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An Extension of Raney's Algorithm for Transducing Continued Fractions

Thesis B.Sc. Mathematics

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Contents

1 Introduction

In 1695, John Wallis formally defined a method of approximating real numbers with rational numbers using large embedded fractions. He named these approximations "continued fractions". Despite this being the first formal definition of continued fractions, this method of approximating real numbers had already been used for centuries by other mathematicians. Nowadays, continued fractions are used in various areas of mathematics.

In essence, a continued fraction is a fraction of the form

$$x = a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \cfrac{1}{\ddots}}}} \tag{1}$$

where all coefficients a_i are whole numbers.

The use of continued fractions to approximate real numbers has several advantages over the use of decimal approximation. In most cases, continued fractions converge to their appropriate real number a little faster than decimal approximations do. Furthermore, continued fractions of rational numbers are always finite, much unlike their decimal notations, and continued fractions of quadratic irrational numbers are always periodic [1].

Despite these advantages, continued fractions have one very clear disadvantage to decimal approximation that often makes them impractical to use. When simple arithmetic expressions are applied to real numbers, their continued fractions drastically change, to the degree that no simple methods exist to predict the outcome of such expressions, given the continued fraction of the input.

In 1891, Adolf Hurwitz devised an algorithm to compute the continued fraction of a real number $\beta = 2\alpha$, given the continued fraction of α. In 1947, Marshall Hall devised a generalised algorithm that computes the continued fraction of any Möbius transformation applied to another continued fraction [2]. This method, however, turned out to be highly impractical for anything but the simplest Möbius transformations. Eventually, in 1973, George N. Raney found a more efficient method to replace Hall's [7].

Raney's algorithm makes use of sequence transducers that automatically transform input continued fractions into different continued fractions. Given a Möbius transformation μ, a transducer $\mathcal{T}_\mu$ is constructed that transforms the continued fraction of any real number α into the continued fraction of $\beta = \mu(\alpha)$. The advantage of this method over Hall's is that in Raney's algorithm, only a single transducer needs to be constructed for any Möbius transformation, whereas in Hall's algorithm, a whole number of difficult computations are required for every α, even if the Möbius transformation remains the same.

What makes Raney's algorithm especially strong is that it can be stopped at any point to give an accurate truncated continued fraction. For continued fractions to be used in practice, they will need to be truncated, otherwise they cannot be stored in finite space.

Next to the continued fraction in equation 1, also known as the "regular continued fraction", many other types of continued fractions exist. Two popular ones are the “nearest integer continued fraction” and its complex analogue; the “Hurwitz continued fraction”. Especially the Hurwitz continued fraction is used often when approximating complex numbers with continued fractions. The different algorithms mentioned above all only work for regular continued fractions. Several attempts have been made to devise similar algorithms for these types of continued fractions with varying success. So far, Raney’s algorithm has not been successfully extended to either of these continued fractions.

This thesis gives an in-depth explanation of how Raney’s algorithm works, followed by an assessment of the problems with attempting to directly extend Raney’s algorithm to work for nearest integer continued fractions. In 2011, a direct extension of the algorithm was attempted for the similar Hurwitz continued fraction with no success [5], but an algorithm for the simpler nearest integer continued fraction might give a new perspective on how to extend it to Hurwitz continued fractions.

2 Initial Definitions

2.1 Disambiguation

This thesis uses a few sets, functions and notations that are ambiguously defined within mathematics. This section serves to explain which definitions and notations are used within this thesis.

The set $\mathbb{N}$ is used to denote all non-negative integers; i.e., $\mathbb{N} = \{0, 1, 2, 3, ...\}$.

For rounding real numbers x to integers, the following three methods are used:

- $\lfloor x \rfloor = \max\{a \in \mathbb{Z} \mid a \leq x\}$
- $\lceil x \rceil = \min\{a \in \mathbb{Z} \mid a \geq x\}$
- $\lfloor x \rceil$ is the element $a \in \{\lfloor x \rfloor, \lceil x \rceil\}$ that minimises $|x - a|$. If $|x - \lfloor x \rfloor| = |x - \lceil x \rceil|$, we set $\lfloor x \rceil$ to be equal to the $a \in \{\lfloor x \rfloor, \lceil x \rceil\}$ for which $|a|$ is smallest.

2.2 Continued Fractions

Continued fractions can be defined over many different fields. In this thesis, however, we will only be looking at continued fractions for the set of real numbers.

Definition 2.1. Let $x \in \mathbb{R}$. A *generalised continued fraction* of x is a tuple of sequences $(\{a_n\}_{n=0}^{\infty}, \{b_n\}_{n=1}^{\infty})$ where $a_n \in \mathbb{Z}, b_n \in \mathbb{Z}\backslash\{0\}$ for all indices n, and $a_n \neq 0$ for infinitely many n, such that:

$$x = a_0 + \cfrac{b_1}{a_1 + \cfrac{b_2}{a_2 + \ddots}}$$

If $x \in \mathbb{Q}$, a *finite continued fraction* of x is a tuple of sequences $(\{a_n\}_{n=0}^{N}, \{b_n\}_{n=1}^{N})$ where $a_n \in \mathbb{Z}, b_n \in \mathbb{Z}\backslash\{0\}$ for all indices n, and $a_N \neq 0$, such that:

$$x = a_0 + \cfrac{b_1}{a_1 + \cfrac{b_2}{\ddots + \cfrac{b_N}{a_N}}}$$

Furthermore, a *standard continued fraction* is a generalised continued fraction for which $b_n = 1$ for every n.

We write the generalised continued fraction with sequences $\{a_n\}_{n=0}^{\infty}$ and $\{b_n\}_{n=1}^{\infty}$ as $[a_0; b_1/a_1, b_2/a_2, ...]$. If the continued fraction is a standard continued fraction, we simply write it as $[a_0; a_1, a_2, ...]$.

We write the finite generalised continued fraction with sequences $\{a_n\}_{n=0}^{N}$ and $\{b_n\}_{n=1}^{N}$ as $[a_0; b_1/a_1, ..., b_N/a_N]$. If the finite continued fraction is a finite standard continued fraction, we write it as $[a_0; a_1, ..., a_N]$.

Remark. For any standard continued fraction $[a_0; a_1, a_2, ...]$ where $a_i = 0$ for some $i \neq 0$, the continued fraction can be rewritten to $[a_0; ..., a_{i-2}, a_{i-1} + a_{i+1}, a_{i+2}, ...]$. This can easily be checked since

$$a_{i-1} + \cfrac{1}{0 + \cfrac{1}{a_{i+1} + \cfrac{1}{\ddots}}} = a_{i-1} + \left(\left(a_{i+1} + \frac{1}{\ddots}\right)^{-1}\right)^{-1} = a_{i-1} + a_{i+1} + \frac{1}{\ddots}$$

From here on out, the term *continued fraction* will always refer to a standard continued fraction.

Definition 2.2. A continued fraction $[a_0; a_1, a_2, ...]$ is called periodic if an index $N \geq 1$ and a period $p \geq 1$ exist such that for every $i \geq N$, the following holds: $a_i = a_{i+p}$. If $[a_0; a_1, a_2, ...]$ is periodic, we denote the continued fraction as $[a_0; ..., a_{N-1}, \overline{a_N, ..., a_{N+p-1}}]$.

Of the standard continued fractions, two in particular are very broadly used; the *regular continued fraction (RCF)* and the *nearest integer continued fraction (NICF)*

Definition 2.3. Let $\alpha \in \mathbb{R}\backslash\mathbb{Q}$. Then the regular continued fraction $[a_0; a_1, a_2, ...]$ for α is defined as follows:

- $\phi(x) = \frac{1}{x - \lfloor x \rfloor}$
- $a_n = \lfloor \phi^n(\alpha) \rfloor$

If $\alpha \in \mathbb{Q}$, then the regular continued fraction $[a_0; a_1, ..., a_N]$ is defined in the same way, where N is the least index for which $\lfloor \phi^N(\alpha) \rfloor \in \mathbb{Z}$.

Definition 2.4. Let $\alpha \in \mathbb{R}\backslash\mathbb{Q}$. Then the nearest integer continued fraction $[a_0; a_1, a_2, ...]$ for α is defined as follows:

- $\psi(x) = \frac{1}{x - \lfloor x \rceil}$
- $a_n = \lfloor \psi^n(\alpha) \rceil$

If $\alpha \in \mathbb{Q}$, then the regular continued fraction $[a_0; a_1, ..., a_N]$ is defined in the same way, where N is the least index for which $\lfloor \phi^N(\alpha) \rceil \in \mathbb{Z}$.

Theorem 2.5. *Suppose $\alpha \in \mathbb{R}\backslash\mathbb{Q}$. Let $[a_0; a_1, a_2, ...]$ be the RCF of α and let $[a'_0; a'_1, a'_2, ...]$ be the NICF of α. Then the following holds:*

For all $n \in \mathbb{N}_{>0}$

- $a_n > 0$
- $a'_n \notin \{-1, 0, 1\}$
- *If $|a'_n| = 2$, then $\texttt{sgn}(a'_{n+1}) = \texttt{sgn}(a'_n)$.*

Proof. A proof for this can be found in Section 39 of [6]. □

Remark. If $\alpha \in \mathbb{Q}$ with RCF $[a_0; a_1, ..., a_N]$ and NICF $[a'_0; a'_1, ..., a'_N]$, the conditions on a_n and a'_n in Theorem 2.5 hold as well.

