. 61].

Theorem 5.9. *Let G be a game and $t \geq 0$ a number.*

(i) $\mathbf{LS}(G_t) \leq \mathbf{LS}(G) \leq \mathbf{LS}(G_t) + t$

(ii) $\mathbf{RS}(G_t) \geq \mathbf{RS}(G) \geq \mathbf{RS}(G_t) - t$

Corollary 5.10. *For all games G and numbers $t \geq 0$ we have*

(i) $t(G) \geq \frac{1}{2}(\mathbf{LS}(G) - \mathbf{RS}(G))$

(ii) $\mathbf{RS}(G) \leq m(G) \leq \mathbf{LS}(G)$

Proof of Theorem 5.8. If G is a number, $n/2^i$ say, then $r = 1/2^i$ clearly has the desired properties.

Otherwise, suppose the result is true for each G^L and G^R and we have associated numbers r_L and r_R, respecitvely. Now let r be the lowest number among all r_L and r_R, divided by two. Fix some $\delta = 1/2^j \leq r$. Define

$$L_u = \max_{G^L} \mathbf{RS}({G^L}_u) - u$$
$$R_u = \min_{G^R} \mathbf{LS}({G^R}_u) + u$$

and observe that, by Corollary 5.6(ii) and Proposition 5.5, we have $\mathbf{LS}(G_u) = L_u$ and $\mathbf{RS}(G_u) = R_u$ whenever $L_u \geq R_u$. Also note that if $t \geq 0$ is an integer multiple of 2δ, so are L_t and R_t. Fix t to be just such a multiple.

If $t \geq t(G)$ then $\mathbf{LS}(G_{t+x}) = \mathbf{RS}(G_{t+x}) = m(G)$ for all $x \geq 0$, so that (a) and (c) are satisfied. Now suppose that $t < t(G)$, so that $L_t > R_t$.

Consider the $G^{L'}$ for which $\mathbf{RS}({G^{L'}}_t) = \max_{G^L} \mathbf{RS}({G^L}_t)$. If any such $G^{L'}$ is of type (d) on the interval from t to $t + 2\delta$, then clearly $L_{t+x} = L_t$ for all non-negative $x \leq 2\delta$.

If, on the other hand, all $G^{L'}$ satisfy (c), then write $\mathbf{RS}({G^{L'}}_t) = n \cdot 2\delta$ for some integer n. Any G^L of type (d) on the interval from t to $t + 2\delta$ must then have $\mathbf{RS}({G^L}_t) \leq (n-1) \cdot 2\delta$, so that $\mathbf{RS}({G^L}_{t+x}) \leq n \cdot 2\delta$ for all non-negative $x \leq 2\delta$. Thus $L_{t+x} = L_t - x$ for all non-negative $x \leq 2\delta$, in this case.

We can obtain analogous results for R_{t+x}. Since L_t and R_t are multiples of 2δ and $L_t > R_t$, we must either have $L_t \geq R_t + 4\delta$ or $L_t = R_t + 2\delta$. In the first case, it is possible to have $L_{t+x} \leq R_{t+x}$ for non-negative $x \leq 2\delta$ only at $x = 2\delta$, since L_{t+x} and R_{t+x} have slope at least -1 and at most 1, respectively. If $L_t = R_t + 2\delta$ then L_{t+x} and R_{t+x} may also intersect at $x = \delta$. When this crossover occurs, we've reached the freezing point of G, whereafter the cooled game and its Left and Right stops remain constant. We conclude that δ satisfies (ii) and (iii).

