Complex Continued Fraction Algorithms

A thesis presented in partial fulfilment
of the requirements for the degree of
Master of Mathematics

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1 Introduction

In this chapter we will briefly look into the history of continued fractions. Subsequently, the outline of this thesis is explained and finally we will look at some basic definitions and notation that are used in this thesis.

1.1 History

The origin of real continued fractions can be traced back to the Euclidean algorithm, which was introduced around 300 BCE. The Euclidean algorithm is a procedure for finding the greatest common divisor of two natural numbers m and n, but generates as a by-product a continued fraction of $\frac{m}{n}$. Still, it is not clear whether Euclid and his contemporaries recognised this phenomenon. Another early appearance can be found in the 6th century, when Aryabhata used continued fractions to solve linear diophantine equations. Although continued fractions often appear in the history of mathematics, it was not until the end of the 17th century that the theory of continued fractions was generalised. Before this date continued fractions only emerged in specific examples, e.g., Bombelli calculated a continued fraction of $\sqrt{13}$ in 1579. In 1695, Wallis was the first to build up the basic groundwork for the theory of continued fractions. Huygens used the convergents of a continued fraction to find the best rational approximations for gear ratios, which he used to build a mechanical planetarium. In the 18th century Euler, Lambert and Lagrange made huge contributions to the theory of continued fractions. Euler showed that every rational number can be expressed as a finite continued fraction. Lambert was the first to prove that π is irrational by using continued fractions and Lagrange proved that the continued fraction of a quadratic irrational number is periodic.

Along with real continued fractions, complex continued fractions have also been studied. In 1887, A. Hurwitz generalised the nearest integer continued fraction expansion to the complex numbers, where the partial quotients are Gaussian integers. Fifteen years later, his brother J. Hurwitz devised two complex continued fraction expansions where the partial quotients are to be taken from the set $(1 + i)\mathbb{Z}[i]$. Another famous complex continued fraction is due to A. L. Schmidt, which he defined in 1975, but his approach is fairly different from that of the brothers Hurwitz. In 1979, J. O. Shallit devised a complex continued fraction expansion that generalises the regular continued fraction expansion. It is notable that only some of the nice properties of real continued fractions also hold for complex continued fractions.

To this day, the theory of continued fractions is a flourishing field in mathematics and has multiple applications in other fields.

1.2 Thesis outline

In Chapter 2 of this thesis we will define what a complex continued fraction is and have a quick look at the theory of complex continued fractions. In Chapter 3 we give the definition of a complex continued fraction algorithm and we will extensively examine the properties of these algorithms, such as convergence and periodicity. In Chapter 4 we will see that the algorithm of A. Hurwitz fits in our general framework of complex continued fraction algorithms. Additionally, we will study specific properties of this algorithm. In Chapter 5 it is shown that also the algorithms of J. Hurwitz suit this framework, and we will look at some special properties of his algorithms. In Chapter 6 the algorithm of J. O. Shallit is defined. We will see that also this algorithm fits in our framework and we will examine some particular properties of this algorithm. In Chapter 7 we will investigate the relation between complex continued fraction algorithms and real continued fraction algorithms.

1.3 Basic definitions and notation

In this section we will provide some useful definitions and notation. Most of this is standard, but given for completeness.

Let $\mathbb{N} = \{0, 1, 2, 3, \ldots\}$ be the set of all natural numbers. Let $\mathbb{N}_{>0}$ be the set of natural numbers which are strictly greater than 0. Let $\mathbb{Z} = \{\ldots, -2, -1, 0, 1, 2, \ldots\}$ be the set of all integers. Let $\mathbb{Q} = \{\frac{p}{q} \mid p \in \mathbb{Z}, q \in \mathbb{N}_{>0}\}$ be the set of all real rational numbers. Let $\mathbb{R}$ be the set of all real numbers. Then we have $\mathbb{N} \subseteq \mathbb{Z} \subseteq \mathbb{Q} \subseteq \mathbb{R}$.

Let i be the imaginary unit, where $i^2 = -1$. Let $\mathbb{Z}[i] = \{a + bi \mid a, b \in \mathbb{Z}\}$ be the set of the Gaussian integers. Let $\mathbb{Q}[i] = \{\frac{p}{q} \mid p \in \mathbb{Z}[i],\ q \in \mathbb{Z}[i] \setminus \{0\}\}$ be the set of all complex rational numbers. Let $\mathbb{C}$ be the set of all complex numbers. Here we have $\mathbb{Z}[i] \subseteq \mathbb{Q}[i] \subseteq \mathbb{C}$. Moreover, we obtain $\mathbb{Z} \subseteq \mathbb{Z}[i]$, $\mathbb{Q} \subseteq \mathbb{Q}[i]$ and $\mathbb{R} \subseteq \mathbb{C}$.

Let $z \in \mathbb{C}$; we call z *rational* if $z \in \mathbb{Q}[i]$, and we call z *irrational* if $z \in \mathbb{C} \setminus \mathbb{Q}[i]$. We call z *quadratic irrational* if z is irrational and there exists A, B and $C \in \mathbb{Z}[i]$ such that $Az^2 + Bz + C = 0$, where $A \neq 0$.

Let $2\mathbb{Z} = \{2a \mid a \in \mathbb{Z}\}$ and let $(1+i)\mathbb{Z}[i] = \{(1+i)a \mid a \in \mathbb{Z}[i]\}$. Note that $(1+i)\mathbb{Z}[i] = \{a + bi \mid a, b \in \mathbb{Z},\ a + b \equiv 0 \pmod 2\}$.

Let $z \in \mathbb{C}$, $z \neq 0$. Note that $y^2 = z$ has two solutions in $\mathbb{C}$, and if y_0 is a solution, then $-y_0$ also is a solution. We define the square root $\sqrt{z}$ of z to be the number y such that $y^2 = z$ and $y \in \{x \in \mathbb{C} \mid \operatorname{Re}(x) > 0 \text{ or } \operatorname{Re}(x) = 0 \wedge \operatorname{Im}(x) \geq 0\}$. Of course, $\sqrt{0} := 0$.

We can write any complex number $z \in \mathbb{C}$ in the form $z = x + yi$, where x, $y \in \mathbb{R}$. We call x the real part of z, and denote this by $\operatorname{Re}(z)$. Similarly, we call y the imaginary part of z, and denote this by $\operatorname{Im}(z)$. The conjugate of z is denoted by $\overline{z}$ and given by $\overline{z} = x - yi$. The modulus of z is denoted by $|z|$ and given by $|z| = \sqrt{x^2 + y^2}$.

Let a, $b \in \mathbb{Z}[i]$, $b \neq 0$. We define the greatest common divisor $\gcd(a, b)$ of a and b up to units. That is, we define $\gcd(a, b)$ to be a Gaussian integer g such that $g \mid a$ and $g \mid b$ and there is no $h \in \mathbb{Z}[i]$ such that $|h| > |g|$ and $h \mid a$ and $h \mid b$. As $\{-1, 1, -i, i\}$ is the set of units in $\mathbb{Z}[i]$ we have that if $g = \gcd(a, b)$, then also $\sigma g = \gcd(a, b)$ for $\sigma \in \{-1, 1, -i, i\}$.

For $p \in \mathbb{C}$, $r > 0$, we define $B_r(p) := \{x \in \mathbb{C} \mid |x - p| < r\}$. We define $\overline{B_r(p)}$ to be the closure of $B_r(p)$, so $\overline{B_r(p)} := \{x \in \mathbb{C} \mid |x - p| \leq r\}$.

Let $c_n \in \mathbb{C}$ for every $n \in \mathbb{N}$. We say that $\lim_{n\to\infty} c_n = \infty$ if $\lim_{n\to\infty} \frac{1}{c_n} = 0$. Let $z \in \mathbb{C}$ and let $A \subseteq \mathbb{C}$ be a finite subset of $\mathbb{C}$. We define $d(z, A) := \min_{a \in A} |z - a|$.

For $x \in \mathbb{R}$ we define $\lfloor x \rfloor$ to be the floor of x, that is: the largest integer $a \in \mathbb{Z}$ such that $a \leq x$. Note that $|x - \lfloor x \rfloor| < 1$. We also have the following: $|x - \lfloor x + \frac{1}{2} \rfloor| \leq \frac{1}{2}$.

We end this section with a proposition.

Proposition 1.1. *Let $z \in \mathbb{C}$. Then: z is quadratic irrational if and only if $z = \frac{p+q\sqrt{r}}{s}$ for some p, q, r, $s \in \mathbb{Z}[i]$, $q \neq 0$, $s \neq 0$ and r not a square.*

Proof. Suppose z is quadratic irrational. Accordingly: let A, B, $C \in \mathbb{Z}[i]$ such that $A \neq 0$ and $Az^2 + Bz + C = 0$. Then we have by the quadratic formula:

$$z = \frac{-B + \sqrt{B^2 - 4AC}}{2A} \quad \text{or} \quad z = \frac{-B - \sqrt{B^2 - 4AC}}{2A}.$$

It follows that $B^2 - 4AC$ is not a square, otherwise z would be rational. So we can write $z = \frac{p+q\sqrt{r}}{s}$ for some p, q, r, $s \in \mathbb{Z}[i]$, $q \neq 0$, $s \neq 0$ and r not a square.

For the other direction, let p, q, r, $s \in \mathbb{Z}[i]$ such that $q \neq 0$, $s \neq 0$ and r not a square. Let $z := \frac{p+q\sqrt{r}}{s}$. As $q \neq 0$ and r not a square, we have that z is irrational. Now consider the polynomial $s^2x^2 - 2psx + (p^2 - q^2r)$. We see: s^2, $-2ps$, $p^2 - q^2r \in \mathbb{Z}[i]$ and $s^2 \neq 0$. Now:

$$\begin{aligned}
s^2z^2 - 2psz + (p^2 - q^2r) &= s^2\Big(\frac{p + q\sqrt{r}}{s}\Big)^2 - 2ps\frac{p + q\sqrt{r}}{s} + (p^2 - q^2r) \\
&= p^2 + 2pq\sqrt{r} + q^2r - 2p^2 - 2pq\sqrt{r} + p^2 - q^2r \\
&= 0.
\end{aligned}$$

As z is irrational we conclude by the last equation: z is quadratic irrational. □

2 Complex continued fractions

In this chapter we define what a complex continued fraction is. We will also study these complex continued fractions by proving some results about their properties. As we distinguish between a finite and an infinite complex continued fraction, we first give the definition of a finite complex continued fraction.

Definition 2.1. Let $n \in \mathbb{N}$. A *finite complex continued fraction* is a tuple $\langle A, E \rangle$, where $A = (a_0, a_1, \ldots, a_n)$ is a finite sequence with $a_k \in \mathbb{Z}[i]$ for every $k \in \{0, \ldots, n\}$, and $E = (e_1, e_2, \ldots, e_n)$ is a finite sequence with $e_k \in \{-1, 1, -i, i\}$ for every $k \in \{1, \ldots, n\}$. We will also use the notation $[a_0, e_1/a_1, \ldots, e_n/a_n]$ for a finite complex continued fraction.

The definition of an infinite complex continued fraction is similar.

Definition 2.2. An *infinite complex continued fraction* is a tuple $\langle A, E \rangle$, where $A = (a_0, a_1, a_2, \ldots)$ is an infinite sequence with $a_k \in \mathbb{Z}[i]$ for every $k \in \mathbb{N}$, and $E = (e_1, e_2, \ldots)$ is an infinite sequence with $e_k \in \{-1, 1, -i, i\}$ for every $k \in \mathbb{N}_{>0}$. We will also use the notation $[a_0, e_1/a_1, e_2/a_2, \ldots]$ for an infinite complex continued fraction.

In a finite or infinite complex continued fraction the e_{k+1} are called the *partial numerators*, and the a_k are called the *partial quotients*, for every $k \in \mathbb{N}$. We define the *length* of a complex continued fraction as follows: $|[a_0, e_1/a_1, \ldots, e_n/a_n]| = n+1$ for a finite complex continued fraction, and $|[a_0, e_1/a_1, e_2/a_2, \ldots]| = \infty$ for an infinite complex continued fraction. In order to perform calculations with complex continued fractions, one more definition is needed.

Definition 2.3. Let $n \in \mathbb{N}$. A *pseudo continued fraction* is a tuple $\langle A, E \rangle$, where $A = (a_0, a_1, \ldots, a_n)$ is a finite sequence with $a_n \in \mathbb{C}$, $a_k \in \mathbb{Z}[i]$ for every $k \in \{0, \ldots, n-1\}$, and $E = (e_1, e_2, \ldots, e_n)$ is a finite sequence with $e_k \in \{-1, 1, -i, i\}$ for every $k \in \{1, \ldots, n\}$. We will also use the notation $[a_0, e_1/a_1, \ldots, e_n/a_n]$ for a pseudo continued fraction.