Definition 2.6. Given a standard continued fraction $[a_0; a_1, a_2, ...]$, we define two sequences $\{p_i\}_{i=-2}^{\infty}$ and $\{q_i\}_{i=-2}^{\infty}$ as follows:

$$\begin{aligned} p_{-2} &= 0, & q_{-2} &= 1, \\ p_{-1} &= 1, & q_{-1} &= 0, \\ p_{n+2} &= a_{n+2}p_{n+1} + p_n & q_{n+2} &= a_{n+2}q_{n+1} + q_n \end{aligned}$$

Theorem 2.7. *Let $[a_0; a_1, a_2, ...]$ be a standard continued fraction. Then for every $n \in \mathbb{N}$, the following statement holds: $[a_0; ..., a_n] = \frac{p_n}{q_n}$.*

Proof. First of all, it is easy to see that $\frac{p_0}{q_0} = a_0 = [a_0]$ and $\frac{p_1}{q_1} = a_0 + \frac{1}{a_1} = [a_0; a_1]$. Suppose $\frac{p_n}{q_n} = [a_0; ..., a_n]$ holds for a certain $n \in \mathbb{N}$. This means that:

$$\frac{p_{n+1}}{q_{n+1}} = \frac{a_{n+1}p_n + p_{n-1}}{a_{n+1}q_n + q_{n-1}} = \frac{[a_n; a_{n+1}]p_{n-1} + p_{n-2}}{[a_n; a_{n+1}]q_{n-1} + q_{n-2}}$$

$$= [a_0; ..., a_{n-1}, [a_n; a_{n+1}]] = [a_0; ..., a_{n+1}]$$

Thus, by induction we can conclude that $[a_0; ..., a_n] = \frac{p_n}{q_n}$ for every $n \in \mathbb{N}$.

□

Remark. According to Proposition 1.1.2 of [4], the following holds: $[a_0; a_1, a_2, ...] = \lim_{n\to\infty}[a_0; a_1, ..., a_n]$. Thus, $[a_0; a_1, a_2, ...] = \lim_{n\to\infty} \frac{p_n}{q_n}$.

Definition 2.8. Given $x \in \mathbb{R}$. An improper continued fraction of x is a finite sequence of numbers $[a_0; a_1, ..., a_n, \alpha]$ with $a_0, ..., a_n \in \mathbb{Z}$ and $\alpha \in \mathbb{R}$, such that

$$x = a_0 + \cfrac{1}{a_1 + \cfrac{1}{\ddots + \cfrac{1}{a_n + \cfrac{1}{\alpha}}}}$$

2.3 Sequence Transducers

Raney's algorithm extensively uses finite-state sequence transducers to generate new continued fractions using other continued fractions. This section explains what a finite-state sequence transducer is and how it can be used to generate sequences from other sequences.

Definition 2.9. Given a finite, non-empty set Σ, we call any finite sequence of elements of Σ a word of alphabet Σ. The set of all words of alphabet Σ is called Σ^*. We simply write the word $\{\sigma_i\}_{i=0}^{n}$ as $\sigma_0\sigma_1 \cdots \sigma_n$. Furthermore, we write the empty word $\{\}$ as ϵ.

The set of infinite sequences of elements of Σ is written as Σ^∞.

Definition 2.10. Given two words $V = V_0 \cdots V_m$ and $W = W_0 \cdots W_n$, the concatenation $V \parallel W$ of V and W is defined as $V \parallel W = V_0 \cdots V_m W_0 \cdots W_n$.

Likewise, given a word $V = V_0 \cdots V_m$ and a sequence $S = S_0S_1 \cdots$, the concatenation of V and S is defined as $V \parallel S = V_0 \cdots V_m S_0 S_1 \cdots$.

Definition 2.11. Given a word $W = W_0 \cdots W_n \in \Sigma^*$, a prefix of V is a word $V = V_0 \cdots V_m \in \Sigma^*$ for which $m \leq n$ and $V_i = W_i$ for all $0 \leq i \leq m$. We write this as $V \mid W$.

Likewise, a prefix V of a sequence S is a word in $V = V_0 \cdots V_m \in \Sigma^*$ for which $V_i = S_i$ for all $0 \leq i \leq m$. We also write this as $V \mid S$.

Given a sequence $S = S_1 S_2 \cdots$ and a word $V = V_1 \cdots V_m$ with $V \mid S$, we define S/V as the sequence where prefix V has been removed. That is to say, $S/V = S_{m+1} S_{m+2} \cdots$.

Definition 2.12. Given an alphabet Σ, a base of Σ^∞ is a set $\mathcal{B} \subseteq \Sigma^*$ such that the following holds:

- $\mathcal{B}$ is finite.
- Any sequence in Σ^∞ can be written as a unique concatenation of elements of $\mathcal{B}$.

Remark. Since every sequence in Σ^∞ can be written as a *unique* concatenation of elements of $\mathcal{B}$, no base $\mathcal{B}$ of Σ^∞ may contain the empty word ϵ.

Definition 2.13. A finite-state sequence transducer is a tuple $(Q, \Sigma, \Gamma, \delta, q_0)$ defined as follows:

- Σ and Γ are finite alphabets. We call these the "input alphabet" and the "output alphabet" respectively.
- Q is a finite, non-empty set of "states".
- q_0 is a state in the set Q. We call this state the "initial state".
- δ is a partial function $\delta : Q \times \Sigma^* \to Q \times \Gamma^*$ with a finite domain of definition (DOD), such that for every state $q \in Q$, the set $\mathcal{B}_q = \{W \in \Sigma^* : (q, W) \text{ is in the DOD of } \delta\}$ is a base of Σ^∞.

Given a finite-state sequence transducer $\mathcal{T} = (Q, \Sigma, \Gamma, \delta, q_0)$ and an input sequence $S \in \Sigma^\infty$, we can generate an output sequence $S' \in \Gamma^\infty$ as follows:

1. The current state q is set to equal q_0 and S' is set to be an empty word $\epsilon \in \Gamma^*$.
2. Since the set $\mathcal{B}_q$ is a base for Σ^∞, a unique, non-empty word $W = W_0 \cdots W_n \in \mathcal{B}_q$ exists such that $W \mid S$.
3. The function δ is applied to (q, W) to result in a new tuple $(q', W') = \delta((q, W))$.
4. The word S' is replaced by $S' \parallel W'$, the sequence S is replaced by S/W, and q is set to equal q'.
5. This process is infinitely repeated from step 2.

Sequence transducers can be schematically represented with a diagram where every state in Q is represented by a circle, and the transition function δ is represented by arrows in such a way that if $\delta((q, W)) = (q', W')$, an arrow will point from state q to state q' with a label "W / W'". Furthermore, an arrow is added to signify which state is the initial state. An example of this is shown in Figure 1.

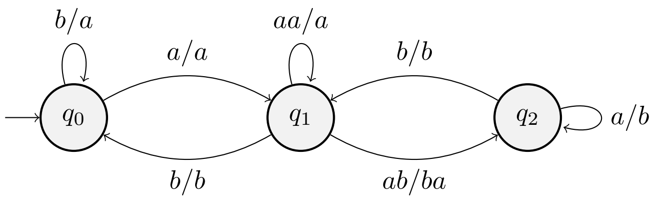

Figure 1: Example of a finite-state sequence transducer.

3 Raney's algorithm

It has been shown to be far from trivial to perform arithmetic operations on continued fractions. Throughout history, several different algorithms have been devised to compute different kinds of functions applied to continued fractions. Among these, a popular one is the calculation of Möbius transforms of continued fractions:

Given $\alpha = [a_0; a_1, a_2, ...] \in \mathbb{R}$ and given a Möbius transformation $\mu(x) = \frac{ax+b}{cx+d}$. Can we find a continued fraction $[b_0; b_1, b_2, ...]$ such that $[b_0; b_1, b_2, ...] = \beta = \mu(\alpha)$?

George N. Raney developed an algorithm to solve this problem for regular continued fractions [7]. This section aims to explain how his algorithm works, so it can be expanded upon for the nearest integer continued fraction.

3.1 Matrix representation of regular continued fractions

In his algorithm, Raney uses a different representation for RCFs than the typical sequence of integers defined in 2.1. For his representation, Raney defines two matrices:

$$L = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \quad \text{and} \quad R = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

It is important to note that for any value n:

$$L^n = \begin{pmatrix} 1 & 0 \\ n & 1 \end{pmatrix} \quad \text{and} \quad R^n = \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix}$$

Using these two matrices L and R, Raney is able to represent regular continued fractions with infinite sequences of a finite alphabet. Since these two matrices will be used both as matrices and as characters in an alphabet, some variation in notation is required to make clear when these are used as matrices and when they are used as characters. In the case of matrix multiplication, we will simply denote the product $L^{n_1} \cdot R^{n_2} \cdots L^{n_{k-1}} \cdot R^{n_k}$ as $L^{n_1} R^{n_2} \cdots L^{n_{k-1}} R^{n_k}$, while the word $\{L, \overset{n_1}{\dots}, L, R, \overset{n_2}{\dots}, R, ..., L, \overset{n_{k-1}}{\dots}, L, R, \overset{n_k}{\dots}, R\}$ is denoted as '$L^{n_1} R^{n_2} \cdots L^{n_{k-1}} R^{n_k}$'.

Definition 3.1. Let $v = \xi \left(\begin{smallmatrix} \alpha \\ 1 \end{smallmatrix} \right) \in \mathbb{R}^2$ with $\xi > 0, \alpha \geq 0$. We say that v *accepts* a sequence $S =$ '$R^{a_0} L^{a_1} R^{a_2} \cdots$' $\in \{L, R\}^\infty$ if a sequence of vectors $\{v_k\}_{k=0}^\infty$ exists such that

$$v = \prod_{i=0}^{k} S_i \cdot v_k \quad \text{for all } k \in \mathbb{N}.$$

Theorem 3.2. *Let $v = \xi \left(\begin{smallmatrix} \alpha \\ 1 \end{smallmatrix} \right) \in \mathbb{R}^2$ with $\xi > 0, \alpha \geq 0$.*

- *If α is irrational with $\alpha = [a_0; a_1, a_2, ...]$, then v only accepts the sequence '$R^{a_0} L^{a_1} R^{a_2} \cdots$' $\in \{L, R\}^\infty$.*

- *If α is rational with $\alpha = [a_0; a_1, ..., a_n]$, then v only accepts the sequences '$R^{a_0} L^{a_1} \cdots L^{a_n} R^\infty$' and '$R^{a_0} L^{a_1} \cdots L^{a_n - 1} R L^\infty$' if n is odd or the sequences '$R^{a_0} L^{a_1} \cdots R^{a_n} L^\infty$' and '$R^{a_0} L^{a_1} \cdots R^{a_n - 1} L R^\infty$' if n is even.*

Raney refers to A. Hurwitz [3]. Since this article is very old and not available in English, a proof will follow of the first part of this theorem. The second part of the theorem is proven quite similarly to the first, but is of little interest for this thesis, so an explicit proof of this part is omitted.