Since $L_0 = \mathbf{LS}(G)$ and $R_0 = \mathbf{RS}(G)$ are integer multiples of δ, (i) follows immediately from (ii) and (iii). ■

Proposition 5.11. *Every short game G has $t(G) < +\infty$.*

Proof. By induction. Suppose G isn't a number (all numbers have temperature zero) and that we have $t(G^L) < +\infty$ and $t(G^R) < +\infty$ for all G^L and G^R. Since each option has a finite temperature, they all have a finite mean value as well. Take some positive real number T strictly greater than all $t(G^L)$, $t(G^R)$, $m(G^L)$ and $-m(G^R)$. Then by Theorem 1.4

$$G(T) = \{m(G^L) - T \mid m(G^R) + T\} = 0$$

Thus $\mathbf{LS}(G(T)) = \mathbf{RS}(G(T))$ and we conclude $t(G) \leq T$. (I propose calling this method "freezing to absolute zero".) ■

It is very difficult to give a more useful upper bound of a short game's temperature, as a game's temperature can vary quite a bit during play. Take for instance the tepid game $G = \{0 \mid \{0 \mid -2\}\}$, where Right's only move increases the temperature from 0 to 1.

It immediate follows from Theorem 5.8 and Proposition 5.11 that the temperature of a game is always a dyadic rational number, whereby we've finally established (inductively!) the consistency of Definition 5.7.

Theorem 5.12. *For all games G, H and numbers $t, u \geq 0$ the following are true.*

(i) $(G + H)_t = G_t + H_t$

(ii) $G_{t+u} = (G_t)_u$

For a proof, see Conway [Con76, pp. 106–107] or Albert, Nowakowski, and Wolfe [ANW07, p. 172].

Corollary 5.13. *If $G = H$ then $G_t = H_t$ for all $t \geq 0$.*

Proof. $G_t - H_t = (G - H)_t = 0_t = 0$. ■

We will use the following result in our analysis of Go.

Proposition 5.14 (Cf. [BW94, lemma 1.4])**.** *Let $r = 1/2^i$ for some positive integer i, and G a game which only has stopping positions that are multiples of $2r$.*

(i) $t(G) = 0$ *or* $t(G) \geq r$.

(ii) *G_r's stopping positions are multiples of r.*

(iii) *If $m(G)$ is not a multiple of r, then $t(G_r) > 0$.*

Proof. Suppose G has temperature > 0, then necessarily $\mathbf{LS}(G) - \mathbf{RS}(G) \geq 2r$. By Corollary 5.10(i) we have $t(G) \geq 2r$.

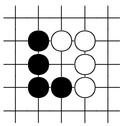
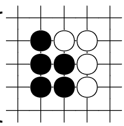
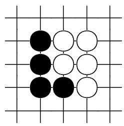
Part (ii) follows from a simple inductive argument, and part (iii) follows immediately from (i) and (ii). ■

6 Chilling and warming in Go

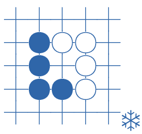
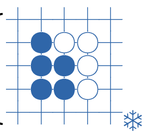
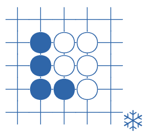
6.1 Chilling

The most important operation in the combinatorial analysis of Go is to chill. That is, to impose on unsettled positions a move tax of one point. This corresponds almost exactly to "cooling by 1 point", except when the temperature of the position happens to be less than one. As we'll see, playing a game of Go is just about equivalent to playing the chilled variant. If G is our Go position, we shall write $f(G)$ to refer to the chilled position.[1]

Let's see what chilling a position looks like in practice. Consider the following position.

 $= \Big\{$  $\Big|$  $\Big\} = *$

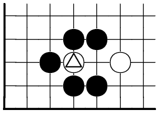
We see quickly (using Theorem 1.4) that the chilled game is

 $= \Big\{$  $-1 \Big|$  $+1 \Big\} = 0$

Which, if anything, is a more sensible value for a dame point.

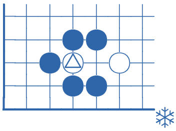
[1] For a formal implementation of the chilling operator we have to wait for a bit, on account of it being tailor-made to map canonical forms to canonical forms. The informal definition will suffice for now.