Definition 2.4. Let $\mathcal{P}$ be the set of all pseudo continued fractions. We inductively define $\mathrm{Val} : \mathcal{P} \to \mathbb{C} \cup \{\perp\}$.

$$\begin{cases} \mathrm{Val}([a_0]) & := & a_0, & \\ \mathrm{Val}([a_0, e_1/a_1, \ldots, e_n/a_n]) & := & a_0 + \frac{e_1}{V} & \text{if } V \notin \{0, \perp\}, \\ \mathrm{Val}([a_0, e_1/a_1, \ldots, e_n/a_n]) & := & \perp & \text{if } V \in \{0, \perp\}, \\ \text{where } V := \mathrm{Val}([a_1, e_2/a_2, \ldots, e_n/a_n]). & & & \end{cases}$$

The function Val can be thought of as a function giving a finite complex continued fraction a value. We use the symbol $\perp$ to denote 'undefined', as not all finite complex continued fractions have a value in $\mathbb{C}$. An example is given by $[2, 1/i, 1/i]$. To compute $\mathrm{Val}([2, 1/i, 1/i])$, it is required to compute $\mathrm{Val}([i, 1/i])$ first. As

$$\mathrm{Val}([i, 1/i]) = i + \frac{1}{\mathrm{Val}([i])} = i + \frac{1}{i} = i - i = 0,$$

we obtain: $\mathrm{Val}([2, 1/i, 1/i]) = \perp$. The function Val gives rise to the following definition.

Definition 2.5. We call a pseudo continued fraction $[a_0, e_1/a_1, \ldots, e_n/a_n]$ *proper* if $\mathrm{Val}([a_0, e_1/a_1, \ldots, e_n/a_n]) \neq \perp$.

Note that if $[a_0, e_1/a_1, \ldots, e_n/a_n]$ is proper, then $[a_k, e_{k+1}/a_{k+1}, \ldots, e_n/a_n]$ is also proper for every $0 \leq k \leq n$. The following proposition clarifies why we call $[a_0, e_1/a_1, \ldots, e_n/a_n]$ a complex continued fraction.

Proposition 2.6. *Let $[a_0, e_1/a_1, \ldots, e_n/a_n]$ be a proper complex continued fraction. Then*

$$\mathrm{Val}([a_0, e_1/a_1, \ldots, e_n/a_n]) = a_0 + \cfrac{e_1}{a_1 + \cfrac{e_2}{\ddots + \cfrac{e_n}{a_n}}}.$$

Proof. We prove this by induction. By definition: $\mathrm{Val}([a_n]) = a_n$. Now suppose for $0 \leq k < n$:

$$\mathrm{Val}([a_{k+1}, e_{k+2}/a_{k+2}, \ldots, e_n/a_n]) = a_{k+1} + \cfrac{e_{k+2}}{a_{k+2} + \cfrac{e_{k+3}}{\ddots + \cfrac{e_n}{a_n}}}.$$

As $[a_0, e_1/a_1, \ldots, e_n/a_n]$ is proper, we have $\mathrm{Val}([a_{k+1}, e_{k+2}/a_{k+2}, \ldots, e_n/a_n]) \notin \{0, \perp\}$, therefore:

$$\begin{aligned}\mathrm{Val}([a_k, e_{k+1}/a_{k+1}, \ldots, e_n/a_n]) &= a_k + \frac{e_{k+1}}{\mathrm{Val}([a_{k+1}, e_{k+2}/a_{k+2}, \ldots, e_n/a_n])} \\ &= a_k + \cfrac{e_{k+1}}{a_{k+1} + \cfrac{e_{k+2}}{\ddots + \cfrac{e_n}{a_n}}},\end{aligned}$$

and this ends the proof. □

From this result we easily obtain the following consequence.

Proposition 2.7. *Let $[a_0, e_1/a_1, \ldots, e_n/a_n]$ be a proper finite complex continued fraction. Then:* $\mathrm{Val}([a_0, e_1/a_1, \ldots, e_n/a_n]) \in \mathbb{Q}[i]$.

Proof. As $a_k \in \mathbb{Z}[i]$ for every $k \in \{0, \ldots, n\}$ and $e_k \in \{-1, 1, -i, i\}$ for every $k \in \{1, \ldots, n\}$, this follows directly from Proposition 2.6. □

Let $[a_0, e_1/a_1, e_2/a_2, \ldots]$ be an infinite complex continued fraction. Note that for every $k \in \mathbb{N}$ we have that $[a_0, e_1/a_1, \ldots, e_k/a_k]$ is a finite complex continued fraction. We call $[a_0, e_1/a_1, \ldots, e_k/a_k]$ a *prefix* of $[a_0, e_1/a_1, e_2/a_2, \ldots]$. Similarly, for a finite complex continued fraction $[a_0, e_1/a_1, \ldots, e_n/a_n]$ we call $[a_0, e_1/a_1, \ldots, e_k/a_k]$ an *prefix* for every $k \in \{0, \ldots, n\}$.

From now on, we will sometimes drop the word 'complex' from 'complex finite continued fraction' and 'complex infinite continued fraction' and write 'finite continued fraction' and 'infinite continued fraction' respectively.

Definition 2.8. Let $[a_0, e_1/a_1, e_2/a_2, \ldots]$ be an infinite continued fraction. We define for every $k \in \mathbb{N}$:

$$c_k := \mathrm{Val}([a_0, e_1/a_1, \ldots, e_k/a_k])$$

and call c_k the *k-th convergent* of the continued fraction. For a finite continued fraction $[a_0, e_1/a_1, \ldots, e_n/a_n]$ we define c_k in the same way, with the obvious restriction $k \leq n$.

Definition 2.9. Let $[a_0, e_1/a_1, e_2/a_2, \ldots]$ be an infinite continued fraction. We inductively define two infinite sequences $(p_k)_{k\geq -1}$ and $(q_k)_{k\geq -1}$, by

$$\begin{array}{lll} p_{-1} := 1 & p_0 := a_0 & p_k := a_k p_{k-1} + e_k p_{k-2}, \\ q_{-1} := 0 & q_0 := 1 & q_k := a_k q_{k-1} + e_k q_{k-2}. \end{array}$$

For a finite continued fraction $[a_0, e_1/a_1, \ldots, e_n/a_n]$ we define the two finite sequences $(p_k)_{-1\leq k\leq n}$ and $(q_k)_{-1\leq k\leq n}$ in the same way, with the obvious restriction $k \leq n$.

Now we will formulate some useful properties of the sequences $(p_k)_{k\geq -1}$ and $(q_k)_{k\geq -1}$.

Proposition 2.10. *Let $[a_0, e_1/a_1, e_2/a_2, \ldots]$ be an infinite continued fraction, and let $(p_k)_{k\geq -1}$ and $(q_k)_{k\geq -1}$ as in Definition 2.9. Then for every $n \in \mathbb{N}$:*

i. $p_n q_{n-1} - p_{n-1} q_n = (-1)^{n-1} \cdot \prod_{k=1}^{n} e_k$,

ii. $\gcd(p_n, q_n) = 1$.

Proof. We prove this by induction. We find: $p_0 q_{-1} - p_{-1} q_0 = 0 - 1 = (-1)^{-1} \cdot \prod_{k=1}^{0} e_k$. Now suppose $p_n q_{n-1} - p_{n-1} q_n = (-1)^{n-1} \cdot \prod_{k=1}^{n} e_k$. Then:

$$\begin{aligned} p_{n+1} q_n - p_n q_{n+1} &= (a_{n+1} p_n + e_{n+1} p_{n-1}) q_n - p_n (a_{n+1} q_n + e_{n+1} q_{n-1}) \\ &= -e_{n+1} (p_n q_{n-1} - p_{n-1} q_n) \\ &= -e_{n+1} (-1)^{n-1} \cdot \prod_{k=1}^{n} e_k \\ &= (-1)^n \cdot \prod_{k=1}^{n+1} e_k \end{aligned}$$

and this proves the first statement. For the second statement: suppose $g \mid p_n$ and $g \mid q_n$. Then: $g \mid p_n q_{n-1} - p_{n-1} q_n = (-1)^{n-1} \cdot \prod_{k=1}^{n} e_k$, so $g \mid 1$. Therefore we conclude: $\gcd(p_n, q_n) = 1$. □

Proposition 2.11. *Let $[a_0, e_1/a_1, \ldots e_m/a_m]$ be a finite continued fraction, and let $(p_n)_{-1\leq n\leq m}$ and $(q_n)_{-1\leq k\leq m}$ as in Definition 2.9. Then for every $0 \leq n \leq m$:*

i. $p_n q_{n-1} - p_{n-1} q_n = (-1)^{n-1} \cdot \prod_{k=1}^{n} e_k$,

ii. $\gcd(p_n, q_n) = 1$.

Proof. The proof is similar to the proof of Proposition 2.10 and is therefore left to the reader. □

Note that it is possible that $q_n = 0$ for some $n > 0$. Here is an example: consider $[2, 1/i, 1/i]$. Then $q_0 = 1$, $q_1 = i$ and $q_2 = 0$. For completeness: $p_0 = 2$, $p_1 = 1 + 2i$ and $p_2 = i$. We see that it makes no sense to consider $\frac{p_2}{q_2}$ in this case. Note that we already found that $\mathrm{Val}([2, 1/i, 1/i]) = \perp$. However, for a proper continued fraction, we have the following result.

Proposition 2.12. *Let $[a_0, e_1/a_1, e_2/a_2, \ldots]$ be an infinite continued fraction and suppose $[a_0, e_1/a_1, \ldots, e_k/a_k]$ is proper for every $k \in \mathbb{N}$. Let c_k be the k-th convergent of this fraction, and let p_k and q_k as in Definition 2.9. Then: $c_k = \frac{p_k}{q_k}$ for every $k \in \mathbb{N}$.*

Proof. According to the definitions we have $c_0 = \mathrm{Val}([a_0]) = a_0 = \frac{a_0}{1} = \frac{p_0}{q_0}$ and $c_1 = \mathrm{Val}([a_0, e_1/a_1]) = a_0 + \frac{e_1}{a_1} = \frac{a_0 a_1 + e_1}{a_1} = \frac{p_1}{q_1}$. Suppose $c_j = \frac{p_j}{q_j}$ for every $j \leq k$. We have to prove: $c_{k+1} = \frac{p_{k+1}}{q_{k+1}}$.

<u>Claim:</u> $\mathrm{Val}([a_0, e_1/a_1, \ldots, e_k/a_k, e_{k+1}/a_{k+1}]) = \mathrm{Val}([a_0, e_1/a_1, \ldots, e_k/(a_k + \frac{e_{k+1}}{a_{k+1}})])$.

<u>Proof of claim:</u> This follows because $\mathrm{Val}[a_k, e_{k+1}/a_{k+1}] = a_k + \frac{e_{k+1}}{a_{k+1}} = \mathrm{Val}([a_k + \frac{e_{k+1}}{a_{k+1}}])$. ■

Note that for $[a_0, e_1/a_1, \ldots, e_k/a_k, e_{k+1}/a_{k+1}]$ and $[a_0, e_1/a_1, \ldots, e_k/a_k + \frac{e_{k+1}}{a_{k+1}}]$, both the sequences $(p_j)_{-1 \leq j \leq k-1}$ and $(q_j)_{-1 \leq j \leq k-1}$ are the same. Together with the claim this gives:

$$\begin{aligned}
c_{k+1} &= \mathrm{Val}([a_0, e_1/a_1, \ldots, e_k/a_k, e_{k+1}/a_{k+1}]) \\
&= \mathrm{Val}([a_0, e_1/a_1, \ldots, e_k/(a_k + \tfrac{e_{k+1}}{a_{k+1}})]) \\
&= \frac{(a_k + \frac{e_{k+1}}{a_{k+1}})p_{k-1} + e_k p_{k-2}}{(a_k + \frac{e_{k+1}}{a_{k+1}})q_{k-1} + e_k q_{k-2}} \\
&= \frac{(a_k p_{k-1} + e_k p_{k-2}) + \frac{e_{k+1}}{a_{k+1}} p_{k-1}}{(a_k q_{k-1} + e_k q_{k-2}) + \frac{e_{k+1}}{a_{k+1}} q_{k-1}} \\
&= \frac{a_{k+1} p_k + e_{k+1} p_{k-1}}{a_{k+1} q_k + e_{k+1} q_{k-1}} \\
&= \frac{p_{k+1}}{q_{k+1}},
\end{aligned}$$

and this ends the proof. □

Proposition 2.13. *Let $[a_0, e_1/a_1, \ldots, e_n/a_n]$ be a finite continued fraction and suppose $[a_0, e_1/a_1, \ldots, e_k/a_k]$ is proper for every $k \leq n$. Let c_k be the k-th convergent of this fraction, and let p_k and q_k as in Definition 2.9. Then: $c_k = \frac{p_k}{q_k}$ for every $k \leq n$.*

Proof. This proof is similar to the proof of Proposition 2.12 and is therefore left to the reader. □

Proposition 2.12 and Proposition 2.13 turn out to be false if we do not assume that $[a_0, e_1/a_1, \ldots, e_k/a_k]$ is proper for every k. An example is given by $[4, 1/2, 1/i, 1/i, \ldots]$ and $[4, 1/2, 1/i, 1/i]$ respectively. In both cases we find $c_3 = \mathrm{Val}([4, 1/2, 1/i, 1/i]) = \perp$, and on the other hand: $\frac{p_3}{q_3} = \frac{4i}{i} = 4$. Therefore we have $c_3 \neq \frac{p_3}{q_3}$.