Proof. Let $v = \xi \left(\begin{smallmatrix} \alpha \\ 1 \end{smallmatrix}\right) \in \mathbb{R}^2$ with $\xi > 0, \alpha \geq 0$. Furthermore, let us define a function $d : \mathbb{R}^2 \backslash \{\left(\begin{smallmatrix} x \\ 0 \end{smallmatrix}\right) : x \in \mathbb{R}\} \to \mathbb{R}$ as $d\left(\begin{smallmatrix} x_1 \\ x_2 \end{smallmatrix}\right) = \frac{x_1}{x_2}$.

Suppose $d(v) = \alpha = [a_0; a_1, a_2, ...]$. As remarked after Theorem 2.7, the following holds: $\alpha = \lim_{n\to\infty} \frac{p_n}{q_n}$. With these p and q sequences, it is important to note the following:

$$\begin{pmatrix} p_{n-1} & p_{n-2} \\ q_{n-1} & q_{n-2} \end{pmatrix} R^{a_n} = \begin{pmatrix} p_{n-1} & p_n \\ q_{n-1} & q_n \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} p_{n-2} & p_{n-1} \\ q_{n-2} & q_{n-1} \end{pmatrix} L^{a_n} = \begin{pmatrix} p_n & p_{n-1} \\ q_n & q_{n-1} \end{pmatrix}$$

Furthermore, we can note that

$$\begin{pmatrix} p_{-1} & p_{-2} \\ q_{-1} & q_{-2} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

From this, we can deduce the following two equations:

$$\begin{aligned} \begin{pmatrix} p_{n-1} & p_n \\ q_{n-1} & q_n \end{pmatrix} &= R^{a_0} L^{a_1} \cdots R^{a_n} \quad \text{if } n \text{ is even, and} \\ \begin{pmatrix} p_n & p_{n-1} \\ q_n & q_{n-1} \end{pmatrix} &= R^{a_0} L^{a_1} \cdots L^{a_n} \quad \text{if } n \text{ is odd.} \end{aligned}$$

We will name these matrices $\left(\begin{smallmatrix} p_{n-1} & p_n \\ q_{n-1} & q_n \end{smallmatrix}\right) = P_n$ if n is even, and $\left(\begin{smallmatrix} p_n & p_{n-1} \\ q_n & q_{n-1} \end{smallmatrix}\right) = P_n$ if n is odd.

Let k be any integer greater than or equal to 0.

Suppose k is even. Let $w = \left(\begin{smallmatrix} \alpha_k \\ 1 \end{smallmatrix}\right) \in \mathbb{R}^2$ be the vector with $\alpha_k = [a_k; a_{k+1}, a_{k+2}, ...]$. Then, we can conclude that

$$\begin{aligned} d\left(P_{k-1} w\right) &= d\left(\begin{pmatrix} \alpha_k p_{k-1} + p_{k-2} \\ \alpha_k q_{k-1} + q_{k-2} \end{pmatrix}\right) = \frac{p_{k-1} + p_{k-2}/\alpha_k}{q_{k-1} + q_{k-2}/\alpha_k} \\ &= [a_0; ..., a_{k-1} + \frac{1}{\alpha_k}] = [a_0; ..., a_{k-1}, \alpha_k] = \alpha \end{aligned}$$

So a $\xi_k \in \mathbb{R}$ exists such that $v = P_{k-1}\xi_k w = R^{a_0} L^{a_1} \cdots L^{a_{k-1}} \xi w$.

Suppose k is odd. Let $w = \left(\begin{smallmatrix} 1 \\ \alpha_k \end{smallmatrix}\right) \in \mathbb{R}^2$ be the vector with $\alpha_k = [a_k; a_{k+1}, a_{k+2}, ...]$. Then, we can conclude that

$$\begin{aligned} d\left(P_{k-1} w\right) &= d\left(\begin{pmatrix} p_{k-1} + p_{k-2}/\alpha_k \\ q_{k-1} + q_{k-2}/\alpha_k \end{pmatrix}\right) = \frac{p_{k-1} + p_{k-2}/\alpha_k}{q_{k-1} + q_{k-2}/\alpha_k} \\ &= [a_0; ..., a_{k-1} + \frac{1}{\alpha_k}] = [a_0; ..., a_{k-1}, \alpha_k] = \alpha \end{aligned}$$

So a $\xi_k \in \mathbb{R}$ exists such that $v = P_{k-1}\xi_k w = R^{a_0} L^{a_1} \cdots L^{a_{k-1}} \xi w$.

Thus, for every $k \in \mathbb{N}$, a vector $v_k \in \mathbb{R}^2$ exists such that $v = R^{a_0} L^{a_1} \cdots R^{a_k} v_k$ or $v = R^{a_0} L^{a_1} \cdots L^{a_k} v_k$, so we can deduce that v accepts the sequence '$R^{a_0} L^{a_1} R^{a_2} \cdots$'.

Now suppose a vector $0 \neq v \in \mathbb{R}^2_{\geq 0}$ accepts the sequence '$R^{a_0} L^{a_1} R^{a_2} \cdots$'. This means that vectors v'_k exist such that $v = P_k v'_k$ for every $k \in \mathbb{N}$. This means that for every v'_k, we can state that

$$\begin{aligned} d(v) = d(P_k v'_k) &= d\left(\begin{pmatrix} d(v'_k) p_k + p_{k-1} \\ d(v'_k) q_k + q_{k-1} \end{pmatrix}\right) = [a_0; ..., a_k + \frac{1}{d(v'_k)}] \quad \text{if } n \text{ is even, and} \\ d(v) = d(P_k v'_k) &= d\left(\begin{pmatrix} p_k + d(v'_k) p_{k-1} \\ q_k + d(v'_k) q_{k-1} \end{pmatrix}\right) = [a_0; ..., a_k + d(v'_k)] \quad \text{if } n \text{ is odd.} \end{aligned}$$

Thus, if v accepts the sequence '$R^{a_0}L^{a_1}R^{a_2}\cdots$', then $\alpha = [a_0; a_1, a_2, ...]$.

Thus, we can conclude that $v = \xi \left(\begin{smallmatrix} \alpha \\ 1 \end{smallmatrix}\right) \in \mathbb{R}^2$ with $\xi > 0, \alpha = [a_0; a_1, a_2, ...] \geq 0$ only accepts the sequence '$R^{a_0}L^{a_1}R^{a_2}\cdots$' $\in \{L, R\}^\infty$.

$\square$

With this result, we can now start representing regular continued fractions $[a_0; a_1, a_2, ...]$ with sequences of matrices '$R^{a_0}L^{a_1}R^{a_2}\cdots \in \{L, R\}^\infty$.

Now, if we want to find an RCF $[b_0; b_1, b_2, ...]$ for $\beta = \mu(\alpha)$ with $\mu(x) = \frac{ax+b}{cx+d}$, it suffices to find the accepted sequence '$R^{b_0}L^{b_1}R^{b_2}\cdots$' for $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)\left(\begin{smallmatrix} \alpha \\ 1 \end{smallmatrix}\right) = \xi \left(\begin{smallmatrix} \beta \\ 1 \end{smallmatrix}\right)$, provided $\beta \geq 0$ and $\xi > 0$.

3.2 Balanced matrices

In his article, Raney defines three different types of matrices; *row-balanced matrices*, *column-balanced matrices* and *doubly-balanced matrices*. These matrices are used in an important theorem that forms the basis for Raney's algorithm.

Definition 3.3. Let $M = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathbb{Z}^{2x2}$ with $a, b, c, d \geq 0$ and $\det(M) \geq 1$.

- M is called "row-balanced" if $a > c$ and $d > b$.
- M is called "column-balanced" if $a > b$ and $d > c$.
- M is called "doubly-balanced" if it is both row-balanced and column-balanced.

Given an $n \in \mathbb{Z}_{\geq 1}$, we denote the sets of row-balanced, column-balanced and doubly-balanced matrices of determinant n with $\mathcal{RB}_n$, $\mathcal{CB}_n$ and $\mathcal{DB}_n$ respectively.

Suppose $M = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathcal{RB}_n$, then we can define $r(M) = \left(\begin{smallmatrix} d-b \\ a-c \end{smallmatrix}\right)$. Since M is row-balanced, the integers $d-b$ and $a-c$ are both positive. This means we can create a unique "generating word" for $r(M)$; that is to say, a word $W_M \in \{L, R\}^*$ such that $r(M) = \prod_i (W_M)_i \cdot \left(\begin{smallmatrix} \gcd(d-b, a-c) \\ \gcd(d-b, a-c) \end{smallmatrix}\right)$. With this word W_M, we can finally define a set of "immediate offshoots" $B_M = \{W \in \{L, R\}^* : W \nmid W_M, \text{ but } V|W \Rightarrow V|W_M \text{ for every } V \in \{L, R\}^* \backslash \{W\}\}$.

Since every matrix M has a unique generating word W_M, this means that by extension, it also always has a unique set B_M.