A somewhat less straightforward example is the following position (a *corridor*, albeit a rather short one)

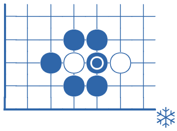


$$= \{2 \mid 0\} \tag{6.1}$$

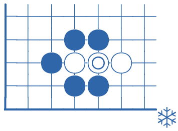
If we don't chill the position, Black has no problem capturing the marked stone in *gote* (losing initiative), since two points is two points. When we chill, however, both players have to pay one point for giving up the initiative.



$$= \Bigg\{$$



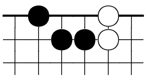
$$-1 \;\Bigg|$$



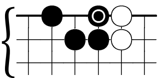
$$+1\Bigg\}$$

$$= \{1 \mid 1\} = 1 + *$$

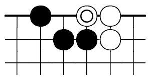
In some sense, the tepid game $1 + *$ is easier to handle than the hot game $\{2 \mid 0\}$. It turns out many positions frequently occurring in the endgame chill down to relatively simple tepid games. In fact, some hot positions even chill to cold games. Consider



$$= \Big\{$$



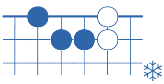
$$\Big|$$



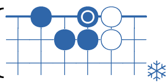
$$\Big\} \tag{6.2}$$

$$= \{1 \mid *\}$$

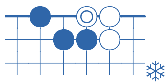
Then



$$= \Big\{$$



$$-1 \;\Big|$$



$$+1\Big\}$$

$$= \{0 \mid 1\} = \tfrac{1}{2}$$

And indeed, many Go players count such a point in gote as "half a point".

If you take away anything from the application of CGT to Go, it should be that chilling provides a very useful method of counting certain endgame positions. It can simplify the evaluation of one's position, especially in tight endgames.

6.2 Empty corridors

Before rigorously establishing the power of chilling in Go, we'll look at the chilled values of some general classes of frequently occurring endgame positions, namely *corridors*.

The simplest kind of corridor is a closed, empty corridor of a given length. In equation 6.2 we saw such a corridor of length two, and saw that it chilled to $\frac{1}{2}$.

The pattern of play in such corridors, of length n say, is quite easy to figure out. One player can take $n-1$ points in gote, and the other can shrink the corridor by one intersection. These options dominate all other possible moves. In the abstract, we can define a corridor of length n in favour of Black, for which I shall use the notation $\mathrm{Corr}(n)$, as follows.

$$\begin{aligned} \mathrm{Corr}(0) &= 0 \\ \mathrm{Corr}(n+1) &= \{n \mid \mathrm{Corr}(n)\} \end{aligned}$$

Theorem 6.1. *$f(\mathrm{Corr}(n)) = n - 2 + (1/2)^{n-1}$ for all non-negative integers n.*

Proof. By induction on n.

Since we're dealing with natural induction, we actually need to concern ourselves with a base case $n = 0$. We have $f(\mathrm{Corr}(0)) = 0 = -2 + (1/2)^{-1}$, so all is good.

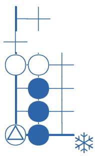
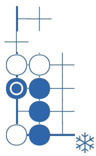
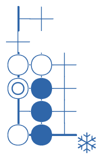
As for the induction step,

$$\begin{aligned} f(\mathrm{Corr}(n+1)) &= \{f(n) - 1 \mid f(\mathrm{Corr}(n)) + 1\} \\ &= \left\{n - 1 \,\middle|\, n - 1 + (1/2)^{n-1}\right\} \\ &= n - 1 + (1/2)^n \end{aligned}$$ ■

6.3 Corridors with gold

Before we move on to our next general result, we should briefly introduce some more infinitesimal games—that is, a game G with $\mathbf{LS}(G) = \mathbf{RS}(G) = 0$—and see how they appear in (chilled) Go. The only such game we've given a name so far is $* = \{0 \mid 0\}$, which is neither positive nor negative, and confused with 0.