The following lemmas will be useful in the next chapter.

Lemma 2.14. *Let $[a_0, e_1/a_1, \dots, e_n/a_n]$ be a finite continued fraction. Let $0 \le k \le n-1$. Then* $\mathrm{Val}([a_{k+1}, e_{k+2}/a_{k+2}, \dots, e_n/a_n]) = 0$ *iff* $\mathrm{Val}([a_n, e_n/a_{n-1}, \dots, e_{k+2}/a_{k+1}]) = 0$.

Proof. Note that in the following equations we have for every $j \in \{2, \dots, n-k\}$ that

$$\cfrac{e_{k+j}}{a_{k+j} + \cfrac{e_{k+j+1}}{\ddots + \cfrac{e_n}{a_n}}} \neq 0.$$

Therefore:

$$\begin{aligned}
\mathrm{Val}([a_{k+1}, e_{k+2}/a_{k+2}, \dots, e_n/a_n]) = 0 &\Leftrightarrow a_{k+1} + \cfrac{e_{k+2}}{a_{k+2} + \cfrac{e_{k+3}}{\ddots + \cfrac{e_n}{a_n}}} = 0 \\
&\Leftrightarrow a_{k+2} + \cfrac{e_{k+3}}{a_{k+3} + \cfrac{e_{k+4}}{\ddots + \cfrac{e_n}{a_n}}} = -\frac{e_{k+2}}{a_{k+1}} \\
&\Leftrightarrow a_{k+3} + \cfrac{e_{k+4}}{a_{k+4} + \cfrac{e_{k+5}}{\ddots + \cfrac{e_n}{a_n}}} = -\cfrac{e_{k+3}}{a_{k+2} + \cfrac{e_{k+2}}{a_{k+1}}} \\
&\vdots \\
&\Leftrightarrow a_n = -\cfrac{e_n}{a_{n-1} + \cfrac{e_{n-1}}{\ddots + \cfrac{e_{k+2}}{a_{k+1}}}} \\
&\Leftrightarrow 0 = -a_n - \cfrac{e_n}{a_{n-1} + \cfrac{e_{n-1}}{\ddots + \cfrac{e_{k+2}}{a_{k+1}}}} \\
&\Leftrightarrow 0 = -\mathrm{Val}([a_n, e_n/a_{n-1}, \dots, e_{k+2}/a_{k+1}]),
\end{aligned}$$

as had to be shown. □

Lemma 2.15. *Let $n \in \mathbb{N}_{>0}$ and let $[a_0, e_1/a_1, \dots, e_n/a_n]$ be a finite continued fraction. Let $(q_k)_{-1 \le k \le n}$ as in Definition 2.9. Suppose $q_k \neq 0$ for every $k \in \{0, \dots, n-1\}$. Then:* $\frac{q_n}{q_{n-1}} = \mathrm{Val}([a_n, e_n/a_{n-1}, \dots, e_2/a_1])$.

Proof. We prove this by induction. For $n = 1$ it follows that $\frac{q_1}{q_0} = \frac{a_1}{1} = a_1 = \mathrm{Val}([a_1])$. Now suppose the claim is true for n, so $\frac{q_n}{q_{n-1}} = \mathrm{Val}([a_n, e_n/a_{n-1}, \dots, e_2/a_1])$. Note that we have $\mathrm{Val}([a_n, e_n/a_{n-1}, \ldots, e_2/a_1]) \notin \{0, \perp\}$, as $\frac{q_n}{q_{n-1}} \neq 0$. Therefore:

$$\begin{aligned} \frac{q_{n+1}}{q_n} &= \frac{a_{n+1}q_n + e_{n+1}q_{n-1}}{q_n} \\ &= a_{n+1} + \frac{e_{n+1}}{(\frac{q_n}{q_{n-1}})} \\ &= a_{n+1} + \frac{e_{n+1}}{\mathrm{Val}([a_n, e_n/a_{n-1}, \ldots, e_2/a_1])} \\ &= \mathrm{Val}([a_{n+1}, e_{n+1}/a_n, \ldots, e_2/a_1]), \end{aligned}$$

and this ends the proof. □

We end this chapter by giving two important definitions regarding infinite complex continued fractions.

Definition 2.16. Let $z \in \mathbb{C}$ and let $[a_0, e_1/a_1, e_2/a_2, \ldots]$ be an infinite continued fraction. Let $(c_n)_{n\in\mathbb{N}}$ be the convergents of $[a_0, e_1/a_1, e_2/a_2, \ldots]$. We say that this continued fraction *converges to z* if

$$\lim_{k\to\infty} c_k = z.$$

Here we use the convention that $|z - \perp| = 1$ for every $z \in \mathbb{C}$.

Definition 2.17. Let $[a_0, e_1/a_1, e_2/a_2, \ldots]$ be an infinite complex continued fraction. We call this continued fraction *periodic* if there exists $N \in \mathbb{N}$ and $m \in \mathbb{N}_{>0}$ such that $(a_n, e_{n+1}) = (a_{n+m}, e_{n+m+1})$ for every $n \geq N$. We call a continued fraction *purely periodic* if this holds with $N = 0$. We say that m is the *length of the period* if there exists no $0 < k < m$ such that $(a_n, e_{n+1}) = (a_{n+k}, e_{n+k+1})$ for every $n \geq N$. We will use the following notation for a periodic continued fraction: let $[a_0, e_1/a_1, e_2/a_2, \ldots]$ be an infinite complex fraction such that $(a_n, e_{n+1}) = (a_{n+m_0}, e_{n+m_0+1})$ for every $n \geq N_0$. Then we will denote this fraction by

$$[a_0, e_1/a_1, \ldots, e_{N_0-1}/a_{N_0-1}, e_{N_0}/\overline{a_{N_0}, e_{N_0+1}/a_{N_0+1}, \ldots, e_{N_0+m_0-1}/a_{N_0+m_0-1}, e_{N_0+m_0}/}].$$

Remark 2.18. This definition of a periodic continued fraction generalizes both the definition of a periodic continued fraction by A. Hurwitz and the definition of a periodic continued fraction by J. Hurwitz. That is: if one of the brothers Hurwitz calls a continued fraction periodic, then it is also periodic according to Definition 2.17.

3 Complex continued fraction algorithms

In this chapter we will define what a complex continued fraction algorithm is. Furthermore, we prove some general properties of a continued fraction algorithm and we will find a relation with the greatest common divisor algorithm. We will also study convergence and periodicity of continued fractions which are obtained by a complex continued fraction algorithm.

3.1 Floor functions & sign functions

Before we can give a formal definition of a complex continued fraction algorithm, we will first consider two special kinds of complex functions.

Definition 3.1. Let $f : \mathbb{C} \to \mathbb{Z}[i]$ be a function. We call f a *floor function* if $|f(z)-z| < 1$ for every $z \in \mathbb{C}$.

Definition 3.2. We call $f : \mathbb{C} \to \mathbb{Z}[i]$ a *shift floor function* if f is a floor function and $f(z+\alpha) = f(z) + \alpha$ for every $z \in \mathbb{C}$, $\alpha \in \mathbb{Z}[i]$.

Example 3.3. Define $f : \mathbb{C} \to \mathbb{Z}[i]$ as follows: $f(z) := \lfloor \mathrm{Re}(z) + \frac{1}{2} \rfloor + \lfloor \mathrm{Im}(z) + \frac{1}{2} \rfloor i$. Then f is a floor function, and moreover, it is a shift floor function. For the proof of this, see Proposition 4.3.

Lemma 3.4. *Let f be a floor function. If $z \in \mathbb{Z}[i]$, then $f(z) = z$.*

Proof. Let $z \in \mathbb{Z}[i]$. By definition we have: $f(z) \in \mathbb{Z}[i]$ and $f(z) \in \{x \,\big|\, |x-z| < 1\}$. Therefore: $f(z) \in \mathbb{Z}[i] \cap \{x \,\big|\, |x-z| < 1\} = \{z\}$, and we obtain: $f(z) = z$. □

A floor function f gives rise to a tiling of the complex plane. For every $\alpha \in \mathbb{Z}[i]$ we define the *tile* T_α^f of α by $T_\alpha^f := \{z \in \mathbb{C} \,\big|\, f(z) = \alpha\}$. As f is a function, every $z \in \mathbb{C}$ lies in exactly one tile. We have by definition that $f(z) = \alpha$ for every $z \in T_\alpha^f$. For every $\alpha \in \mathbb{Z}[i]$ we define the set $U_\alpha^f \subseteq \mathbb{C}$ by $U_\alpha^f := \{z - \alpha \,\big|\, z \in T_\alpha^f\}$. We define

$$\Delta_f := \bigcup_{\alpha \in \mathbb{Z}[i]} U_\alpha^f$$

and call Δ_f the *fundamental domain* of f.

Proposition 3.5. *Let f be a floor function. Then*

i. $z - f(z) \in \Delta_f$ *for every* $z \in \mathbb{C}$*,*

ii. $\Delta_f \subseteq B_1(0)$*.*

Proof. First, let $z \in \mathbb{C}$ and let $\alpha := f(z)$. As $z \in T_\alpha^f$, we have $z - f(z) = z - \alpha \in U_\alpha^f$. Therefore: $z - f(z) \in \Delta_f$.
For the second statement: let $\alpha \in \mathbb{Z}[i]$ and consider U_α^f. We have $x \in U_\alpha^f$ iff $x = z - \alpha$ for some $z \in T_\alpha^f$. As $z - \alpha = z - f(z)$ for every $z \in T_\alpha^f$, and $|z - f(z)| < 1$, we have that $|x| < 1$, so $x \in B_1(0)$. Therefore: $U_\alpha^f \subseteq B_1(0)$. As α is arbitrary, we obtain: $\Delta_f = \cup_{\alpha \in \mathbb{Z}[i]} U_\alpha^f \subseteq B_1(0)$. □

Proposition 3.6. *Let f be a shift floor function. Then:* $\Delta_f = \{z \in \mathbb{C} \mid f(z) = 0\} = T_0^f$.

Proof. Let $\alpha \in \mathbb{Z}[i]$, then: $z \in U_0^f$ iff $z \in T_0^f$ iff $f(z) = 0$ iff $f(z)+\alpha = \alpha$ iff $f(z+\alpha) = \alpha$ iff $z+\alpha \in T_\alpha^f$ iff $z \in U_\alpha^f$. Consequently: $U_0^f = U_\alpha^f$. As α is arbitrary: $\Delta_f = \cup_{\alpha\in\mathbb{Z}[i]} U_\alpha^f = U_0^f$. Now: $U_0^f = T_0^f = \{z \in \mathbb{C} \mid f(z) = 0\}$, and therefore: $\Delta_f = \{z \in \mathbb{C} \mid f(z) = 0\}$. $\square$

Definition 3.7. We call any function $g : \mathbb{C} \to \{-1, 1, -i, i\}$ a *sign function.*

Let f be a floor function and g be a sign function. We define $V_\alpha^{f,g} \subseteq \mathbb{C}$ for every $\alpha \in \mathbb{Z}[i]$ by $V_\alpha^{f,g} := \{\frac{g(z)}{z-f(z)} \mid z \in T_\alpha^f, z - f(z) \neq 0\}$. We define

$$\Gamma_{f,g} := \bigcup_{\alpha\in\mathbb{Z}[i]} V_\alpha^{f,g}$$

and call $\Gamma_{f,g}$ the *fundamental codomain* of f and g.

Proposition 3.8. *Let f be a floor function and g be a sign function. Then:*

i. $\frac{g(z)}{z-f(z)} \in \Gamma_{f,g}$ *for every* $z \in \mathbb{C}$, $z - f(z) \neq 0$,

ii. $\Gamma_{f,g} \subseteq \mathbb{C} \setminus \overline{B_1(0)}$

Proof. First let $z \in \mathbb{C}$ such that $z - f(z) \neq 0$ and let $\alpha := f(z)$. Then by definition: $\frac{g(z)}{z-f(z)} \in V_\alpha^{f,g}$, therefore: $\frac{g(z)}{z-f(z)} \in \Gamma_{f,g}$.
For the second statement, let $\alpha \in \mathbb{Z}[i]$ and let $x \in V_\alpha^{f,g}$. Now we have that $x = \frac{g(z)}{z-f(z)}$ for a $z \in \mathbb{C}$, $z - f(z) \neq 0$. As $|g(z)| = 1$ and $|z - f(z)| < 1$ we have that $|x| = \left|\frac{g(z)}{z-f(z)}\right| > 1$. Therefore: $V_\alpha^{f,g} \subseteq \mathbb{C} \setminus \overline{B_1(0)}$ and we conclude: $\Gamma_{f,g} = \bigcup_{\alpha\in\mathbb{Z}[i]} V_\alpha^{f,g} \subseteq \mathbb{C} \setminus \overline{B_1(0)}$. $\square$

Proposition 3.9. *Let f be a floor function and g be a sign function. If g is a constant function, then* $\Gamma_{f,g} = \{\frac{g(z)}{z} \mid z \in \Delta_f, z \neq 0\}$.