Example 3.4. Consider the set $\mathcal{DB}_3 = \{A = \left(\begin{smallmatrix} 3 & 0 \\ 0 & 1 \end{smallmatrix}\right), A' = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 3 \end{smallmatrix}\right), B = \left(\begin{smallmatrix} 2 & 1 \\ 1 & 2 \end{smallmatrix}\right)\}$. For these matrices, we will find the sets of immediate offshoots. We start by finding $r(M)$ for every $M \in \mathcal{DB}_3$.

$$r(A) = \begin{pmatrix} 1 \\ 3 \end{pmatrix}, \quad r(A') = \begin{pmatrix} 3 \\ 1 \end{pmatrix}, \quad r(B) = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

Next, we find that $r(A) = L^2 \left(\begin{smallmatrix} 1 \\ 1 \end{smallmatrix}\right)$ and $r(A') = R^2 \left(\begin{smallmatrix} 1 \\ 1 \end{smallmatrix}\right)$, thus we have:

$$W_A = L^2, \quad W_{A'} = R^2, \quad W_B = \epsilon$$

This gives us the following immediate offshoots:

$$B_A = \{L^3, L^2R, LR, R\}, \quad B_{A'} = \{R^3, R^2L, RL, L\}, \quad B_B = \{L, R\}$$

Theorem 3.5. *Let $M \in \mathcal{RB}_n$ and $W \in B_M$. Then a non-empty word $W' \in \{L, R\}^*$ and a matrix $M' \in \mathcal{DB}_n$ exist such that $MW = W'M'$.*

The proof of this theorem is quite long and technical and would take up several pages. The proof can be found in [7] in sections 4 and 5.

3.3 Creating a transducer

Using Theorem 3.5, we can note the following: Suppose $M \in \mathcal{DB}_n$. Since M is row-balanced, for every $W \in B_M$ a non-empty word $W' \in \{L, R\}^*$ and a matrix $M' \in \mathcal{DB}_n$ exist such that $MW = W'M'$. Using the definition of B_M, it is easy to see that B_M is a base for the set of sequences Σ^∞. Thus, for every sequence $S \in \Sigma^\infty$, we can find a prefix $W_S \in B_M$ and a sequence $S' \in \Sigma^*$ such that $S = W_S \parallel S'$. This means that

$$\begin{aligned} M \prod_i S_i &= M \cdot \prod_i (W_S)_i \cdot \prod_i S'_i \\ &= \prod_i (W'_S)_i \cdot M' \cdot \prod_i S'_i \end{aligned} \tag{2}$$

for a certain $W'_S \in \{L, R\}^*$.

Suppose now that $v = \xi \left(\begin{smallmatrix} \alpha \\ 1 \end{smallmatrix}\right) \in \mathbb{R}^2$ with $\xi > 0, \alpha = [a_0; a_1, a_2, ...] \geq 0$, $M = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathcal{DB}_n$ with $n \geq 1$ and $u = Mv \in \mathbb{R}^2$. It is important to note that, since $M \in \mathbb{Z}^{2\times 2}$, $\alpha \geq 0$ and $\xi > 0$, we can write u as $u = \xi' \left(\begin{smallmatrix} \beta \\ 1 \end{smallmatrix}\right)$ for certain $\xi' > 0$ and $\beta \geq 0$. Thus, as stated at the end of Section 3.1, if we can find the accepted sequence for u, we can deduce a regular continued fraction $[b_0; b_1, b_2, ...]$ for $\beta = \frac{a\alpha + b}{c\alpha + d}$. Using Equation (2) and Theorem 3.5, we can describe a procedure for finding the accepted sequence for u, using the accepted sequence S of v:

1. Since v accepts the sequence S, a unique $W \in B_M$ exists such that $W|S$. This means that a vector v' exists such that $v = W \cdot v'$.

2. For u, the following holds: $u = Mv = M \cdot W \cdot v'$. Since $W \in B_M$, a non-empty word W' and a doubly-balanced matrix M' exist such that $M \cdot W = W' \cdot M'$, so $u = M \cdot W \cdot v' = W' \cdot M' \cdot v' = W' \cdot u'$ for a certain $u' = M' \cdot v'$. Thus, u accepts the finite sequence W'.

3. The vectors v, u are set to $v := v', u := u'$, the matrix M is set to $M := M'$ and the sequence S is set to $S := S'$, where S' is the sequence for which $S = W||S'$.

4. The process is repeated from step 1., while every W' acquired in step 2. is concatenated to form an infinite sequence accepted by u.

With induction, it is easy to see that u accepts the sequence acquired from this process. We can write this sequence as '$R^{b_0} L^{b_1} R^{b_2} \cdots$' and with it deduce the regular continued fraction $[b_0; b_1, b_2, ...] = \beta$.

If we view every doubly-balanced matrix M as a state, and every equation $M \cdot W = W' \cdot M'$ as a transition, it becomes clear that this process can essentially be viewed as a sequence transducer. Since the set $\mathcal{DB}_n$ is finite for every $n \in \mathbb{Z}_{\geq 1}$, as stated in claim (3.1) of [7], this sequence transducer is a finite-state sequence transducer.

Definition 3.6. Let $g, n \in \mathbb{Z}_{\geq 1}$ with $g^2|n$ and let $M \in \mathcal{DB}_n$ where the greatest common divisor of the elements is equal to g. We define the finite-state sequence transducer $\mathcal{T}_{n,g}(M) = (Q, \Sigma, \Gamma, \delta, q_0)$ as follows:

- $Q = \{\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathcal{DB}_n | \gcd(a, b, c, d) = g\}$
- $\Sigma = \Gamma = \{L, R\}$
- $\delta(q, W)$ with $q \in Q, W \in \Sigma^*$ is defined if and only if $W \in B_q$ and is defined as $\delta(q, W) = (q', W')$ where $q \cdot W = W' \cdot q'$.

- $q_0 = M$

With this definition, it is clear that the transducers $\mathcal{T}_{n,g}(M)$ precisely execute the process described above. The set Q is limited to $Q = \{\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathcal{DB}_n |\, \gcd(a,b,c,d) = g\}$ rather than $Q = \mathcal{DB}_n$, because multiplication with L or R does not change the greatest common divisor of the elements of a matrix.

Example 3.7. Let $\alpha = 1 + \sqrt{7} = [3; \overline{1,1,1,4}], \beta = \frac{2\alpha+1}{\alpha+2}$. This means that the vector $v = \left(\begin{smallmatrix} \alpha \\ 1 \end{smallmatrix}\right) \in \mathbb{R}^2$ accepts the sequence '$R^3LRLR^4\cdots$'. Since $\beta = \frac{2\alpha+1}{\alpha+2}$, we define the matrix $M = \left(\begin{smallmatrix} 2 & 1 \\ 1 & 2 \end{smallmatrix}\right)$, so that Mv accepts the sequence corresponding to the RCF of β. With this matrix M, we can define the transducer $\mathcal{T}_{3,1}(M)$ shown in Figure 2.

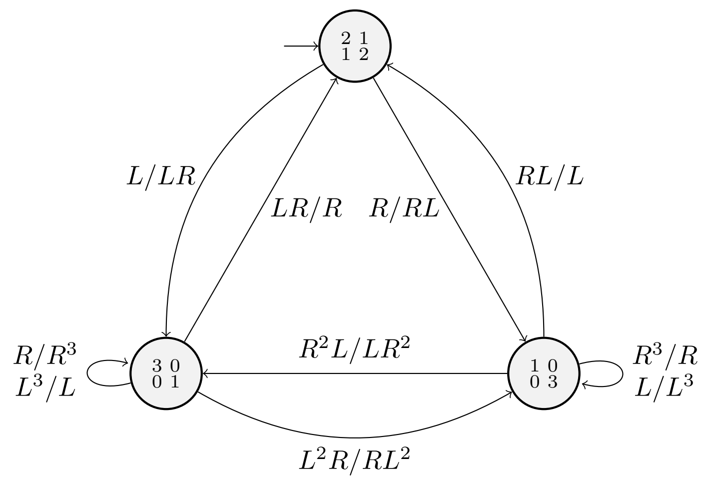

Figure 2: Transducer $\mathcal{T}_{3,1}$ with initial state $\left(\begin{smallmatrix} 2 & 1 \\ 1 & 2 \end{smallmatrix}\right)$.

Using $\mathcal{T}_{3,1}(M)$, we can transduce the accepted sequence for v into the accepted sequence for Mv. This results in $S_{Mv} =$ '$RL^2R^7L^2R^7\cdots$', so we can conclude that $\beta = \frac{2\alpha+1}{2+\alpha} = [1; \overline{2,7}]$. We can compute that indeed $[1; \overline{2,7}] = \frac{3+2\sqrt{7}}{3+\sqrt{7}} = \frac{2\alpha+1}{\alpha+2}$.

Using Raney's proof of Theorem 3.5, we can determine the size of the set of states Q for any transducer $\mathcal{T}_{n,g}$. First of all, we can note that if $g \neq 1$, then the transducer $\mathcal{T}_{n,g}$ is isomorphic to $\mathcal{T}_{\frac{n}{g^2},1}$. After all, when $\gcd(M) = \gcd(M') = g$, then $MW = W'M'$ holds if and only if $(\frac{1}{g}M)W = W'(\frac{1}{g}M')$. This means that it suffices to determine the sizes of $Q_{\mathcal{T}_{n,1}}$ for every n.