Another important infinitesimal is *up*, written $\uparrow := \{0 \mid *\}$, which is a positive game smaller than any positive surreal number. Its negative is called *down*, $(-\uparrow) =: \downarrow$. These appear in the analysis positions such as the one below, which is an example of a *corridor with gold*, the gold here referring to the endangered marked stone.[2]


$= \Big\{$  $-1 \;\Big|\;$  $+1 \Big\}$

$$= \{2 \mid 2 + *\} = 2 + \uparrow$$

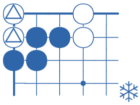
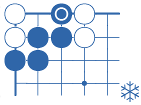
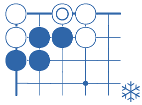
Other positive infinitesimals appear as we increase the amount of gold or the length of our corridor. The final class of infinitesimals we shall name are the *tinies* and the *minies*. For a given parameter x, the game *tiny* x is given as $+_x := \{0 \mid \{0 \mid -x\}\}$. It's negative, the *miny*, has the symbol $-_x$. I cite Berlekamp and Wolfe:

> How do tinies compare with other games like numbers, $\uparrow$ and $*$? Let x be any positive number (or, for that matter, any game exceeding a positive number – like $x = \frac{1}{2}$ or $x = \frac{1}{2}*$). Then $+_x$ is a positive infinitesimal that's less than $\uparrow$. In fact, $+_x$ is much less than $\uparrow$; no matter how

[2] That the position is depicted in the corner doesn't matter of course, it is equivalent to a straight corridor on the centre of the board.

> many $+$'s you add together, you'll never add up enough to get bigger than $\uparrow$. You could say that $+_x$ is infinitesimal with respect to $\uparrow$, just as $\uparrow$ is infinitesimal with respect to positive numbers. [BW94, p. 22]

For example

 $= \Big\{$  $- 1 \Big|$  $+ 1 \Big\}$

$$= \{4 \mid \{3 \mid 1\} + 1\} = \{4 \mid \{4 \mid 2\}\}$$
$$= 4 + +_2$$

It turns out that—in general—as the corridors get longer and the gold is buried deeper, we need infinitesimals with more and more nested zeros to describe the chilled game. For the corridor where the the gold was 1 intersection deep (equation 6.1), we needed the infinitesimal $* = \{0 \mid 0\}$, and when 2 deep we needed $\uparrow = \{0 \mid \{0 \mid 0\}\}$ and $+_2 = \{0 \mid \{0 \mid -2\}\}$. This leads to the definition of the following operator.

Definition 6.2. The operator "$0^n \mid$" for a given non-negative integer n is given recursively as follows.

$$0^0 \mid G = G$$
$$0^{n+1} \mid G = \{0 \mid (0^n \mid G)\}$$

For example $0^3 \mid G = \{0 \mid \{0 \mid \{0 \mid G\}\}\}$, and $0^2 \mid -2 = +_2$.

We can model a corridor with s stones of gold "buried" at a depth of d intersections as

$$\mathrm{Corr}_s(0) = 0$$
$$\mathrm{Corr}_s(d+1) = \{d + 2s \mid \mathrm{Corr}_s(d)\}$$

Theorem 6.3. $f(\mathrm{Corr}_s(d)) = d + 2(s-1) + (0^d \mid -2(s-1))$ *for all non-negative integers d and s.*

Proof. By induction on d.

For our base case we have $f(\mathrm{Corr}_s(0)) = 0 = 2(s-1) + (0^0 \mid -2(s-1))$ for all $s \geq 0$.