Proof. Suppose that g is constant, say $g(z) = \rho$ for every $z \in \mathbb{C}$.
First suppose $x \in \Gamma_{f,g}$, then $x = \frac{\rho}{y-f(y)}$ for some $y \in \mathbb{C}$, $y - f(y) \neq 0$. Consequently: as $y - f(y) \in \Delta_f$, we have that $x = \frac{\rho}{y-f(y)} \in \{\frac{g(z)}{z} \mid z \in \Delta_f, z \neq 0\}$.
Now suppose $x \in \{\frac{g(z)}{z} \mid z \in \Delta_f, z \neq 0\}$. Then $x = \frac{\rho}{y}$ for some $y \in \Delta_f$, $y \neq 0$. As $y \in \Delta_f$, there exists some $z \in \mathbb{C}$ such that $y = z - f(z)$. So $x = \frac{\rho}{z-f(z)} \in \Gamma_{f,g}$.
We obtain $x \in \Gamma_{f,g}$ iff $x \in \{\frac{g(z)}{z} \mid z \in \Delta_f, z \neq 0\}$, and this completes the proof. $\square$

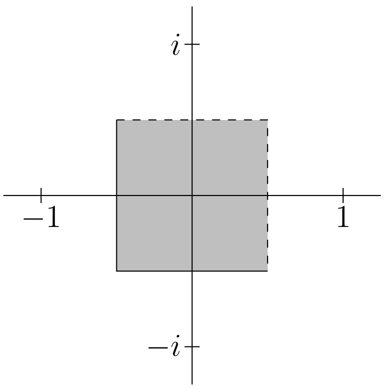

Figure 1: *Fundamental domain Δ_f, with f as in Example 3.3.*

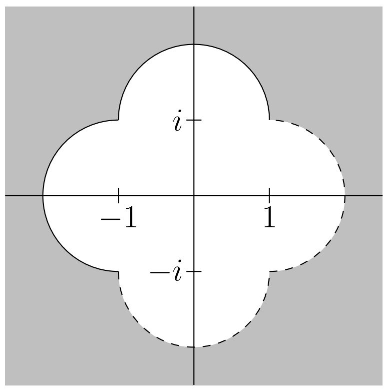

Figure 2: *Fundamental codomain $\Gamma_{f,g}$, with f as in Example 3.3 and $g(z) = 1$ for every $z \in \mathbb{C}$.*

3.2 Definition of a complex continued fraction algorithm

Now we have defined what a floor function and a sign function is, we can give the definition of a complex continued fraction algorithm.

Definition 3.10. A *complex continued fraction algorithm* is a floor function $\mathit{fl} : \mathbb{C} \to \mathbb{Z}[i]$ and a sign function $sg : \mathbb{C} \to \{-1, 1, -i, i\}$, together with the following sequence of transformations

$$\begin{cases} x_0 & := & x \\ a_n & := & \mathit{fl}(x_n) \\ e_{n+1} & := & sg(x_n) \\ x_{n+1} & := & \frac{e_{n+1}}{x_n - a_n}. \end{cases}$$

These transformations recursively define a_n, e_{n+1} and x_{n+1} for n starting at 0. The input of such an algorithm should be a complex number $x \in \mathbb{C}$. The algorithm halts if $a_k = x_k$ for some $k \in \mathbb{N}$, and outputs the finite continued fraction $[a_0, e_1/a_1, \ldots, e_k/a_k]$. Otherwise the algorithm will not halt. However, we could say it then produces an infinite list $[a_0, e_1/a_1, e_2/a_2, \ldots]$, which is indeed an infinite continued fraction. In this case we will say that the algorithm outputs an infinite continued fraction. We call $(x_n)_{n \in \mathbb{N}}$ the *complete quotients* of x under the continued fraction algorithm.

Remark 3.11. As we study infinite continued fractions, it is convenient to consider them as one object. Therefore we say that a complex continued fraction algorithm can output an infinite continued fraction, although this is not fully in accordance with the usual idea of an algorithm.

This general definition of a complex continued fraction algorithm will serve as a framework for specific complex continued fraction algorithms. When results are proven for this general definition, they apply to any instance of a complex continued fraction algorithm. In the following chapters we will look at some well-known algorithms concerning complex continued fractions, and we will see that they fit in this framework. In the following sections, we will prove results for this general definition of a complex continued fraction algorithm.

Remark 3.12. A famous algorithm concerning complex continued fractions is due to A. L. Schmidt [8]. Unfortunately, his algorithm is not an instance of a complex continued fraction algorithm as given in Definition 3.10. In contrast, the idea of this algorithm is to partition the complex plane in areas bounded by arcs of circles and line segments. Then successively refine this partition, and consider the areas to which the input $x \in \mathbb{C}$ belongs. These areas correspond to rational convergents which are the output of the algorithm, rather than a complex continued fraction.

3.3 General properties

In this section and in the following sections of this chapter, let CF be a complex continued fraction algorithm. Let fl be the floor function and sg be the sign function of CF. Next, we will prove some general properties of a complex continued fraction algorithm.

Proposition 3.13. *Let $z \in \mathbb{Z}[i]$, then $\mathsf{CF}(z) = [z]$.*

Proof. By Lemma 3.4 we have: $z_0 = z = \mathit{fl}(z) = \mathit{fl}(z_0) = a_0$. Therefore $z_0 = a_0$ and we obtain $\mathsf{CF}(z) = [z]$. $\square$

Lemma 3.14. *Let $x \in \mathbb{C}$, $n \in \mathbb{N}$ and let x_{n+1} be the $(n+1)$-th complete quotient and a_{n+1} be the $(n+1)$-th partial quotient of x under CF. Then*

i. $|x_{n+1}| > 1$,

ii. $a_{n+1} \neq 0$.

Proof. As fl is a floor function, we have $|x_n - \mathit{fl}(x_n)| < 1$. Therefore it follows that $|x_{n+1}| = \frac{|e_{n+1}|}{|x_n - a_n|} = \frac{1}{|x_n - \mathit{fl}(x_n)|} > 1$. For the second statement: suppose $a_{n+1} = 0$. Then we have $1 > |x_{n+1} - \mathit{fl}(x_{n+1})| = |x_{n+1} - a_{n+1}| = |x_{n+1} - 0| = |x_{n+1}| > 1$. So $1 > 1$. This is a contradiction and we conclude that $a_{n+1} \neq 0$. $\square$

The next result shows how the fundamental domain Δ_{fl} and the fundamental codomain $\Gamma_{\mathit{fl},sg}$ are related to the complex continued fraction algorithm CF.

Proposition 3.15. *Let $x \in \mathbb{C}$, $n \in \mathbb{N}$ and let x_n be the n-th complete quotient, a_n the n-th partial quotient and x_{n+1} the $(n+1)$-th complete quotient of x under CF. Then*

i. $x_n - a_n \in \Delta_{\mathit{fl}}$,

ii. $x_{n+1} \in \Gamma_{\mathit{fl},sg}$.

Proof. We have $x_n - a_n = x_n - \mathit{fl}(x_n)$ and by Proposition 3.5: $x_n - a_n \in \Delta_{\mathit{fl}}$. Furthermore: $x_{n+1} = \frac{e_{n+1}}{x_n - a_n} = \frac{sg(x_n)}{x_n - \mathit{fl}(x_n)}$ and by Proposition 3.8 we conclude that $x_{n+1} \in \Gamma_{\mathit{fl},sg}$. $\square$

Now we will see that the type of the output of CF depends on whether the input is rational or irrational.

Proposition 3.16. *If $x \in \mathbb{C}$ is rational, then $\mathsf{CF}(x)$ is a finite continued fraction.*

Proof. Let $x \in \mathbb{Q}[i]$, so we can set $x = \frac{r}{s}$, with $r, s \in \mathbb{Z}[i]$. Suppose for the sake of contradiction: $\mathsf{CF}(x)$ is an infinite continued fraction. Let $\mathsf{CF}(x) = [a_0, e_1/a_1, e_2/a_2, \ldots]$. Now define four sequences in $\mathbb{Z}[i]$:

$$\begin{aligned} r_0 &:= r, & r_{n+1} &:= s_n f_{n+1}, \\ s_0 &:= s, & s_{n+1} &:= r_n - s_n b_n, \\ b_n &:= \mathit{fl}(\tfrac{r_n}{s_n}), \\ f_{n+1} &:= sg(\tfrac{r_n}{s_n}). \end{aligned}$$

<u>Claim:</u> For every $n \in \mathbb{N}$: $x_n = \frac{r_n}{s_n}$, $a_n = b_n$ and $e_{n+1} = f_{n+1}$.

<u>Proof of claim:</u> We prove this by induction: $x_0 = x = \frac{r}{s} = \frac{r_0}{s_0}$, $a_0 = \mathit{fl}(x_0) = \mathit{fl}(\frac{r_0}{s_0}) = b_0$ and $e_1 = sg(x_0) = sg(\frac{r_0}{s_0}) = f_1$ Now:

$$x_{n+1} = \frac{e_{n+1}}{x_n - a_n} = \frac{f_{n+1}}{\frac{r_n}{s_n} - b_n} = \frac{s_n f_{n+1}}{r_n - s_n b_n} = \frac{r_{n+1}}{s_{n+1}},$$

$a_{n+1} = \mathit{fl}(x_{n+1}) = \mathit{fl}(\frac{r_{n+1}}{s_{n+1}}) = b_{n+1}$ and $e_{n+2} = sg(x_{n+1}) = sg(\frac{r_{n+1}}{s_{n+1}}) = f_{n+2}$, and the claim follows. ■

From the claim we obtain: $x_{n+1} = \frac{r_{n+1}}{s_{n+1}} = \frac{s_n f_{n+1}}{s_{n+1}}$, for every $n \in \mathbb{N}$. As $|x_{n+1}| > 1$ and $|f_{n+1}| = 1$, we have $|s_{n+1}| < |s_n|$ for every $n \in \mathbb{N}$. Therefore, as $s_n \in \mathbb{Z}[i]$, we have that there exists $N \in \mathbb{N}$ such that $s_{N+1} = 0$. So: $0 = s_{N+1} = r_N - s_N b_N = r_N - s_N a_N$, and consequently: $x_N - a_N = \frac{r_N}{s_N} - a_N = 0$. Therefore the algorithm terminates after N transformations, and outputs a finite continued fraction. This contradicts our assumption and we conclude: $\mathsf{CF}(x)$ is a finite continued fraction. □

Proposition 3.17. *If $x \in \mathbb{C}$ is irrational, then $\mathsf{CF}(x)$ is an infinite continued fraction.*

Proof. Let x be irrational. Investigating the algorithm CF, we see that the algorithm outputs a finite continued fraction iff $x_n - \mathit{fl}(x_n) = 0$ for some $n \in \mathbb{N}$. As x_0 is irrational and $\mathit{fl}(x_0)$ is rational, we have $x_0 - \mathit{fl}(x_0) \neq 0$. Now suppose x_k is irrational, then also $x_{k+1} = \frac{sg(x_k)}{x_k - \mathit{fl}(x_k)}$ is irrational. Therefore $x_{k+1} - \mathit{fl}(x_{k+1}) \neq 0$. Consequently, for every $n \in \mathbb{N}$: $x_n - \mathit{fl}(x_n) \neq 0$ and we conclude: $\mathsf{CF}(x)$ is an infinite continued fraction. □

Theorem 3.18. *Let $x \in \mathbb{C}$, then:*

i. x is rational iff $\mathsf{CF}(x)$ is a finite continued fraction,

ii. x is irrational iff $\mathsf{CF}(x)$ is an infinite continued fraction.

Proof. This follows directly from Proposition 3.16 and Proposition 3.17. □

Next we will see that CF and Val are closely related.

Proposition 3.19. *Let $x \in \mathbb{Q}[i]$, then:* $\mathrm{Val}(\mathsf{CF}(x)) = x$.

Proof. Let $\mathsf{CF}(x) = [a_0, e_1/a_1, \ldots, e_n/a_n]$.

Claim: $\mathrm{Val}([a_{n-k}, e_{n-k+1}/a_{n-k+1}, \ldots, e_n/a_n]) = x_{n-k}$, for every $k \in \{0, \ldots, n\}$.