In his proof of Theorem 3.5, Raney states that for every n, a finite number of combinations (g, s, s') exists such that $g(g + s + s') = n$. Furthermore, he states that a one-to-one correlation exists between (g, s, s')-tuples and doubly-balanced matrices in $\mathcal{DB}_n$. That is to say, $\#\{(g, s, s') : g(g + s + s') = n\} = \#\mathcal{DB}_n$. The number of different (g, s, s')-tuples for any given n can be found using the formula

$$\#\{(g, s, s') : g(g + s + s') = n\} = \sum_{d|n} \left(\frac{n}{d} - d + 1\right)$$

If n has no square divisors, then $Q_{\mathcal{T}_{n,1}} = \mathcal{DB}_n$. If n does have square divisors, then $\mathcal{DB}_n$ is strictly larger than $Q_{\mathcal{T}_{n,1}}$. Since the set $\{Q_{\mathcal{T}_{n,g}} : g^2 \text{ divides } n\}$ is a partition of $\mathcal{DB}_n$, we can find the size of $Q_{\mathcal{T}_{n,1}}$ using the formula

$$\#Q_{\mathcal{T}_{n,1}} = \#\mathcal{DB}_n - \sum_{g^2|n} \#Q_{\mathcal{T}_{\frac{n}{g^2},1}} \tag{3}$$

A list of sizes of $\mathcal{DB}_n$ and $Q_{\mathcal{T}_{n,1}}$ for $n \in \{1, ..., 50\}$ can be found in appendix A.

3.4 Reducing to the doubly-balanced case

The transducers created in 3.3 are very useful if the matrix representation of the Möbius transformation is a doubly-balanced matrix, but the algorithm no longer holds up if this is not the case. At the end of his article, Raney introduces a method to reduce the general problem $\beta = \frac{a\alpha+b}{c\alpha+d}$ to a problem $\beta' = \frac{a'\alpha'+b'}{c'\alpha'+d'}$ where the matrix $\left(\begin{smallmatrix} a' & b' \\ c' & d' \end{smallmatrix}\right) \in \mathcal{DB}_n$ for a certain $n \in \mathbb{Z}_{\geq 1}$. This method is as follows:

Algorithm 3.8. Let $\alpha = [a_0; a_1, a_2, ...], \beta = \frac{a\alpha+b}{c\alpha+d}, M = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$.

1. If $a_0 < 0$; set $M \coloneqq \left(\begin{smallmatrix} a & b+a_0a \\ c & d+a_0c \end{smallmatrix}\right)$, $\alpha \coloneqq \alpha - a_0 = [0; a_1, a_2, ...]$ and $\beta \coloneqq \beta$.
2. If $\det(M) < 0$; set $M \coloneqq \left(\begin{smallmatrix} b & a \\ d & c \end{smallmatrix}\right)$, $\alpha \coloneqq \frac{1}{\alpha} = [0; a_0, a_1, ...]$ and $\beta \coloneqq \beta$.
3. If $\alpha < 1$, and $ab < 0$ or $cd < 0$; set $M \coloneqq ML$, $\alpha \coloneqq \frac{\alpha}{1-\alpha} = [a_0; a_1 - 1, a_2, ...]$ and $\beta \coloneqq \beta$.
   If instead $\alpha \geq 1$, and $ab < 0$ or $cd < 0$; set $M \coloneqq MR$, $\alpha \coloneqq \alpha - 1 = [a_0 - 1; a_1, a_2, ...]$ and $\beta \coloneqq \beta$.
   This step is repeated until both $ab \geq 0$ and $cd \geq 0$.
4. If $c < 0$ or $d < 0$; set $M \coloneqq -M$, $\alpha \coloneqq \alpha$ and $\beta \coloneqq \beta$.
5. If $a < 0$ or $b < 0$; let $m = -\min(\lfloor \frac{a}{c} \rfloor, \lfloor \frac{b}{d} \rfloor)$. Set $M \coloneqq R^m M$, $\alpha \coloneqq \alpha$, $\beta \coloneqq \beta + m$
6. If $\det(M) = n$ but $M \notin \mathcal{RB}_n$; two matrices P, Q exist with $Q \in \mathcal{RB}_n$, $P = \left(\begin{smallmatrix} x & y \\ z & w \end{smallmatrix}\right) = \prod_i V_i$ for some $V \in \{L, R\}^*$, and $M = PQ$. Let T be the accepted sequence corresponding to the RCF of β. Set $M \coloneqq Q$, $\alpha \coloneqq \alpha$ and $\beta = \frac{w\beta - y}{-z\beta + x}$. Here, the accepted sequence corresponding to the RCF of the new β is equal to T/V.
7. If $M \in \mathcal{RB}_n$ but $M \notin \mathcal{DB}_n$; let S be the accepted sequence corresponding to the RCF of α, let T be the accepted sequence corresponding to the RCF of β, and let $W \in B_M$ such that $W \mid S$. As per Theorem 3.5, a word $W' \in \{L, R\}^*$ and a matrix $M' \in \mathcal{DB}_n$ exist such that $MW = W'M'$. Let $\left(\begin{smallmatrix} x_1 & y_1 \\ z_1 & w_1 \end{smallmatrix}\right) = \prod_i W_i$ and $\left(\begin{smallmatrix} x_2 & y_2 \\ z_2 & w_1 \end{smallmatrix}\right) = \prod_i W'_i$ Set $M \coloneqq M'$, set α to correspond to the accepted sequence S/V and set β to correspond to the accepted sequence T/V'.

When all these steps have been executed, the matrix $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ corresponding to the new problem $\beta = \frac{a\alpha+b}{c\alpha+d}$ is doubly-balanced, so the original problem has been reduced to a problem that can be solved with a transducer $\mathcal{T}_{n,g}$ from section 3.3.

Remark. Since steps 1, 3, 4, 5, 6 and 7 of Algorithm 3.8 do not change the determinant of matrix M, and since step 2 changes the sign of the determinant of M if and only if $\det M < 0$, the determinant of the new matrix is the absolute value of that of the original matrix.

3.5 Example

Let $\alpha = [1; \overline{2,3,4,1}]$ and $\beta = \mu(\alpha) = \frac{-37\alpha+43}{-73\alpha+85}$. The Möbius transformation μ has a determinant of -6, so we will have to reduce this problem to a problem $\beta' = \frac{a\alpha'+b}{c\alpha'+d}$ with $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathcal{DB}_6$.

Let us define $M = \left(\begin{smallmatrix} -37 & 43 \\ -73 & 85 \end{smallmatrix}\right)$. Following the steps from Algorithm 3.8, we get the following:

1. $a_0 = 1 \geq 0$, so M, α and β remain unchanged.
2. $\det(M) = -6 < 0$, so we set $M := \left(\begin{smallmatrix} 43 & -37 \\ 85 & -73 \end{smallmatrix}\right)$ and $\alpha := [0; \overline{1,2,3,4}]$.
3. Repeated application of this step results in:
   - $M := \left(\begin{smallmatrix} 6 & -37 \\ 12 & -73 \end{smallmatrix}\right)$, $\alpha = [0; 0, \overline{2,3,4,1}] = [2; \overline{3,4,1,2}]$.
   - $M := \left(\begin{smallmatrix} 6 & -31 \\ 12 & -61 \end{smallmatrix}\right)$, $\alpha = [1; \overline{3,4,1,2}]$.
   - $M := \left(\begin{smallmatrix} 6 & -25 \\ 12 & -49 \end{smallmatrix}\right)$, $\alpha = [0; \overline{3,4,1,2}]$.
   - $M := \left(\begin{smallmatrix} -19 & -25 \\ -37 & -49 \end{smallmatrix}\right)$, $\alpha = [0; 2, \overline{4,1,2,3}]$.
4. $c < 0$ and $d < 0$, so we set $M := \left(\begin{smallmatrix} 19 & 25 \\ 37 & 49 \end{smallmatrix}\right)$.
5. $a \geq 0$ and $b \geq 0$, so M, α and β remain unchanged.
6. $M \notin \mathcal{RB}_6$, but $M = LRL^{18} \left(\begin{smallmatrix} 1 & 1 \\ 0 & 6 \end{smallmatrix}\right)$ with $\left(\begin{smallmatrix} 1 & 1 \\ 0 & 6 \end{smallmatrix}\right) \in \mathcal{RB}_6$, so we set $M := \left(\begin{smallmatrix} 1 & 1 \\ 0 & 6 \end{smallmatrix}\right)$ and $\beta := \beta / LRL^{18}$.
7. $M \notin \mathcal{DB}_6$, but $ML = L^3 \left(\begin{smallmatrix} 2 & 1 \\ 0 & 3 \end{smallmatrix}\right)$ with $\left(\begin{smallmatrix} 2 & 1 \\ 0 & 3 \end{smallmatrix}\right) \in \mathcal{DB}_6$, so we set $M := \left(\begin{smallmatrix} 2 & 1 \\ 0 & 3 \end{smallmatrix}\right)$, $\alpha := [0; 1, \overline{4,1,2,3}]$ and $\beta := \beta / L^3$.

To construct the the transducer $\mathcal{T}_{6,1}$, we need to know the elements of $\mathcal{DB}_6$. Since 6 has no square divisors, all matrices in $\mathcal{DB}_6$ are states in $\mathcal{T}_{6,1}$. We can easily find the following eight doubly-balanced matrices of determinant 6:

$$A = \begin{pmatrix} 6 & 0 \\ 0 & 1 \end{pmatrix} \qquad A' = \begin{pmatrix} 1 & 0 \\ 0 & 6 \end{pmatrix}$$
$$B = \begin{pmatrix} 3 & 0 \\ 0 & 2 \end{pmatrix} \qquad B' = \begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix}$$
$$C = \begin{pmatrix} 3 & 1 \\ 0 & 2 \end{pmatrix} \qquad C' = \begin{pmatrix} 2 & 0 \\ 1 & 3 \end{pmatrix}$$
$$D = \begin{pmatrix} 3 & 0 \\ 1 & 2 \end{pmatrix} \qquad D' = \begin{pmatrix} 2 & 1 \\ 0 & 3 \end{pmatrix}$$

Using Equation 3, we see that $\#Q_{\mathcal{T}_{6,1}} = 8$, so these are all the states of transducer $\mathcal{T}_{6,1}$.