As for the induction step, we have

$$\begin{aligned}
f(\mathrm{Corr}_s(d+1)) &= \{d + 2s - 1 \mid f(\mathrm{Corr}_s(d)) + 1\} \\
&= \{d + 1 - 2(s-1) \mid d + 1 - 2(s-1) + (0^d \mid -2(s-1))\} \\
&= d + 1 - 2(s-1) + \{0 \mid (0^d \mid -2(s-1))\} \qquad (\dagger) \\
&= d + 1 - 2(s-1) + (0^{d+1} \mid -2(s-1))
\end{aligned}$$

Note that step † is rather subtle. If $s > 0$ we can simply use Theorem 2.8, but not for $s = 0$. In that case, however, we notice that $0^d \mid -2$ is simply $(1/2)^{d-1}$, and see that the logic still works. ■

6.4 Warming justifies chilling

We closely follow the argumentation used in Berlekamp and Wolfe [BW94, §3.6].

Let's first establish some definitions and make some observations about the game of Go.

Definition 6.4. A Go position is *even* (or *odd*) if the number of empty intersections plus the number of prisoners captured is even (resp. odd).

This parity property has natural properties with regards to addition of positions, and it alternates during play.

By definition, the stopping positions of an elementary Go position, as long as it's expressed in canonical form, are all integers. The following also follows immediately from the definitions.

Lemma 6.5. *A stopping position of an elementary Go position in canonical form is even if and only if its value is even.*

The next result is of fundamental importance if we want to understand the structure of Go. It's quite intuitive when expressed slightly more informally: a Go position in which neither player has any points left to gain, consists only of settled stones, territory and perhaps some dame points.

Proposition 6.6. *Let G be an elementary Go position in canonical form. If G has $t(G) = 0$, then it is of the form n or $n + *$, for some integer n.*

Proof. Take $n = \mathbf{LS}(G) = \mathbf{RS}(G)$. We'll suppose that G is even; the argument for odd G is analogous.

If n is an even number, then consider $G - n$. Since G is even, its stopping position must be reached after an even number of moves. By the number avoidance theorem, we may assume that play only occurs in G, and after a certain even number of plays the second player moves G to the number n, i.e. moves $G - n$ to 0, winning the game. Thus $G - n = 0$.

If n is an odd number, then consider $G - n*$. Two moves in a row by either player in G will guarantee a stopping position at least as favourable to them as n, so both players will keep playing in G. Since G is even, its stop n is reached after an odd number of plays, after which the second player moves $*$ to 0, winning the game. So $G - n* = 0$. ■

Now let's formally introduce the chilling operator.

$$f(G) := \begin{cases} n & \text{if } G = n \text{ or } G = n + * \\ \{f(G^L) - 1 \mid f(G^R) + 1\} & \text{otherwise} \end{cases}$$

Chilling, like cooling, is linear.

Definition 6.7. *Warming* a game G (notation $\int G$) is the conceptual opposite of chilling it. The operation is defined as follows.

$$\int G := \begin{cases} G & \text{if } G \text{ is an even integer} \\ G* & \text{if } G \text{ is an odd integer} \\ \{1 + \int G^L \mid -1 + \int G^R\} & \text{otherwise} \end{cases}$$

Warming is a linear and order-preserving operation [BW94, p. 52].

Note that for arbitrary games G, it is not necessarily true that $\int G_1 = G$, since cooling is generally a many-to-one operation. That this *is* the case for Go (up to addition of $*$) will be our main result. To this end we shall first demonstrate that for even elementary Go positions in canonical form, we have $G = \int f(G)$, and then show that $f(G) = G_1$.

Lemma 6.8. *If A and B are elementary Go positions in canonical form with the same parity, then $f(A) \geq f(B) \implies A \geq B$.*

This is a powerful result in and of itself, essentially guaranteeing that a good move in the chilled game of Go is also a good move in the unchilled game (provided there are no kos, etc.).

Proof. Suppose G is an *even* elementary Go position in canonical form. It suffices to show that we have $G \geq 0$ whenever $f(G) \geq 0$.

If G has $t(G) = 0$ then $G = n$ for some even n, or $G = n + *$ for some odd n (Proposition 6.6). Since in the first case $G \geq 0$ if and only if $n \geq 0$, and in the second case $G \geq 0$ if and only if $n \geq 1$, our hypothesis is satisfied.