Proof of claim: We prove this by induction. For $k = 0$ we have $\mathrm{Val}([a_n]) = a_n = x_n$. Now assume that $\mathrm{Val}([a_{n-k+1}, e_{n-k+2}/a_{n-k+2}, \ldots, e_n/a_n]) = x_{n-k+1}$. Note that $|x_{n-k+1}| > 1$ for every $k \in \{1, \ldots, n\}$. So: $\mathrm{Val}([a_{n-k+1}, e_{n-k+2}/a_{n-k+2}, \ldots, e_n/a_n]) \in \mathbb{Q}[i] \setminus \{0\}$. Therefore:

$$\begin{aligned}
\mathrm{Val}([a_{n-k}, e_{n-k+1}/a_{n-k+1}, \ldots, e_n/a_n]) &= a_{n-k} + \frac{e_{n-k+1}}{\mathrm{Val}([a_{n-k+1}, e_{n-k+2}/a_{n-k+2}, \ldots, e_n/a_n])} \\
&= a_{n-k} + \frac{e_{n-k+1}}{x_{n-k+1}} \\
&= x_{n-k}
\end{aligned}$$

and this proves the claim. ■

From the claim immediately follows: $x = x_0 = \mathrm{Val}([a_0, e_1/a_1, \ldots, e_n/a_n]) = \mathrm{Val}(\mathsf{CF}(x))$, and this completes the proof. □

Corollary 3.20. *Let $x \in \mathbb{Q}[i]$, then: $\mathsf{CF}(x)$ is proper.*

Proof. According to Proposition 3.19: $\mathrm{Val}(\mathsf{CF}(x)) = x \neq \perp$. $\square$

In the remaining part of this section, we will more closely examine the complete quotients x_n which occur when we compute $\mathsf{CF}(x)$, for some $x \in \mathbb{C}$.

Proposition 3.21. *Let $x \in \mathbb{C}$ be irrational. Let $\mathsf{CF}(x) = [a_0, e_1/a_1, e_2/a_2, \ldots]$ and let x_n be the n-th complete quotient of $\mathsf{CF}(x)$. Then: $x = \mathrm{Val}([a_0, e_1/a_1, \ldots, e_{n-1}/a_{n-1}, e_n/x_n]$ for every $n \in \mathbb{N}$.*

Proof. This proof is by induction. We have $x = x_0 = \mathrm{Val}([x_0])$. Now suppose $x = \mathrm{Val}([a_0, e_1/a_1, \ldots, e_{n-1}/a_{n-1}, e_n/x_n]$. As $x_{n+1} = \frac{e_{n+1}}{x_n - a_n}$ we have:

$$\mathrm{Val}([x_n]) = x_n = a_n + \frac{e_{n+1}}{x_{n+1}} = a_n + \frac{e_{n+1}}{\mathrm{Val}([x_{n+1}])} = \mathrm{Val}([a_n, e_{n+1}/x_{n+1}]).$$

Therefore:

$$\begin{aligned} x &= \mathrm{Val}([a_0, e_1/a_1, \ldots, e_{n-1}/a_{n-1}, e_n/x_n]) \\ &= \mathrm{Val}([a_0, e_1/a_1, \ldots, e_{n-1}/a_{n-1}, e_n/a_n, e_{n+1}/x_{n+1}]) \end{aligned}$$

and this completes the proof. $\square$

Proposition 3.22. *Let $x \in \mathbb{C}$ be rational. Let $\mathsf{CF}(x) = [a_0, e_1/a_1, \ldots, e_n/a_n]$ and let x_k be the k-th complete quotient of $\mathsf{CF}(x)$. Then: $x = \mathrm{Val}([a_0, e_1/a_1, \ldots, e_{k-1}/a_{k-1}, e_k/x_k]$ for every $k \in \{0, \ldots, n\}$.*

Proof. This proof is similar to the proof of Proposition 3.21 and is therefore left to the reader. $\square$

Corollary 3.23. *Let $x \in \mathbb{C}$ and $n \in \mathbb{N}$. Let x_n be the n-th complete quotient, let e_k be the k-th partial numerator and a_k be the k-th partial quotient of x under CF, for every $k \leq n$. Then: $[a_0, e_1/a_1, \ldots, e_{n-1}/a_{n-1}, e_n/x_n]$ is proper.*

Proof. According to Proposition 3.21 and Proposition 3.22 we have that

$$\mathrm{Val}([a_0, e_1/a_1, \ldots, e_{n-1}/a_{n-1}, e_n/x_n]) = x \neq \perp$$

and therefore we conclude: $[a_0, e_1/a_1, \ldots, e_{n-1}/a_{n-1}, e_n/x_n]$ is proper. $\square$

Proposition 3.24. *Let $x \in \mathbb{C}$ be irrational and let $\mathsf{CF}(x) = [a_0, e_1/a_1, e_2/a_2, \ldots]$. Let x_n be the n-th complete quotient of $\mathsf{CF}(x)$. Then: $\mathsf{CF}(x_n) = [a_n, e_{n+1}/a_{n+1}, e_{n+2}/a_{n+2}, \ldots]$ for every $n \in \mathbb{N}$.*

Proof. Let $n \in \mathbb{N}$, and let $y := x_n$. Let $\mathsf{CF}(y) = [b_0, f_1/b_1, f_2/b_2, \ldots]$ and let y_k be the k-th complete quotient of y under CF.

Claim: $y_k = x_{n+k}$, $b_k = a_{n+k}$ and $f_{k+1} = e_{n+k+1}$ for every $k \in \mathbb{N}$.

Proof of claim: We prove this by induction. By assumption: $y_0 = x_n$, therefore $b_0 = \mathit{fl}(y_0) = \mathit{fl}(x_n) = a_n$ and $f_1 = sg(y_0) = sg(x_n) = e_{n+1}$. Now suppose $y_k = x_{n+k}$, $b_k = a_{n+k}$ and $f_{k+1} = e_{n+k+1}$. Then:

$$y_{k+1} = \frac{f_{k+1}}{y_k - b_k} = \frac{e_{n+k+1}}{x_{n+k} - a_{n+k}} = x_{n+k+1},$$

and

$$b_{k+1} = fl(y_{k+1}) = fl(x_{n+k+1}) = a_{n+k+1},$$

and

$$f_{k+1+1} = sg(y_{k+1}) = sg(x_{n+k+1}) = e_{n+k+1+1},$$

and this proves the claim. ■

By the claim we conclude:

$$\mathsf{CF}(x_n) = \mathsf{CF}(y) = [b_0, f_1/b_1, f_2/b_2, \ldots] = [a_n, e_{n+1}/a_{n+1}, e_{n+2}/a_{n+2}, \ldots]$$

and this ends the proof. □

Proposition 3.25. *Let $x \in \mathbb{C}$ be rational and let $\mathsf{CF}(x) = [a_0, e_1/a_1, \ldots, e_n/a_n]$. Let x_k be the k-th complete quotient of $\mathsf{CF}(x)$. Then: $\mathsf{CF}(x_k) = [a_k, e_{k+1}/a_{k+1}, \ldots, e_n/a_n]$ for every $k \in \{0, \ldots, n\}$.*

Proof. This proof is similar to the proof of Proposition 3.24 and therefore left to the reader. □

Let $x \in \mathbb{C}$ be irrational and let $\mathsf{CF}(x) = [a_0, e_1/a_1, e_2/a_2, \ldots]$. By Proposition 3.21 we have: $x = \mathrm{Val}([a_0, e_1/a_1, \ldots, e_{n-1}/a_{n-1}, e_n/x_n])$. According to Proposition 3.24 it follows that $\mathsf{CF}(x_n) = [a_n, e_{n+1}/a_{n+1}, e_{n+2}/a_{n+2}, \ldots]$. This justifies to write $\mathsf{CF}(x) = [a_0, e_1/a_1, \ldots, e_{n-1}/a_{n-1}, e_n/\mathsf{CF}(x_n)]$. Obviously, with the same reasoning we obtain a similar result for x rational.

3.4 Greatest common divisor algorithm

In this section we will discover a relation between a continued fraction algorithm and a greatest common divisor algorithm.

Example 3.26. Consider $7 - 61i$ and $26 - 36i$. Note that $\gcd(7 - 61i, 26 - 36i) = 7 - 3i$. $\mathsf{CF}_{\mathsf{JS}}$ is a continued fraction algorithm, which we will define in Section 6.1. When we compute $\mathsf{CF}_{\mathsf{JS}}(\frac{7-61i}{26-36i})$ we find: $\mathsf{CF}_{\mathsf{JS}}(\frac{7-61i}{26-36i}) = [1 - i, 1/1 - 2i, 1/2 + i]$. This continued fraction gives rise to the following equations of the form $x_k = a_k + (x_k - a_k)$, for $k \in \{0, 1, 2\}$:

$$\begin{aligned} \tfrac{7-61i}{26-36i} &= 1 - i + \tfrac{17+i}{26-36i}, \\ \tfrac{26-36i}{17+i} &= 1 - 2i + \tfrac{7-3i}{17+i}, \\ \tfrac{17+i}{7-3i} &= 2 + i. \end{aligned}$$

When we multiply each of the lines above of the form $\frac{b}{d} = a + \frac{c}{d}$ by d we obtain:

$$\begin{aligned} 7 - 61i &= (1 - i)(26 - 36i) + (17 + i), \\ 26 - 36i &= (1 - 2i)(17 + i) + (7 - 3i), \\ 17 + i &= (2 + i)(7 - 3i). \end{aligned}$$

These equations look like those of a greatest common divisor algorithm. On the last line, we find $7 - 3i$, which is a greatest common divisor of $7 - 61i$ and $26 - 36i$. This is not a coincidence, as is shown by Proposition 3.27.

Proposition 3.27. *Let* CF *be a complex continued fraction algorithm. Let* $r,\ s \in \mathbb{Z}[i]$, *and* $s \neq 0$. *Let* $\mathsf{CF}(\frac{r}{s}) = [a_0, e_1/a_1, \ldots, e_n/a_n]$. *Now define two finite sequences* $(r_k)_{0\le k\le n}$ *and* $(s_k)_{0\le k\le n}$ *as follows:*

$$\begin{aligned} r_0 &:= r, \qquad & r_{k+1} &:= s_k e_{k+1}, \\ s_0 &:= s, \qquad & s_{k+1} &:= r_k - s_k a_k. \end{aligned}$$

Then: $\gcd(r, s) = s_n$.

Proof. Claim 1: For every $k \in \{0, \ldots, n\}$: $\gcd(r, s) = \gcd(r_k, s_k)$.

Proof of claim 1: We prove this by induction. By assumption: $\gcd(r, s) = \gcd(r_0, s_0)$. Now suppose $\gcd(r, s) = \gcd(r_k, s_k)$. Let $g = \gcd(r_k, s_k)$, so $g \mid r_k$ and $g \mid s_k$. Therefore $g \mid r_{k+1} = s_k e_{k+1}$ and $g \mid s_{k+1} = r_k - s_k a_k$. Now suppose $h \in \mathbb{Z}[i]$ such that $h \mid r_{k+1}$ and $h \mid s_{k+1}$. Then $h \mid s_k e_{k+1}$ and $h \mid r_k - s_k a_k$. Consequently: $h \mid s_k a_k$, therefore: $h \mid r_k - s_k a_k + s_k a_k = r_k$. So $h \mid r_k$ and $h \mid s_k$, therefore $h \mid \gcd(r_k, s_k) = g$. We conclude: $\gcd(r, s) = g = \gcd(r_{k+1}, s_{k+1})$. ■

Claim 2: Let x_k be the k-th complete quotient of $\mathsf{CF}(\frac{r}{s})$. Then for every $k \in \{0, \ldots, n\}$: $x_k = \frac{r_k}{s_k}$.

Proof of claim 2: We prove this by induction. By assumption: $x_0 = x = \frac{r}{s} = \frac{r_0}{s_0}$. Now:

$$x_{k+1} = \frac{e_{k+1}}{x_k - a_k} = \frac{e_{k+1}}{\frac{r_k}{s_k} - a_k} = \frac{s_k e_{k+1}}{r_k - s_k a_k} = \frac{r_{k+1}}{s_{k+1}},$$

and this proves the claim. ■

As $\frac{r_n}{s_n} - a_n = x_n - a_n = 0$, it follows that $r_n = s_n a_n$. Therefore: $s_n \mid r_n$. As $s_n \mid s_n$ we have that $s_n \mid \gcd(r_n, s_n)$. Consequently: $s_n = \gcd(r_n, s_n) = \gcd(r, s)$. □

We will now see that a complex continued fraction algorithm gives rise to a greatest common divisor algorithm. Let $r,\ s \in \mathbb{Z}[i]$, $s \neq 0$. Compute $\mathsf{CF}(\frac{r}{s}) = [a_0, e_1/a_1, \ldots, e_n/a_n]$. Then compute the sequence $(s_0, \ldots, s_n)$, which is defined in Proposition 3.27. Finally, return s_n. From Proposition 3.27 it follows that $s_n = \gcd(r, s)$.

3.5 Convergence

In this section we will prove that the complex continued fraction of an irrational number x which is obtained by a complex continued fraction algorithm, converges to x. Before this, we prove some lemmas which will be useful in this and in the next section. The following two results are subsequent to the ideas of Section 3 of [2].