Finding all transitions requires some work, but is easily accomplished. For $M = A$ and $W = L^2R \in B_A$ we find:

$$AL^2R = \begin{pmatrix} 6 & 6 \\ 2 & 3 \end{pmatrix} = R \begin{pmatrix} 4 & 3 \\ 2 & 3 \end{pmatrix} = R^2 \begin{pmatrix} 2 & 0 \\ 2 & 3 \end{pmatrix} = R^2 LB'$$

Continuing this for every matrix and every immediate offshoot, we find that the transducer $\mathcal{T}_{6,1}$ is as shown in Table 1 and Figure 3.

The states in the left column are the starting states, and the states in the top row are the destination states of the transitions.

Table 1: Transitions of $\mathcal{T}_{6,1}$

	A	B	C	D	D'	C'	B'	A'
A	L^6/L, R/R^6	L^3R/RLR		LR/R^3	L^4R/RL^2		L^2R/R^2L	L^5R/RL^5
B	LRL/LR^3		R/R	L^2/L				LR^2/RL^2
C	L^3/LR	R/R^2			LR/RL			L^2R/RL^4
D		L/L				R/R		
D'			L/L				R/R	
C'	R^2L/LR^4			RL/LR			L/L^2	R^3/RL
B'	RL^2/LR^2				R^2/R	L/L		RLR/RL^3
A'	R^5L/LR^5	R^2L/L^2R		R^4L/LR^2	RL/L^3		R^3L/LRL	R^6/R, L/L^6

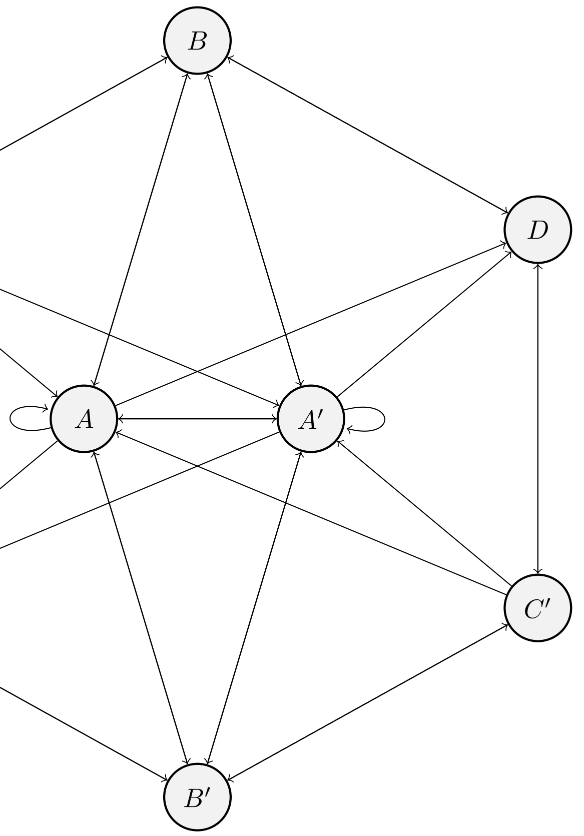

Figure 3: Schematic representation of transducer $\mathcal{T}_{6,1}$ without transition labels

Using the reduced problem $\beta' = \frac{2\alpha+1}{3}$ with $\alpha = [0; 1, \overline{4, 1, 2, 3}]$, we find that $\beta' = [0; 1, 7, \overline{1, 1, 4, 1, 4, 1, 1, 6, 1, 17, 1, 6}]$. Tracing back steps 7 through 1 of the algorithm, we deduce that the RCF for the original β is equal to $[0; 1, 1, 22, 7, \overline{1, 1, 4, 1, 4, 1, 1, 6, 1, 17, 1, 6}]$. This continued fraction evaluates to $\beta = \frac{3809+3\sqrt{(39)}}{7490}$, which is indeed equal to $\mu(\alpha) = \frac{-37\alpha+43}{-73\alpha+85}$.

4 Direct extension to nearest integer continued fractions

This section serves to provide a first step to trying to extend Raney's algorithm to nearest integer continued fractions, followed by the big issues I have come across that make it impossible to develop a direct extension of Raney's algorithm to the NICF case.

4.1 Matrix representation for NICF

If we wish to extend Raney's algorithm to the nearest integer chain fraction, it is evident we need to make a number of additions to the types of matrices used. The original algorithm only allows for positive integers in its continued fractions. Since nearest integer continued fraction can contain negative numbers, the system of writing continued fractions as a sequences of L and R matrices no longer suffices. The most intuitive expansion of the set of L and R matrices would be the introduction of matrices $L^{-1} = \left(\begin{smallmatrix} 1 & 0 \\ -1 & 1 \end{smallmatrix}\right)$ and $R^{-1} = \left(\begin{smallmatrix} 1 & -1 \\ 0 & 1 \end{smallmatrix}\right)$. Furthermore, we will define $\Lambda = \{L, L^{-1}, R, R^{-1}\}$. Theorem 4.3 will prove that this new alphabet suffices and functions in the same way as $\{L, R\}$ does in Raney's original algorithm.

Remark. The matrices L^{-1} and R^{-1} are the respective multiplicative inverse matrices of L and R. As such, for any $n \in \mathbb{N}$, $(L^{-1})^n = L^{-n}$ and $(R^{-1})^n = R^{-n}$. Because of this equality, we will write any word '$(L^{-1})^n$' as 'L^{-n}' and '$(R^{-1})^n$' as 'R^{-n}'.

Lemma 4.1. *The multiplicative group of $\mathbb{Z}^{2\times 2}$-matrices of determinant 1 is generated by the set Λ.*

Proof. The multiplicative group of $\mathbb{Z}^{2\times 2}$-matrices of determinant 1 is generated by the set of all whole elementary matrices of size 2×2 of determinant 1. These are precisely all matrices $\left(\begin{smallmatrix} 1 & n \\ 0 & 1 \end{smallmatrix}\right)$, $\left(\begin{smallmatrix} 1 & 0 \\ n & 1 \end{smallmatrix}\right)$ with $n \in \mathbb{Z}$. These are precisely the matrix products R^n and L^n for $n \in \mathbb{Z}$.

□

Definition 4.2. We call a sequence $S \in \Lambda^\infty$ *simplified* if no non-empty, finite, continuous subsequence $S_N S_{N+1} \cdots S_{M-1} S_M$ exists such that $\prod_{i=N}^{M} S_i = I_{2\times 2}$.

Remark. If a sequence S is simplified, then no L and L^{-1} can occur adjacent to each other, and no R and R^{-1} can occur adjacent to each other. This means that S can be written as $S =$ '$R^{a_0} L^{a_1} R^{a_2} \cdots$' for certain $a_i \in \mathbb{Z}$.

Theorem 4.3. *Let $v = \xi \left(\begin{smallmatrix} \alpha \\ 1 \end{smallmatrix}\right) \in \mathbb{R}^2$. If α is irrational and $\xi > 0$, then the only simplified Λ-sequences $R^{a_0} L^{a_1} R^{a_2} \cdots$ that are accepted by v, are those for which $\alpha = [a_0; a_1, a_2, ...]$. Note: $[a_0; a_1, a_2, ...]$ is not necessarily an NICF.*

Proof. Let $v = \xi \left(\begin{smallmatrix} \alpha \\ 1 \end{smallmatrix}\right) \in \mathbb{R}^2$ with $\xi \neq 0$. Furthermore, let us define a function $d : \mathbb{R}^2 \backslash \{\left(\begin{smallmatrix} x \\ 0 \end{smallmatrix}\right) \mid x \in \mathbb{R}\} \to \mathbb{R}$ as $d\left(\begin{smallmatrix} x_1 \\ x_2 \end{smallmatrix}\right) = \frac{x_1}{x_2}$.

Suppose $d(v) = \alpha = [a_0; a_1, a_2, ...]$ with $a_0 \in \mathbb{Z}$ and $a_1, a_2, ... \in \mathbb{Z}\backslash\{0\}$. Like in the proof for Theorem 3.2, we can note that, even for negative integers a_n

$$\begin{pmatrix} p_{n-1} & p_{n-2} \\ q_{n-1} & q_{n-2} \end{pmatrix} R^{a_n} = \begin{pmatrix} p_{n-1} & p_n \\ q_{n-1} & q_n \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} p_{n-2} & p_{n-1} \\ q_{n-2} & q_{n-1} \end{pmatrix} L^{a_n} = \begin{pmatrix} p_n & p_{n-1} \\ q_n & q_{n-1} \end{pmatrix}$$

Because of this, we can again say that

$$\begin{pmatrix} p_{n-1} & p_n \\ q_{n-1} & q_n \end{pmatrix} = R^{a_0} L^{a_1} \cdots R^{a_n} \quad \text{if } n \text{ is even, and}$$
$$\begin{pmatrix} p_n & p_{n-1} \\ q_n & q_{n-1} \end{pmatrix} = R^{a_0} L^{a_1} \cdots L^{a_n} \quad \text{if } n \text{ is odd.}$$

With this, we can repeat the entirety of the proof for Theorem 3.2 to conclude that the only sequences '$R^{a_0} L^{a_1} R^{a_2} \cdots$' accepted by v are those for which $\alpha = [a_0; a_1, a_2, ...]$, and therefore this also holds for simplified sequences in particular.

□

This theorem is not as strong as Theorem 3.2, since vectors can accept multiple sequences now rather than only those corresponding to the nearest integer continued fraction, but it does show that nearest integer continued fractions can be represented with a sequence of elements of Λ.

Remark. One can trivially see that for every sequence $S \in \Lambda^\infty$ for which the corresponding NICF converges to a value $x \neq 0$, a unique simplified sequence $S' \in \Lambda^\infty$ exists such that the continued fractions corresponding to S' also converges to x, and such that a sequence $\{i_k\}_{k=0}^\infty$ exists such that $S'_k = S_{i_k}$ and $S_{i_k+1} \cdots S_{i_{k+1}-1} = I$ for every $k \in \mathbb{N}$. We call this sequence S' the simplified sequence of S. This is simply the sequence for which every continuous subsequence that derives to I is removed.