Otherwise, a winning strategy for Black moving second in G, is to pretend we're playing $f(G)$. If an even number of moves were made to get to a stopping position of $f(G)$, then the single point adjustments cancel. And since G is even, in the worst case when the stopping value in $f(G)$ is 0, the stopping position in G is also 0. Thus $G \geq 0$.

If an odd number of moves were made to get to a stop in $f(G)$, then we must have arrived at 1 since Black won. For G then, their Black's point advantage consists only of move tax, but they still get the last move to 0, winning the game. ∎

Lemma 6.9. *Let G be an even elementary Go position in canonical form. We have $G = \int f(G)$.*

Proof. If G is of the form n or $n + *$, the lemma holds trivially. If $\mathbf{LS}(G) > \mathbf{RS}(G)$ instead, we need only show that $f(G)$ is in canonical form, after which Theorem 4.3 yields the desired result.

That this is indeed the case follows from Lemma 6.8, showing that an option being dominated in $f(G)$ implies it being dominated in G, which cannot be the case. For reversible options similar. ■

Lemma 6.10. *Let G be an elementary Go position in canonical form. Write $m(G) = i/2^j$, with i odd. Then $t(G) \geq 1 - 1/2^j$.*

Proof. Observe that G has integer-valued stopping positions, has a non-zero temperature (Proposition 6.6), and $m(G)$ is not a multiple of $1/2^k$ for $k < j$. We can therefore iteratively apply Proposition 5.14, cooling first by $1/2$, then further by $1/4$, etc. through to $1/2^j$. This guarantees that G has temperature $\geq 1/2 + 1/4 + \cdots + 1/2^j = 1 - 1/2^j$. ■

Lemma 6.11. *Let G be an elementary Go position in canonical form. We have $f(G) = G_1$.*

Proof. Let G be a counterexample, minimal in the sense that for all G^L and G^R we have $f(G^L) = (G^L)_1$ and $f(G^R) = (G^R)_1$. We must have $t(G) < 1$. Say G has $m(G) = i/2^j$ with i odd, and let $\tau = t(G)$. We know that G_τ has some Left option with Right stop $i/2^j$. And since by the previous lemma we have $\tau \geq 1 - 2^j$, we know that $f(G)$ has a corresponding stopping position $i/2^j$ with Right moving last, or at least $(i-1)/2^j$ with Left moving last. In both cases Black has a winning strategy for the difference game $f(G) - G_1$, White moving first, since the worst White can do is move $-G_1 = \{-(i+1)/2^j \mid -(i-1)/2^j\}$ to $-(i-1)/2^j$. But since by definition of f, all of $f(G)$'s Left options are less than or equal to those of G_1, we must have $f(G) = G_1$. ■

Theorem 6.12. *Let G be an elementary Go position in canonical form. We have $\int G_1 = G$ or $\int G_1 = G + *$.*

Proof. For even positions we've shown that $\int G_1 = \int f(G) = G$. If G is odd and has $t(G) = 0$, the result is also clear. If on the other hand G is odd with $t(G) > 0$, then the result follows from the fact that for hot games, we have $G + * = \{G^L + * \mid G^R + *\}$, and thus that $G + *$ is even if G is odd, with $(G + *)_1 = G_1$.[3] ■

[3]The property $G + * = \{G^L + * \mid G^R + *\}$ for G with $t(G) > 0$ seems intuitive and is stated as fact in Berlekamp and Wolfe [BW94], but I've been unable to prove it myself.