Lemma 3.28. *Let* $R := \{e^{m\pi i/6} \mid m \in \{0, \ldots, 11\}\}$, *the set of all 12th roots of unity in* $\mathbb{C}$. *Then:*

i. for every $z \in \mathbb{C}$, *for every* $a \in \mathbb{Z}[i] \setminus \{0\}$: *if* $|z| = 1$ *and* $|z - a| = 1$ *then* $z \in R$,

ii. for every $\rho \in R$ *there exists an* $a \in \mathbb{Z}[i] \setminus \{0\}$ *such that* $|\rho - a| = 1$,

iii. for every $\rho \in R$, *for every* $a \in \mathbb{Z}[i] \setminus \{0\}$: *if* $|\rho - a| = 1$, *then* $\rho - a \in R$.

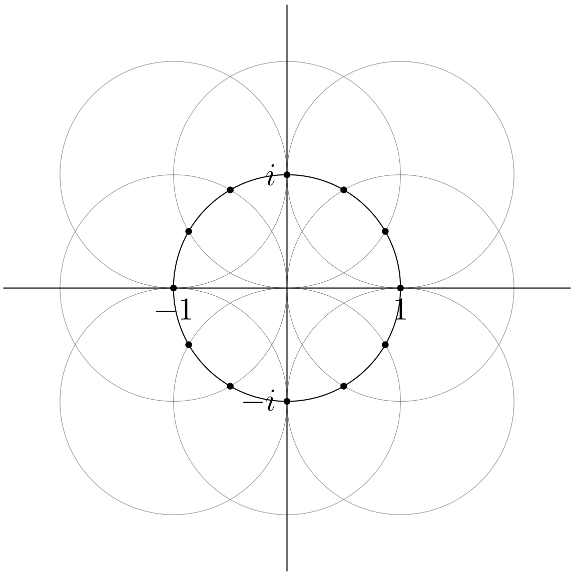

Figure 3: *The complex plane with the unit circle and the set R. The other circles are circles with centre c and radius 1, with $c \in \{1, -1, i, -i, 1+i, 1-i, -1+i, -1-i\}$.*

Proof. We prove the first statement. Let $z \in \mathbb{C}$ such that $|z| = 1$, accordingly: z is on the unit circle. Let $a \in \mathbb{Z}[i] \setminus \{0\}$, if $a \notin \{1, -1, i, -i, 1+i, 1-i, -1+i, -1-i, 2, -2, 2i, -2i\}$ then $|z - a| > 1$. We consider the cases $a = 2$, $a = 1 + i$ and $a = 1$, the other cases are similar. Define for every $\alpha \in \mathbb{Z}[i]$ the circle $S_\alpha := \{z \in \mathbb{C} \mid |z - \alpha| = 1\}$.
If $a = 2$ then $|z| = 1$ and $|z - a| = 1$ iff $z \in S_0 \cap S_a$. So $z \in \{1\}$ and therefore: $z \in R$.
If $a = 1 + i$ then $|z| = 1$ and $|z - a| = 1$ iff $z \in S_0 \cap S_a$. Therefore $z \in \{1, i\}$ and consequently: $z \in R$.
If $a = 1$ then $|z| = 1$ and $|z - a| = 1$ iff $z \in S_0 \cap S_a$. As $|a| = 1$ we know that there are exactly two points in $S_0 \cap S_a$, say p and q. Suppose p is in the upper half plane. As $|1 - 0| = |1 - p| = |p - 0| = 1$, we have that the triangle formed by 0, 1 and p is equilateral. Therefore we have that angles of this triangle measure $\frac{2\pi}{6}$. From this it follows that $p = e^{2\pi i/6} \in R$. With the same reasoning: $q = e^{10\pi i/6} \in R$. As $z \in \{p, q\}$ we conclude: $z \in R$.

Now we prove the second statement. By symmetry, we only consider two cases: $\rho = 1$ and $\rho = e^{\pi i/6}$. The other cases are similar. For $\rho = 1$ we choose $a := 1 + i$, then $|\rho - a| = 1$. For $\rho = e^{\pi i/6}$ we see that i fulfils the property: $|\rho - a| = |e^{\pi i/6} - i| = |e^{11\pi i/6}| = 1$.

Finally we prove the third statement. Again, by symmetry, we only consider two cases: $\rho = 1$ and $\rho = e^{\pi i/6}$. The other cases are similar. Let $\rho = 1$ and $a \in \mathbb{Z}[i] \setminus \{0\}$ such that $|\rho - a| = 1$. Then $a \in \{2, 1+i, 1-i\}$. We see: $\rho - a \in R$ for every $a \in \{2, 1+i, 1-i\}$. Now let $\rho = e^{\pi i/6}$ and $a \in \mathbb{Z}[i] \setminus \{0\}$ such that $|\rho - a| = 1$. Now there is only one possibility for a, namely: $a = i$. We see: $\rho - a = e^{\pi i/6} - i = e^{11\pi i/6} \in R$. This completes the proof. $\square$

We notice that the set R from Lemma 3.28 is precisely the set of points that are both at unit distance from 0 and at unit distance from a for some $a \in \mathbb{Z}[i] \setminus \{0\}$. This fact will be useful in the following proposition.

Proposition 3.29. *Let $z \in \mathbb{C}$ be irrational, and let $(z_{n+1})_{n\in\mathbb{N}}$ be the complete quotients of z under* CF. *Then not:* $\lim_{n\to\infty} |z_n| = 1$.

Proof. Let $\mathsf{CF}(z) = [a_0, e_1/a_1, e_2/a_2, \ldots]$. Let R as in Lemma 3.28. Suppose for the sake of contradiction: $\lim_{n\to\infty} |z_n| = 1$. As $|z_n - a_n| = \frac{|e_{n+1}|}{|z_{n+1}|}$ we also have that $\lim_{n\to\infty} |z_n - a_n| = 1$. Since $a_n \in \mathbb{Z}[i] \setminus \{0\}$ for every $n \geq 1$, we have that $\lim_{n\to\infty} d(z_n, R) = 0$. So, for every $n \in \mathbb{N}$ there exists $\rho_n \in R$ such that $\lim_{n\to\infty} |z_n - \rho_n| = 0$. As $\lim_{n\to\infty} |z_n - a_n| = 1$ and $\lim_{n\to\infty} |z_n - \rho_n| = 0$ we have that $\lim_{n\to\infty} |a_n - \rho_n| = 1$. As $\mathbb{Z}[i]$ and R are discrete, we conclude that there exists $n_0 \in \mathbb{N}$ such that $|a_n - \rho_n| = 1$ for every $n \geq n_0$. By Lemma 3.28 we have that $\rho_n - a_n \in R$ for every $n \geq n_0$. Now define for every $n \in \mathbb{N}$: $B_n := \rho_n - a_n$ and $C_n := z_n - \rho_n$. Then for every $n \geq n_0$: $B_n \in R$ and $z_n - a_n = B_n + C_n$, and $\lim_{n\to\infty} C_n = 0$. Then:

$$z_{n+1} = \frac{e_{n+1}}{z_n - a_n} = \frac{e_{n+1}}{B_n + C_n} = \frac{e_{n+1}}{B_n} + \frac{-C_n e_{n+1}}{B_n(B_n + C_n)}.$$

We have that $\left|\frac{C_n}{B_n+C_n}\right| \leq \frac{|C_n|}{|1-C_n|}$, so $\lim_{n\to\infty} \frac{C_n}{B_n+C_n} = 0$. As $B_n \in R$, also $\frac{1}{B_n} \in R$, therefore for every $n \geq n_0$:

$$\rho_{n+1} = \frac{e_{n+1}}{B_n} \text{ and } C_{n+1} = z_{n+1} - \rho_{n+1} = \frac{-C_n e_{n+1}}{B_n(B_n + C_n)}.$$

Consequently:

$$|C_{n+1}| = \frac{|C_n|}{|B_n + C_n|} = \frac{|C_n|}{|z_n - a_n|} > |C_n|.$$

Therefore $|C_{n+1}| > |C_n|$ for every $n \geq n_0$, but we also have that $\lim_{n\to\infty} C_n = 0$. Therefore we obtain a contradiction and we conclude it is not the case that $\lim_{n\to\infty} |z_n| = 1$. □

The following result is a direct consequence of Proposition 3.29.

Proposition 3.30. *Let $z \in \mathbb{C}$ be irrational and let $(z_{n+1})_{n\in\mathbb{N}}$ be the complete quotients of z under* CF. *Then:* $\lim_{n\to\infty} \prod_{k=0}^{n} z_{k+1} = \infty$.

Proof. According to Proposition 3.29: not $\lim_{n\to\infty} |z_{n+1}| = 1$. As $|z_{n+1}| > 1$ for every $n \in \mathbb{N}$, it follows that

$$\neg \forall \varepsilon > 0\ \exists N \in \mathbb{N}\ \forall n \geq N\ \left[|z_{n+1}| < 1 + \varepsilon\right]$$

or, equivalently:

$$\exists \varepsilon > 0\ \forall N \in \mathbb{N}\ \exists n \geq N\ \left[|z_{n+1}| \geq 1 + \varepsilon\right].$$

Therefore there exist infinitely many indices n such that $|z_{n+1}| \geq 1 + \varepsilon_0$, for a $\varepsilon_0 > 0$. We conclude: $\lim_{n\to\infty} \prod_{k=0}^{n} z_{k+1} = \infty$. □

The following lemmas will be a powerful tool in both this and the following section. They will be useful to write down identities involving the quantities x_0, x_{n+1} and $x_0 q_n - p_n$.

Lemma 3.31. *Let $x \in \mathbb{C}$, $n \in \mathbb{N}$, and let x_{n+1} be the $(n+1)$-th complete quotient of x under* CF. *Let p_n and q_n be as in Definition 2.9 with respect to* $\mathsf{CF}(x)$. *Then:*

$$x_0 = \frac{x_{n+1} p_n + e_{n+1} p_{n-1}}{x_{n+1} q_n + e_{n+1} q_{n-1}}.$$

Proof. This proof is by induction on n. For $n = 0$ we have

$$x_0 = a_0 + \frac{e_1}{x_1} = \frac{x_1 a_0 + e_1}{x_1} = \frac{x_1 p_0 + e_1 p_{-1}}{x_1 q_0 + e_1 q_{-1}}.$$

Again, using Definition 2.9 and the equality $x_{n+1} = \frac{e_{n+1}}{x_n - a_n}$, we obtain:

$$\begin{aligned}
\frac{x_{n+1} p_n + e_{n+1} p_{n-1}}{x_{n+1} q_n + e_{n+1} q_{n-1}} &= \frac{x_{n+1}(a_n p_{n-1} + e_n p_{n-2}) + e_{n+1} p_{n-1}}{x_{n+1}(a_n q_{n-1} + e_n q_{n-2}) + e_{n+1} q_{n-1}} \\
&= \frac{(a_n p_{n-1} + e_n p_{n-2}) + \frac{e_{n+1} p_{n-1}}{x_{n+1}}}{(a_n q_{n-1} + e_n q_{n-2}) + \frac{e_{n+1} q_{n-1}}{x_{n+1}}} \\
&= \frac{(a_n + \frac{e_{n+1}}{x_{n+1}}) p_{n-1} + e_n p_{n-2}}{(a_n + \frac{e_{n+1}}{x_{n+1}}) q_{n-1} + e_n q_{n-2}} \\
&= \frac{x_n p_{n-1} + e_n p_{n-2}}{x_n q_{n-1} + e_n q_{n-2}}.
\end{aligned}$$

□

Lemma 3.32. *Let $x \in \mathbb{C}$, $n \in \mathbb{N}$, and let x_{n+1} be the $(n+1)$-th complete quotient of x under* CF*. Let p_n and q_n be as in Definition 2.9 with respect to* $\mathsf{CF}(x)$*. Then:*

$$x_{n+1} = -e_{n+1} \frac{x_0 q_{n-1} - p_{n-1}}{x_0 q_n - p_n}.$$

Proof. According to Lemma 3.31 we have $x_0(x_{n+1} q_n + e_{n+1} q_{n-1}) = x_{n+1} p_n + e_{n+1} p_{n-1}$. Rearranging the terms gives: $x_{n+1}(x_0 q_n - p_n) = -e_{n+1}(x_0 q_{n-1} - p_{n-1})$ and the claim follows. □

Lemma 3.33. *Let $x \in \mathbb{C}$, $n \in \mathbb{N}$, and let $(x_{k+1})_{k=0}^{n}$ be the first $n+1$ complete quotients of x under* CF*. Let p_n and q_n be as in Definition 2.9 with respect to* $\mathsf{CF}(x)$*. Then:*

$$x_0 q_n - p_n = (-1)^n \cdot \prod_{k=0}^{n} \frac{e_{k+1}}{x_{k+1}}.$$

Proof. According to Lemma 3.32:

$$\prod_{k=0}^{n} x_{k+1} = \prod_{k=0}^{n} -e_{k+1} \frac{x_0 q_{k-1} - p_{k-1}}{x_0 q_k - p_k} = \frac{x_0 q_{-1} - p_{-1}}{x_0 q_n - p_n} \prod_{k=0}^{n} -e_{n+1} = \frac{-\prod_{k=0}^{n} -e_{n+1}}{x_0 q_n - p_n},$$

and we conclude:

$$x_0 q_n - p_n = \frac{-\prod_{k=0}^{n} -e_{n+1}}{\prod_{k=0}^{n} x_{k+1}} = (-1)^n \cdot \prod_{k=0}^{n} \frac{e_{k+1}}{x_{k+1}}.$$

□

The next proposition will be of great value in the remaining part of this section. Recall that $|[a_0, e_1/a_1, \ldots, e_n/a_n]| = n + 1$ and $|[a_0, e_1/a_1, e_2/a_2, \ldots]| = \infty$. We will use the convention $\infty - 1 = \infty$.