4.2 Creating transducers for NICF

Since we always use a single input and output alphabet and since every state in a transducer for Raney's algorithm is a valid initial state, the possible transducers for an extended Raney's algorithm are essentially defined solely by their sets of states and their transitions. Neither of these can unambiguously be translated from standard Raney to NICF.

The first problem already crops up when we try to define a set B_M for a matrix $M \in \mathbb{Z}^{2\times 2}$. In Raney's algorithm, the elements of B_M are used to define transitions between states, so a direct translation from RCF to NICF would require an analogue $B'_M \subseteq \Lambda^*$ such that B'_M is a base for Λ^∞. In Raney's algorithm, the set B_M is defined using the generating word W_M, which is found by applying the Euclidean algorithm to the coefficients of $r(M) = \binom{a}{b}$. Since we only used positive numbers, this resulted in a unique word W_M. If we can use negative numbers for the Euclidean algorithm, there is no longer a unique way to compute $\gcd(a, b)$, which means that we can find different W_M and W'_M such that $r(M) = W_M \begin{pmatrix} \text{gcd} \\ \text{gcd} \end{pmatrix} = W'_M \begin{pmatrix} \text{gcd} \\ \text{gcd} \end{pmatrix}$. This means that B'_M cannot be uniquely defined. Thus, unless we impose restrictions on the application of the Euclidean algorithm, we cannot directly translate the method of finding transitions in Raney's algorithm to a method for finding transitions in an extended algorithm. It should be noted, however, that the algorithm for finding the finite NICF for a fraction $\frac{a}{b}$ induces a very specific application of the Euclidean algorithm $\gcd(a, b)$. This application seems like a good candidate for finding a specific B'_M for every M.

Suppose we impose specific restrictions on the application of the Euclidean algorithm such that we get a unique B'_M for every matrix $M \in \mathbb{Z}^{2\times 2}$ (like the application induced by the NICF algorithm, or something else entirely). We can typically assume that if $r(M) = \binom{a}{a}$ for some matrix M, the result of the algorithm is $W_M = \epsilon$, and thus

$B'_M = \Lambda$. We will find that the set of doubly-balanced matrices $\mathcal{DB}_n$ will no longer suffice for a transducer for transformations of determinant n.

Let $M = \left(\begin{smallmatrix} 2 & 1 \\ 1 & 2 \end{smallmatrix}\right) \in \mathcal{DB}_3$. In this case, $r(M) = \left(\begin{smallmatrix} 1 \\ 1 \end{smallmatrix}\right)$, so B'_M will be equal to Λ. Take $W = L^{-1} \in \Lambda$. If $MW = W'M'$ for certain $W', M' \in \mathbb{Z}^{2\times 2}$, this means that $ML^{-1}(M')^{-1} = W'$. We can check that $ML^{-1}(M')^{-1} \notin \mathbb{Z}^{2\times 2}$ for every $M' \in \mathcal{DB}_3$. Thus, since $\mathcal{DB}_3 \subseteq \mathbb{Z}^{2\times 2}$, we can conclude that no $M' \in \mathcal{DB}_3$ exists such that $MW = W'M'$ for some $W' \in \mathbb{Z}^{2\times 2}$. Thus, the set of doubly-balanced matrices as defined in Definition 3.3 does not suffice as a set of states for most typical applications of the Euclidean algorithm to define transitions.

It is also important to note that it is impossible to use the set of all matrices $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathbb{Z}^{2x2}$ for which $a > b$, $a > c$, $d > b$ and $d > c$, since this set is infinite and can therefore not be used for a *finite-state* transducer.

In order to be able to extend Raney's algorithm to NICF, a solution needs to be found to either or both of these problems. If one wishes to retain the sets $\mathcal{DB}_n$ as sets of states, a set of restrictions must be found on the application of the Euclidean algorithm such that $MW = W'M'$ can always be solved with a matrix $M \in \mathcal{DB}_n$, if such a set of restrictions exists at all. This set of restrictions must be very non-typical, since W_M is is not allowed to be equal to ϵ for $M = \left(\begin{smallmatrix} 2 & 1 \\ 1 & 2 \end{smallmatrix}\right)$, and thus cannot be those induced by the NICF algorithm. Otherwise, a new analogue for doubly-balanced matrices needs to be found.

4.3 Validity of the transducer output

As stated in section 4.2, matrices of determinant 1 can no longer uniquely be written as a product of matrices from our alphabet. This means that, unlike in Raney's algorithm for regular continued fractions, there will no longer be a unique transducer $\mathcal{T}_{n,g}$ for every (n, g) pair. If multiple words $W_1, W_2 \in \Lambda^*$ exist such that $\prod_i (W_1)_i = \prod_i (W_2)_i$, then any transducer $\mathcal{T}$ with a transition $\delta(M, W) = (M', W_1)$ can be replaced by a transducer $\mathcal{T}'$ where $\delta(M, W) = (M', W_2)$, which would generate a different sequence. This means that if $\mathcal{T}$ is a valid transducer for a certain set of matrices, then so is $\mathcal{T}' \neq \mathcal{T}$.

Considering that every $\alpha \in \mathbb{R}$ has a unique NICF $[a_0; a_1, a_2, ...]$, this means that a transducer obtained from the algorithm does not necessarily produce a nearest integer continued fraction, but rather can produce any continued fraction with integer coefficients.

Thus, in order to obtain a NICF from the algorithm, a method is required to transform the output of any given transducer $\mathcal{T}$ into a Λ-sequence that directly corresponds to a NICF.

4.3.1 Transforming a continued fraction into a nearest integer continued fraction

As stated in Theorem 2.5, nearest integer continued fractions $[a_0; a_1, a_2, ...]$ adhere to the following constraints:

- $a_n \notin \{-1, 0, 1\}$ for all $n \in \mathbb{N}_{>0}$.
- If $|a_n| = 2$, then $\texttt{sgn}(a_{n+1}) = \texttt{sgn}(a_n)$ for all $n \in \mathbb{N}_{>0}$.

Suppose a continued fraction $[a_0; a_1, a_2, ...]$ does not adhere to both of these constraints, then we require a method to transform the continued fraction into a continued fraction that does adhere to them.

Proposition 4.4. $[a;1,b,x]=[a+1;-b-1,-x]$ *and* $[a;-1,b,x]=[a-1;-b+1,-x]$.

Proof.

$$[a;1,b,x]=a+\cfrac{1}{1+\cfrac{1}{b+\frac{1}{x}}}=a+\cfrac{1}{\cfrac{b+1+\frac{1}{x}}{b+\frac{1}{x}}}=a+\frac{-b-\frac{1}{x}}{-b-1-\frac{1}{x}}$$
$$=a+1+\frac{1}{-b-1+\frac{1}{-x}}=[a+1;-b-1,-x]$$

$$[a;-1,b,x]=a+\cfrac{1}{-1+\cfrac{1}{b+\frac{1}{x}}}=a+\cfrac{1}{\cfrac{-b+1-\frac{1}{x}}{b+\frac{1}{x}}}=a+\frac{b+\frac{1}{x}}{-b+1-\frac{1}{x}}$$
$$=a-1+\frac{1}{-b+1+\frac{1}{-x}}=[a-1;-b+1,-x]$$

□

Using Proposition 4.4, we can try to fix the continued fraction $[a_0;a_1,a_2,...]$ as follows:

If $a_n=1$ and $a_i\notin\{-1,0,1\}$ for all $i<n$, let $a'_{n-1}=a_{n-1}+1$, $a'_n=-a_{n+1}-1$, $a'_k=a_k$ for every $k<n-1$ and $a'_m=-a_{m+1}$ for every $m>n$. From Proposition 4.4 we see that $[a'_0;a'_1,a'_2,...]=[a_0;a_1,a_2,...]$. We call this transformation of continued fractions "singularisation".

Suppose that $|a_i|=2\implies \mathtt{sgn}(a_{i+1})=\mathtt{sgn}(a_i)$ for all $i\le n-1$. Since $a_n>0$, this means that $a_{n-1}\neq -2$, and therefore $a'_{n-1}\neq -1$. Thus, $a'_k\notin\{-1,0,1\}$ for all $k<n$ in the new continued fraction.

Likewise, if $a_n=-1$ and $a_i\notin\{-1,0,1\}$ for all $i<n$, let $a'_{n-1}=a_{n-1}-1$, $a'_n=-a_{n+1}+1$, $a'_k=a_k$ for every $k<n-1$ and $a'_m=-a_{m+1}$ for every $m>n$. Using Proposition 4.4, we see that $[a'_0;a'_1,a'_2,...]=[a_0;a_1,a_2,...]$.

Suppose again that $|a_i|=2\implies \mathtt{sgn}(a_{i+1})=\mathtt{sgn}(a_i)$ for all $i\le n-1$. Since $a_n<0$, this means that $a_{n-1}\neq 2$, and therefore $a'_{n-1}\neq 1$. Thus, $a'_k\notin\{-1,0,1\}$ for all $k<n$ in the new continued fraction.

Likewise, we can introduce a rule to be able to correct occurrences of $a_n=2$ and $a_{n+1}<-1$, or $a_n=-2$ and $a_{n+1}>1$.

Proposition 4.5. $[a;2,b,x]=[a+1;-2,b+1,x]$ *and* $[a;-2,b,x]=[a-1;2,b-1,x]$.

Proof. Using Proposition 4.4, we find:

$$[a;2,b,x]=[a;1,1,-b-1,-x]=[a+1;-2,b+1,x] \text{ and}$$
$$[a;-2,b,x]=[a;-1,-1,-b+1,-x]=[a-1;2,b-1,x]$$

□

Suppose that $|a_i| = 2 \implies \mathtt{sgn}(a_{i+1}) = \mathtt{sgn}(a_i)$ for all $i \leq n-1$ and suppose that $a_k \notin \{-1, 0, 1\}$ for all $k \in \mathbb{N}$.