Closing thoughts

> Mathematicians have a kind of special dispensation that scientists don't have. They're allowed to stop working when it gets too complicated.
>
> *(Noam Chomsky)*

When I read (on Wikipedia, probably) that Conway's theory of surreal numbers was inspired by the game of Go, I was keen to find out more, and figured the subject would be suitable for my undergraduate thesis. The topic of combinatorial game theory seemed just right for me; being a rather niche subfield of mathematics, it enables one to take take a very broad overview. The application to Go then provided the necessary focus. One of my greatest strengths (in my own estimation, at least) is to very efficiently filter through information, seeking out what's relevant and what's not. And indeed, in that sense this project has been ideal for me. I hope I've succeeded in giving an interesting and concise overview of the field of CGT, focusing primarily on the specific results needed in the analysis of Go, while also appreciating some of the beautiful general theory where possible (in chapter 5 especially).

I initially expected that the link between Conway's surreal numbers and Go was more "direct" than turned out to be the case, suspecting a connection with so-called *miai counting*.[1] It took me a while to properly understand that surreal number theory is but a small subsection of

[1] I've not mentioned miai counting at all in this thesis, since I've been unable to find a sufficiently rigorous standard definition of it. I still suspect however, that there is a fairly direct link between the miai value of a position and what in CGT would be called its mean value.

the broader field of (partizan) combinatorial game theory. After studying the first chapter of Conway [Con76], and learning about the application to Hackenbush, I found myself wondering: "well then what strange kind of game *isn't* equal to a number?!" I've sought to avoid such confusion here by presenting games first, numbers second—the reverse of [Con76].

Everything the application of CGT to Go I've learned from Berlekamp and Wolfe [BW94], but I've sought to be more concise here, while also taking some time in chapter 3 to appreciate the fundamentals. It was never my intention to dive as deeply into Go as possible, but rather to present the big picture. After you've done corridors, you've understood the basic idea and hopefully seen the strengths of CGT as applied to Go. To tabulate several pages of Go configurations ("rooms" in [BW94]) appealed less to me than to discover CGT more generally.

I've deviated from the notation used in [BW94], which uses a rather odd system of adding "markings" to chilled Go positions. I've opted to explicitly add and subtract numbers instead.

Bibliography

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The Rules of Go

1 The board

Go is played on a grid of 19 by 19 lines, called the board or *goban*. A player makes a move by placing a stone of their colour on a grid intersection. One player plays with black stones, one with white. Once a stone is placed on the board, it cannot be moved to a different location on the board.

Black gets to play the first move, after which play alternates.

The first stage of the game is known as *fuseki*, meaning 'scattering of stones'. In the opening, there are certain standard patterns of play in the corners of the board that are considered equal for both players. Such patterns are called *joseki*.

Go can be played on other sizes of boards. 9x9 and 13x13 are other popular board sizes. For the game's early history it was played on a 17x17 board, as it still is in Tibet.

2 Capturing stones

For a single stone, its orthogonally neighbouring empty intersections are called the stone's *liberties*.

Stones of the same colour can be *connected* to each other, forming a *group*. Stones in a group share their liberties.

A group that has only one liberty is said to be in *atari*, and such a group can be captured by the opponent if they fill in the last liberty. The group that was in atari is then taken off the board, now becoming

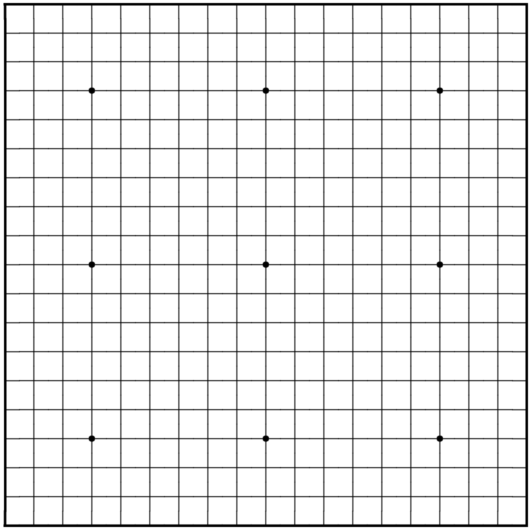

Figure 1: The empty 19x19 go board

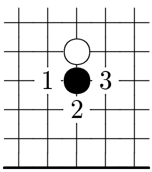

Figure 2: The black stone has three liberties, each marked

their opponent's *prisoners*. As the name implies, getting captured is usually a bad outcome for a player.