Proposition 3.34. *Let $x \in \mathbb{C}$, let $0 \leq n < |\mathsf{CF}(x)|$ and let q_n be as in Definition 2.9 with respect to $\mathsf{CF}(x)$. Then: $q_n \neq 0$.*

Proof. First, let $n = |\mathsf{CF}(x)| - 1$. As $\mathsf{CF}(x)$ is proper by Corollary 3.20, we obtain by Proposition 2.12: $\mathrm{Val}(\mathsf{CF}(x)) = c_n = \frac{p_n}{q_n} \in \mathbb{Q}[i]$, and therefore $q_n \neq 0$.

Now let $0 \leq n < |\mathsf{CF}(x)| - 1$. By Lemma 3.33 we have: $x_0 q_n - p_n = (-1)^n \cdot \prod_{k=0}^{n} \frac{e_{k+1}}{x_{k+1}}$. Assume that $q_n = 0$. Then $|p_n| = |(-1)^n \cdot \prod_{k=0}^{n} \frac{e_{k+1}}{x_{k+1}}| = \frac{1}{|\prod_{k=0}^{n} x_{k+1}|}$. As $|x_{k+1}| > 1$ for every $k \in \{0, \dots, n\}$ we have $0 < \frac{1}{|\prod_{k=0}^{n} x_{k+1}|} < 1$. Therefore: $0 < |p_n| < 1$, which leads to a contradiction as $p_n \in \mathbb{Z}[i]$. We conclude: $q_n \neq 0$. □

Proposition 3.35. *Let $x \in \mathbb{C}$, let $0 \leq n < |\mathsf{CF}(x)|$ and let p_n be as in Definition 2.9 with respect to $\mathsf{CF}(x)$. Then: if $|x| \geq 1$, then $p_n \neq 0$.*

Proof. First, let $n = |\mathsf{CF}(x)| - 1$. Suppose for the sake of contradiction that $p_n = 0$. Then: $0 = \frac{p_n}{q_n} = c_n = \mathrm{Val}([a_0, e_1/a_1, \dots, e_n/a_n]) = x$, so $x = 0$. This is a contradiction as we assumed that $|x| \geq 1$.

Now let $n < |\mathsf{CF}(x)| - 1$. According to Lemma 3.33 we have: $x_0 q_n - p_n = (-1)^n \cdot \prod_{k=0}^{n} \frac{e_{k+1}}{x_{k+1}}$. Now suppose for the sake of contradiction that $p_n = 0$. Then $|x_0 q_n| = |(-1)^n \cdot \prod_{k=0}^{n} \frac{e_{k+1}}{x_{k+1}}| = \frac{1}{|\prod_{k=0}^{n} x_{k+1}|}$. So: $|q_n| = \frac{1}{|\prod_{k=-1}^{n} x_{k+1}|}$. As $n > 0$ and $|x_{k+1}| > 1$ for every $k \in \{0, \dots, n\}$ we have that $0 < \frac{1}{|\prod_{k=-1}^{n} x_{k+1}|} < 1$. Therefore: $0 < |q_n| < 1$, which leads to a contradiction as $q_n \in \mathbb{Z}[i]$. □

The following result will be useful in proving convergence.

Proposition 3.36. *Let $x \in \mathbb{C}$ be irrational and let q_n be as in Definition 2.9 with respect to $\mathsf{CF}(x)$. Then: $\lim_{n \to \infty} q_n = \infty$.*

Proof. Suppose for the sake of contradiction, not: $\lim_{n \to \infty} q_n = \infty$. Consequently, $(q_n)_{n \in \mathbb{N}}$ has a finite accumulation point, therefore there exists $Q \in \mathbb{Z}[i]$ such that $q_n = Q$ for an infinite number of indices n (as $q_n \in \mathbb{Z}[i]$, which is discrete). So, with the use of Lemma 3.33 there are an infinite number of indices n such that $|x_0 Q - p_n| = \frac{1}{|\prod_{k=0}^{n} x_{k+1}|} < 1$. Investigating this last inequality, it follows that there exists $P \in \mathbb{Z}[i]$ such that there are infinite number of indices such that $p_n = P$ and $q_n = Q$. So there are an infinite number of indices n such that $|x_0 q_n - p_n| = |x_0 Q - P|$. However, $1 < |x_{n+1}| = \frac{|x_0 q_{n-1} - p_{n-1}|}{|x_0 q_n - p_n|}$, or $|x_0 q_{n-1} - p_{n-1}| > |x_0 q_n - p_n|$ for every $n \in \mathbb{N}$, and we obtain a contradiction. □

The following result shows that it makes sense to consider the prefixes of a continued fraction obtained by a complex continued fraction algorithm.

Theorem 3.37. *Let $z \in \mathbb{C}$, let a_n be the n-th partial quotient and let e_n be the n-th partial numerator of $\mathsf{CF}(x)$. Then for every $k < |\mathsf{CF}(z)|$: $[a_0, e_1/a_1, \dots e_k/a_k]$ is proper.*

Proof. First let $k = |\mathsf{CF}(z)| - 1$. Then $\mathsf{CF}(z) = [a_0, e_1/a_1, \dots e_k/a_k]$, and according to Corollary 3.20 we obtain: $[a_0, e_1/a_1, \dots e_k/a_k]$ is proper.

Now let $k < |\mathsf{CF}(z)| - 1$. Suppose for the sake of contradiction that $[a_0, e_1/a_1, \dots e_k/a_k]$ is not proper. Examining Definition 2.4 and Definition 2.5 we find that there exists

$j \in \{0, \dots, k-1\}$ such that $\mathrm{Val}([a_{j+1}, e_{j+2}/a_{j+2}, \dots, e_k/a_k]) = 0$. Observe that it is the case that $[a_j, e_{j+1}/a_{j+1}, \dots, e_k/a_k]$ is a finite continued fraction, so we can compute $(q_n)_{-1 \le n \le k-j}$ using Definition 2.9. By Proposition 3.34 we obtain: $q_n \neq 0$ for every $n \in \{0, \dots, k-j\}$. Therefore: $\frac{q_{k-j}}{q_{k-j-1}} \neq 0$. On the other hand we have that $\mathrm{Val}([a_{j+1}, e_{j+2}/a_{j+2}, \dots, e_k/a_k]) = 0$, therefore, by applying Lemma 2.14 we find that $\mathrm{Val}([a_k, e_k/a_{k-1}, \dots, e_{j+2}/a_{j+1}]) = 0$. According to Lemma 2.15 we have that $\frac{q_{k-j}}{q_{k-j-1}} = 0$. Hereby we obtain a contradiction.

In either case we find that $[a_0, e_1/a_1, \dots e_k/a_k]$ is proper and this completes the proof. □

In Definition 2.8 we defined c_k to be the convergents of a finite or infinite continued fraction. In Proposition 2.12 and Proposition 2.13 we proved that $c_k = \frac{p_k}{q_k}$, if $[a_0, e_1/a_1, \dots e_k/a_k]$ is a proper continued fraction. In Theorem 3.37 we saw that that $[a_0, e_1/a_1, \dots e_k/a_k]$ is proper if it is a prefix of the output of a continued fraction algorithm. As from now on we will mainly focus on continued fractions obtained by continued fraction algorithms, we will call $\frac{p_k}{q_k}$ the convergents of a continued fraction, instead of c_k. The benefits are that we can refer to p_k and q_k individually, and that we have an easy way to compute the k-th convergent, using Definition 2.9.

Now we prove that a continued fraction obtained by a complex continued fraction algorithm converges to the input of the algorithm.

Theorem 3.38. *Let $x \in \mathbb{C}$ be irrational and let $\left(\frac{p_n}{q_n}\right)_{n \in \mathbb{N}}$ be the convergents of x under* CF. *Then:* $\lim_{n \to \infty} \frac{p_n}{q_n} = x$.

Proof. According to Lemma 3.33 we have

$$x_0 q_n - p_n = (-1)^n \cdot \prod_{k=0}^{n} \frac{e_{k+1}}{x_{k+1}},$$

with $x_0 = x$. As $q_n \neq 0$ for every $n \in \mathbb{N}$, we can rewrite this to

$$x_0 - \frac{p_n}{q_n} = \frac{(-1)^n}{q_n} \cdot \prod_{k=0}^{n} \frac{e_{k+1}}{x_{k+1}}.$$

Therefore:

$$\left| x_0 - \frac{p_n}{q_n} \right| = \frac{1}{|q_n| \cdot |\prod_{k=0}^{n} x_{k+1}|}.$$

As $\lim_{n \to \infty} q_n = \infty$ (Proposition 3.36), and $\lim_{n \to \infty} \prod_{k=0}^{n} x_{k+1} = \infty$ (Proposition 3.30), we conclude:

$$\lim_{n \to \infty} \left| x_0 - \frac{p_n}{q_n} \right| = 0,$$

and the claim follows. □

From this last result we can prove that the algorithm CF is injective.

Proposition 3.39. *Let x, $y \in \mathbb{C}$. If* $\mathsf{CF}(x) = \mathsf{CF}(y)$, *then $x = y$.*

Proof. Suppose $\mathsf{CF}(x) = \mathsf{CF}(y)$. If both $\mathsf{CF}(x)$ and $\mathsf{CF}(y)$ are finite continued fractions, then x and y are rational and by Proposition 3.19:

$$x = \mathrm{Val}(\mathsf{CF}(x)) = \mathrm{Val}(\mathsf{CF}(y)) = y.$$

Now suppose $\mathsf{CF}(x)$ and $\mathsf{CF}(y)$ are infinite continued fractions. Then x and y are irrational. Let $\frac{p_n}{q_n}$ be the n-th convergent of $\mathsf{CF}(x)$, and $\frac{r_n}{s_n}$ be the n-th convergent of $\mathsf{CF}(y)$. Then we have that $p_n = r_n$ and $q_n = s_n$, and consequently $\frac{p_n}{q_n} = \frac{r_n}{s_n}$ for every $n \in \mathbb{N}$. Therefore by Theorem 3.38:

$$x = \lim_{n\to\infty} \frac{p_n}{q_n} = \lim_{n\to\infty} \frac{r_n}{s_n} = y$$

and this completes the proof. $\square$

The following result gives a simple criterion for a convergent of a continued fraction to be a better approximation than the previous one.

Proposition 3.40. *Let $x \in \mathbb{C}$, $n \in \mathbb{N}_{>0}$ and let $\frac{p_n}{q_n}$ be the n-th convergent of x under CF. If $\left|\frac{q_{n-1}}{q_n}\right| \leq 1$ then $\left|x - \frac{p_n}{q_n}\right| < \left|x - \frac{p_{n-1}}{q_{n-1}}\right|$.*

Proof. According to Lemma 3.32 we have

$$x_{n+1} = -e_{n+1}\frac{x_0 q_{n-1} - p_{n-1}}{x_0 q_n - p_n}.$$

Therefore:

$$\frac{x_0 q_n - p_n}{x_0 q_{n-1} - p_{n-1}} = -\frac{e_{n+1}}{x_{n+1}},$$

and as $\frac{1}{|x_{n+1}|} < 1$ and $\left|\frac{q_{n-1}}{q_n}\right| \leq 1$ we obtain:

$$\frac{\left|x - \frac{p_n}{q_n}\right|}{\left|x - \frac{p_{n-1}}{q_{n-1}}\right|} = \left|\frac{q_{n-1}}{q_n}\right| \cdot \frac{1}{|x_{n+1}|} < 1$$

and this completes the proof. $\square$

We will now define a quantity which more or less estimates the quality of the convergents of a continued fraction.

Definition 3.41. Let $x \in \mathbb{C}$, $n \in \mathbb{N}$ and let $\frac{p_n}{q_n}$ be the n-th convergent of $\mathsf{CF}(x)$. We define $\theta_n := q_n(q_n x - p_n)$. We will call θ_n the *relative error* of x and $\frac{p_n}{q_n}$.