If $a_n = 2$ and $a_{n+1} < -1$, let $a'_{n-1} = a_{n-1} + 1$, $a'_n = -2$, $a'_{n+1} = a_{n+1} + 1$ and let $a'_k = a_k$ for all $k \in \mathbb{N}\backslash\{n-1, n, n+1\}$. Using Proposition 4.5, we find that $[a'_0; a'_1, a'_2, ...] = [a_0; a_1, a_2, ...]$. Since $a_{n-1} \neq 1$, we know that $a'_{n-1} \neq 2$, so $|a'_{n-1}| = 2 \implies \mathtt{sgn}(a'_n) = \mathtt{sgn}(a'_{n-1})$ holds. Furthermore, since $a_{n+1} < -1$, we can say that $\mathtt{sgn}(a'_{n+1}) = \mathtt{sgn}(a'_n)$.

Likewise, if $a_n = -2$ and $a_{n+1} > 1$, let $a'_{n-1} = a_{n-1} - 1$, $a'_n = 2$, $a'_{n+1} = a_{n+1} - 1$ and let $a'_k = a_k$ for all $k \in \mathbb{N}\backslash\{n-1, n, n+1\}$. Using Proposition 4.5, we find that $[a'_0; a'_1, a'_2, ...] = [a_0; a_1, a_2, ...]$. Since $a_{n-1} \neq -1$, we know that $a'_{n-1} \neq -2$, so $|a'_{n-1}| = 2 \implies \mathtt{sgn}(a'_n) = \mathtt{sgn}(a'_{n-1})$ holds. Furthermore, since $a_{n+1} > 1$, we can say that $\mathtt{sgn}(a'_{n+1}) = \mathtt{sgn}(a'_n)$.

On their own, these two methods work well enough, but when combined to try to make an entire continued fraction adhere to both constraints from Theorem 2.5, these methods shed light on a glaring issue with trying to convert a random continued fraction to NICF-form.

Take the continued fraction $[a_0; ..., a_n, -3, 1, -2, 2, a_n - 1, a_{n-1}, ..., a_0, -b_0, -b_1, ...]$, where $a_k \notin \{-1, 0, 1\}$ and $|a_k| = 2 \implies \mathtt{sgn}(a_{k+1}) = \mathtt{sgn}(a_k)$ for all $k \leq n$. Applying 4.4 and 4.5, we get:

$$\begin{aligned}
&[a_0; ..., a_n, -3, 1, -2, 2, a_n - 1, a_{n-1}, ..., a_0, -b_0, -b_1, ...] \\
&= [a_0; ..., a_n, -2, 1, -2, -a_n + 1, -a_{n-1}, ..., -a_0, b_0, b_1, ...] \\
&= [a_0; ..., a_n - 1, 2, 0, -2, -a_n + 1, -a_{n-1}, ..., -a_0, b_0, b_1, ...] \\
&= [a_0; ..., a_n - 1, 0, -a_n + 1, -a_{n-1}, ..., -a_0, b_0, b_1, ...] \\
&\ \vdots \\
&= [b_0; b_1, b_2, ...]
\end{aligned}$$

In essence, in this example the finite continued fraction $[a_0; ..., a_n, -3, 1, -2, 2, a_n - 1, a_{n-1}, ..., a_0]$ evaluates to 0. If a continued fraction $[a_0, a_1, a_2, ...]$ with negative coefficients is not an NICF, the sequence $\{a_i\}_{i=0}^{\infty}$ may contain a finite, continuous subsequence that evaluates to 0. Since these subsequences can be of arbitrary length, at no point while rewriting $[a_0; a_1, a_2, ...]$ using whatever set of rewrite rules, can one say with certainty that no part of their current rewritten prefix $[a'_0; a'_1, ..., a'_n]$ may be part of a subsequence that evaluates to 0. Because of this, the sequence $[a_0; a_1, a_2, ...]$ cannot be used to give a certainly correct truncation of the corresponding NICF within finite time.

To combat this issue, an expansion of Raney's algorithm for NICF would have to directly generate a sequence corresponding to a NICF.

4.3.2 Simplifying Λ-sequences

Another issue with the output from a transducer with output alphabet Λ is that one cannot automatically assume output sequences to be simplified. To be able to get a continued fraction out of the output sequence, the sequence will first need to be simplified. With simplification, a similar issue crops up as in section 4.3.1.

To simplify a Λ-sequence $\{S_n\}_{n\in\mathbb{N}}$, one can use the following algorithm:

1. Let $n := 1$
2. If $S_{n+1} = S_n^{-1}$, let $S_k := S_{k+2}$ for all $k \geq n$ and let $n := n - 1$. Else, let $n := n + 1$.

3. Repeat step 2. indefinitely.

Like with the algorithms in section 4.3.1, since sequences of Λ-matrices that evaluate to 0 can be of arbitrary length, there is no way to say for certain that any section '$S_0 \cdots S_n$' corresponds to the section '$S'_0 \cdots S'_n$' of the simplified sequence S' of S.

Thus, in order to produce a simplified Λ-sequence, the generalised algorithm will have to directly produce a simplified output sequence.

5 Roundabout algorithm

Rather than trying to solve the issues that come with directly adapting Raney's algorithm, one could attempt to circumvent these issues by taking a roundabout way to implement Raney's algorithm. Since there exist ways to convert RCF sequences into NICF sequences and vice versa, it is easy to see that $\beta = \frac{a\alpha+b}{c\alpha+d}$ can be computed by transforming the NICF for α into a RCF, applying Raney's algorithm to this RCF sequence, and then transforming the output of Raney's algorithm into an NICF sequence.

According to section 4.2.2 of [4], applying singularisation on a RCF in a specific order gives precisely the corresponding NICF. Likewise, reverse singularisation can be applied on an NICF to find the corresponding RCF. This can be applied without encountering the problems found in section 4.3.1. This gives us a way to effectively compute $\beta = \frac{a\alpha+b}{c\alpha+d}$ for any NICF without errors. The issue with this approach, however, is its computability.

Rather than using a small, finite alphabet, singularisation directly uses the integers a_i of the continued fractions $[a_0; a_1, a_2, ...]$ it is applied to. This means that it effectively uses the entirety of $\mathbb{Z}$ as its alphabet. This makes it impossible to create a finite state transducer for this roundabout Raney's algorithm, since either an infinite alphabet or an infinite set of states would be required. One can place bounds on the size of the integers a_i to make the alphabet finite, but this still does not make it possible to create a finite state sequence transducer for repeated singularisation over a whole CF sequence.

Take the continued fraction $[a; 1, b]$. Applying singularisation on 1 in this sequence results in the sequence $[a+1, -b-1]$. Since transducers cannot retroactively alter their output, this means that a and b must be processed at the same time as 1 by the transducer to ensure that the input word $a1b$ results in the output word $(a+1)(-b-1)$. Thus, we get the transition $(q, a1b) \mapsto (q', (a+1)(-b-1))$ for some states q and q'.
Let us now consider the contined fraction $[a; 1, ..., 1, b]$ with some arbitrary number of 1's. Applying singularisation once on the first 1 results in $[a+1, -2, -1, ..., -1, -b]$. If we use the same transition as above, this first singularisation would output $(a+1)(-2)$ and leave the input sequence as $1, ..., 1, b$. As stated in [4], singularisation must be applied to every first, third, fifth, etc. occurrence of 1 in a sequence of 1's to obtain the NICF. This means that singularisation must be applied to the first 1 in the remaining sequence $1, ..., 1, b$, since this was the third 1 in the original sequence. This is impossible, however, since we have no character in the remaining input sequence that precedes this 1. Thus, we require a transition for the repeated singularisation of the entire sequence $a, 1, ..., 1, b$ at once.
If we require a different transition for every sequence $a, 1, ..., 1, b$ of arbitrary length, the transition function must be of infinite size. This, however, is not allowed in a finite state sequence transducer, so it is impossible to create such a transducer for repeated singularisation.

That is not to say that this roundabout Raney's algorithm is not at all computable. Unlike a finite state sequence transducer, a Turing machine is able to alter the contents of its output tape at any point in time and thus is able to handle repeated singularisation. Turing machines are more complex than finite state sequence transducers though, so the roundabout algorithm cannot be computed as simply as a direct NICF-extension of Raney's algorithm could.

6 Discussion

In this thesis, we studied how Raney's algorithm can be extended to be used for NICF. We identified the important issues that would have made it unfeasible to find such an extension within the scope of a Bachelor thesis. The question of how or if Raney's algorithm can be expanded to NICF (and subsequently to Hurwitz continued fraction) remains open, and would require further research to be answered. It is important to note that if a solution to the issue in section 4.2 is found, we effectively obtain an algorithm that results in the correct sequence, albeit in infinite time. Although it running in infinite time is not entirely desirable, it is nevertheless an algorithm that does result in a correct NICF.

A Appendix

Sizes of the sets DB_n and $Q_{\mathcal{T}_{n,1}}$ for determinants n from 1 to 50.

n	$\#\mathcal{DB}_n$	$\#Q_{\mathcal{T}_{n,1}}$	n	$\#\mathcal{DB}_n$	$\#Q_{\mathcal{T}_{n,1}}$
1	1	1	26	38	38
2	2	2	27	34	31
3	3	3	28	45	38
4	5	4	29	29	29
5	5	5	30	54	54
6	8	8	31	31	31
7	7	7	32	52	41
8	11	9	33	42	42
9	10	9	34	50	50
10	14	14	35	38	38
11	11	11	36	70	56
12	19	16	37	37	37
13	13	13	38	56	56
14	20	20	39	50	50
15	18	18	40	70	56
16	24	19	41	41	41
17	17	17	42	76	76
18	30	28	43	43	43
19	19	19	44	73	62
20	31	26	45	63	58
21	26	26	46	68	68
22	32	32	47	47	47
23	23	23	48	97	78
24	44	36	49	50	49
25	26	25	50	80	78

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