After placing a stone on the board, any of the opponent's stones left without liberties by the play are removed. Only then are the new stone's liberties counted. If the newly placed stones is then immediately left

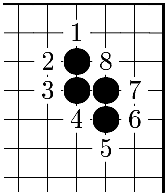

Figure 3: Black's four stones are connected, forming a group with eight liberties

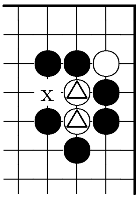

Figure 4: White's marked group is in atari, so Black can capture it by playing at x.

without liberties, the move was *suicidal*. Most rule sets forbid suicide.

A group that cannot be captured is said to be *alive*. A group that cannot be made to live is called *dead*. A group is alive if it has two *eyes*.

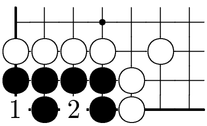

Figure 5: The black group has two eyes. Black need not worry that it is surrounded by White, because unless Black is silly enough to fill in one of their own eyes, White can never capture the Black group.

2.1 Seki

A configuration of stones where neither player has two eyes, but yet both players can't capture the opponent's group without first getting captured themself, is called *seki*, or *mutual life*.

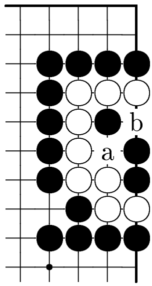

Figure 6: A seki configuration. Neither player wants to play a or b.

3 The Ko Rule

The rule of *ko* exists in many forms across different rule sets. The most mathematically satisfying version is known as *positional superko*. The rule states that a board position may not repeat. Any play must yield a new board position.

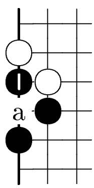

Figure 7: A classic *ko shape*. Black just played 1, capturing a white stone previously at a.

In the above position White may not immediately recapture at a, as that would repeat the board position. White must play elsewhere first.

4 Counting score

A portion of empty board intersections enclosed by a player's stones is said to be that player's *territory*. If we count up the number of empty intersections in the territory, we get the number of *points of territory*.

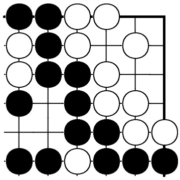

Figure 8: White has 7 points of territory in this corner

When both players have decided all territory is settled, they decide to end the game. This is done after both players pass consecutively. The players decide which groups are dead and which are alive (if there is any disagreement, play continues to settle the matter). Dead groups are then taken off the board and are treated as prisoners. Both players count their points of territory, and subtract from it the number of their stones taken prisoner. White also adds to their tally a number of compensation points for having to play second. This compensation is called *komi*, and is usually 6.5 or 7.5 points (the half point functions as a tie-breaker).

The numbers thus obtained are the players' scores. The player with the highest score wins the game.

The example in figure 9 shows a finished game. Both players agree that the two marked white stones are dead, so these are taken off the board. During the game, Black had six of their stones captured, White had two of theirs captured. Komi for this 9x9 game was 5.5 points. Black has 12 points of territory, White has 19. The intersection marked 'a' is neutral, and isn't added to either player's score.

Adding everything up:

- Black has 12 points of territory - 6 captured stones = 6 points total

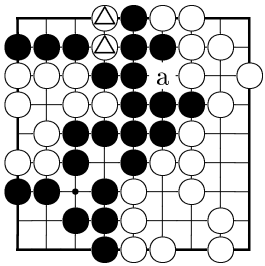

Figure 9: This game has finished.

- White has 19 points of territory - 2 captured stones - 2 dead stones + 5.5 points komi = 20.5 points total

Thus White wins by 14.5 points, notation: W+14.5.