Writing $x - \frac{p_n}{q_n} = \frac{\theta_n}{q_n^2}$ clarifies why we call θ_n the relative error of x and $\frac{p_n}{q_n}$. Note that by Lemma 3.33 we have:

$$\theta_n = q_n(x_0 q_n - p_n) = q_n(-1)^n \cdot \prod_{k=1}^{n+1} \frac{e_k}{x_k}.$$

The following result shows an alternative way to express θ_n.

Lemma 3.42. *Let $x \in \mathbb{C}$, $n \in \mathbb{N}$ and let $\frac{p_n}{q_n}$ be the n-th convergent and x_{n+1} be the $(n+1)$-th complete quotient of $\mathsf{CF}(x)$. Then:*

$$\theta_n = \frac{(-1)^n \cdot \prod_{k=1}^{n+1} e_k}{x_{n+1} + e_{n+1}\frac{q_{n-1}}{q_n}}.$$

Proof. Applying Lemma 3.31 and Proposition 2.10 gives us:

$$\begin{aligned}
\theta_n &= q_n^2\big(x_0 - \frac{p_n}{q_n}\big) \\
&= q_n^2\Big(\frac{x_{n+1}p_n + e_{n+1}p_{n-1}}{x_{n+1}q_n + e_{n+1}q_{n-1}} - \frac{p_n}{q_n} \cdot \frac{x_{n+1} + e_{n+1}\frac{q_{n-1}}{q_n}}{x_{n+1} + e_{n+1}\frac{q_{n-1}}{q_n}}\Big) \\
&= q_n^2 \cdot \frac{\frac{-e_{n+1}}{q_n}(p_n q_{n-1} - p_{n-1}q_n)}{x_{n+1}q_n + e_{n+1}q_{n-1}} \\
&= q_n \cdot \frac{-e_{n+1}(-1)^{n-1} \cdot \prod_{k=1}^{n} e_k}{x_{n+1}q_n + e_{n+1}q_{n-1}} \\
&= \frac{(-1)^n \cdot \prod_{k=1}^{n+1} e_k}{x_{n+1} + e_{n+1}\frac{q_{n-1}}{q_n}},
\end{aligned}$$

and this completes the proof. □

This last result will be useful in the next section.

3.6 Periodicity

In this section we will examine the relation between periodic continued fractions and quadratic irrational numbers. First we will see that if a continued fraction of a number x obtained by a complex continued fraction algorithm is periodic, then x is quadratic irrational. Thereafter we will study the converse of this assertion. First we prove two lemmas.

Lemma 3.43. *Let $x \in \mathbb{C}$ be irrational, and let $\mathsf{CF}(x) = [a_0, e_1/a_1, e_2/a_2, \ldots]$. Let x_m be the m-th complete quotient of x under CF. Let $n \in \mathbb{N}$ and let $\frac{p_k}{q_k}$ be the k-th convergent of $[a_n, e_{n+1}/a_{n+1}, e_{n+2}/a_{n+2}, \ldots]$. Then: $\lim_{k\to\infty} \frac{p_k}{q_k} = x_n$.*

Proof. According to Proposition 3.24: $\mathsf{CF}(x_n) = [a_n, e_{n+1}/a_{n+1}, e_{n+2}/a_{n+2}, \ldots]$. By Theorem 3.38 we conclude: $\lim_{k\to\infty} \frac{p_k}{q_k} = x_n$. □

Lemma 3.44. *Let $x \in \mathbb{C}$ be irrational and let x_n be the n-th complete quotient of x under CF. If $\mathsf{CF}(x)$ is purely periodic with period length m, then $x_0 = x_m$.*

Proof. Let $\mathsf{CF}(x) = [a_0, e_1/a_1, e_2/a_2, \ldots]$ and suppose $\mathsf{CF}(x)$ is purely periodic with period length m. Let $\frac{p_k}{q_k}$ be the k-th convergent of $[a_0, e_1/a_1, e_2/a_2, \ldots]$ and let $\frac{r_k}{s_k}$ be the k-th convergent of $[a_m, e_{m+1}/a_{m+1}, e_{m+2}/a_{m+2}, \ldots]$. As $\mathsf{CF}(x)$ is purely periodic with period length m we have: $a_n = a_{n+m}$, $e_{n+1} = e_{n+m+1}$ for every $n \in \mathbb{N}$. Therefore: $p_k = r_k$ and $q_k = s_k$ and consequently: $\frac{p_k}{q_k} = \frac{r_k}{s_k}$ for every $k \in \mathbb{N}$. By Lemma 3.43 we obtain:

$$x_0 = \lim_{k\to\infty} \frac{p_k}{q_k} = \lim_{k\to\infty} \frac{r_k}{s_k} = x_m$$

and this completes the proof. □

Now we can prove the following result concerning purely periodic continued fractions.

Proposition 3.45. *Let $x \in \mathbb{C}$ be irrational. If $\mathsf{CF}(x)$ is purely periodic, then x is quadratic irrational.*

Proof. Let m be the period length of $\mathsf{CF}(x)$. According to Lemma 3.44: $x_0 = x_m$. By Lemma 3.31 we have:

$$x_0 = \frac{x_m p_{m-1} + e_m p_{m-2}}{x_m q_{m-1} + e_m q_{m-2}} = \frac{x_0 p_{m-1} + e_m p_{m-2}}{x_0 q_{m-1} + e_m q_{m-2}}.$$

Therefore:

$$q_{m-1} x_0^2 + (e_m q_{m-2} - p_{m-1}) x_0 - e_m p_{m-2} = 0.$$

As $m \geq 1$ we have that p_{m-1}, p_{m-2}, q_{m-1}, q_{m-2} and e_m are all well-defined. As $q_{m-1} \in \mathbb{Z}[i]$, $e_m q_{m-2} - p_{m-1} \in \mathbb{Z}[i]$, $e_m p_{m-2} \in \mathbb{Z}[i]$, $q_{m-1} \neq 0$ and x_0 is irrational, we conclude: x_0 is quadratic irrational. □

In order to give a more general result, we need two more lemmas.

Lemma 3.46. *Let $x \in \mathbb{C}$, $a \in \mathbb{Z}[i]$ and $e \in \{-1, 1, -i, i\}$. If x is quadratic irrational, then $\frac{e}{x} + a$ is quadratic irrational.*

Proof. As x is quadratic irrational, it follows by Proposition 1.1 we can write $x = \frac{p+q\sqrt{r}}{s}$, with p, q, r, $s \in \mathbb{Z}[i]$ where $q \neq 0$, $s \neq 0$ and r is not a square. Then:

$$\frac{e}{x} + a = \frac{es}{p + q\sqrt{r}} + a = \frac{es(p - q\sqrt{r})}{p^2 - q^2 r} + \frac{a(p^2 - q^2 r)}{p^2 - q^2 r} = \frac{esp + a(p^2 - q^2 r) - esq\sqrt{r}}{p^2 - q^2 r}.$$

When we set $P := esp + a(p^2 - q^2 r)$, $Q := -esq$, $S := p^2 - q^2 r$ we obtain:

$$\frac{e}{x} + a = \frac{P + Q\sqrt{r}}{S},$$

where P, Q, r, $S \in \mathbb{Z}[i]$ and $Q \neq 0$, $S \neq 0$ and r is not a square. Again by Proposition 1.1 we obtain: $\frac{e}{x} + a$ is quadratic irrational. □

Lemma 3.47. *Let $x \in \mathbb{C}$ be irrational, and let x_n be the n-th complete quotient of x under CF. If x_n is quadratic irrational, then also x_0 is quadratic irrational.*

Proof. Let x_n be quadratic irrational. We have $x_n = \frac{e_n}{x_{n-1} - a_{n-1}}$, or: $x_{n-1} = \frac{e_n}{x_n} + a_{n-1}$. From Lemma 3.46 follows: x_{n-1} is quadratic irrational. By repeating this argument we obtain: x_0 is quadratic irrational. □

Theorem 3.48. *Let $x \in \mathbb{C}$ be irrational. If $\mathsf{CF}(x)$ is periodic, then x is quadratic irrational.*

Proof. Let $n \in \mathbb{N}$ be the least number such that $[a_n, e_{n+1}/a_{n+1}, e_{n+2}/a_{n+2}, \ldots]$ is purely periodic. According to Proposition 3.24 we have: $\mathsf{CF}(x_n) = [a_n, e_{n+1}/a_{n+1}, e_{n+2}/a_{n+2}, \ldots]$. By Proposition 3.45 follows that x_n is quadratic irrational. By Lemma 3.47 we conclude: x_0 is quadratic irrational. □

From here we will study the converse of Theorem 3.48. First we will consider some equivalent formulations of the phrase '$\mathsf{CF}(x)$ is periodic'. In order to do so, we prove two lemmas.

Lemma 3.49. *Let $x \in \mathbb{C}$ be irrational and let $(x_n)_{n\in\mathbb{N}}$ be the complete quotients of x under CF. If there exist r, $s \in \mathbb{N}$, $r < s$ such that $x_r = x_s$, then $\mathsf{CF}(x)$ is periodic.*

Proof. Let $\mathsf{CF}(x) = [a_0, e_1/a_1, e_2/a_2, \ldots]$.

<u>Claim:</u> For every $n \in \mathbb{N}$: $x_{r+n} = x_{s+n}$, $a_{r+n} = a_{s+n}$ and $e_{r+n+1} = e_{s+n+1}$.

<u>Proof of claim:</u> We prove this by induction. For $n = 0$ we have $x_r = x_s$ by assumption, $a_r = fl(x_r) = fl(x_s) = a_s$ and $e_{r+1} = sg(x_r) = sg(x_s) = e_{s+1}$. Now suppose $x_{r+n} = x_{s+n}$, $a_{r+n} = a_{s+n}$ and $e_{r+n+1} = e_{s+n+1}$. Then:

$$x_{r+n+1} = \frac{e_{r+n+1}}{x_{r+n} - a_{r+n}} = \frac{e_{s+n+1}}{x_{s+n} - a_{s+n}} = x_{s+n+1}$$

and

$$a_{r+n+1} = fl(x_{r+n+1}) = fl(x_{s+n+1}) = a_{s+n+1}$$

and

$$e_{r+n+1+1} = sg(x_{r+n+1}) = sg(x_{s+n+1}) = e_{s+n+1+1},$$

as had to be shown. ■

Let $m := s - r$. By the claim immediately follows that $(a_n, e_{n+1}) = (a_{n+m}, e_{n+m+1})$ for every $n \geq r$. We conclude: $[a_0, e_1/a_1, e_2/a_2, \ldots]$ is periodic. □

Lemma 3.50. *Let $x \in \mathbb{C}$ be irrational and let x_n be the n-th complete quotient of x under CF. Let $\mathsf{CF}(x) = [a_0, e_1/a_1, e_2/a_2, \ldots]$. Suppose $\mathsf{CF}(x)$ is periodic, let m be the length of the period and let $N \in \mathbb{N}$ be such that $[a_N, e_{N+1}/a_{N+1}, e_{N+2}/a_{N+2}, \ldots]$ is purely periodic. Then for every $n \geq N$: $x_n = x_{n+m}$.*

Proof. We prove this by induction. Let $n = N$. Let $y := x_n$. Then $\mathsf{CF}(y) = \mathsf{CF}(x_n) = [a_n, e_{n+1}/a_{n+1}, e_{n+2}/a_{n+2}, \ldots]$ is purely periodic with period length m. Therefore we have by Lemma 3.44: $y_0 = y_m$, where y_k is the k-th complete quotient of y under CF. In a similar way as in Proposition 3.24 we obtain $y_m = x_{n+m}$ and therefore we conclude: $x_n = y_0 = y_m = x_{n+m}$.
Now suppose $x_n = x_{n+m}$ for some $n \geq N$. As $\mathsf{CF}(x)$ is periodic with period length m we have: $a_n = a_{n+m}$, $e_{n+1} = e_{n+m+1}$, and therefore:

$$x_{n+1} = \frac{e_{n+1}}{x_n - a_n} = \frac{e_{n+m+1}}{x_{n+m} - a_{n+m}} = x_{n+m+1}$$

and this ends the proof. □

Proposition 3.51. *Let $x \in \mathbb{C}$ be irrational and let x_n be the n-th complete quotient of x under CF. Then the following are equivalent:*

i. $\mathsf{CF}(x)$ *is periodic,*

ii. there exist $N \in \mathbb{N}$, $m \in \mathbb{N}_{>0}$ such that $x_n = x_{n+m}$ for every $n \geq N$, *iii.* $\{x_n \mid n \in \mathbb{N}\}$ *is finite,*

iv. there exist $r, s \in \mathbb{N}$, $r < s$ *such that* $x_r = x_s$.

Proof. We prove (i) ⇒ (ii) ⇒ (iii) ⇒ (iv) ⇒ (i).

(i) ⇒ (ii) This follows directly by Lemma 3.50.

(ii) ⇒ (iii) Let $N \in \mathbb{N}$, $m \in \mathbb{N}_{>0}$ such that $x_n = x_{n+m}$ for every $n \geq N$. Then: $\{x_n \mid n \in \mathbb{N}\} = \{x_n \mid n < N + m\}$, which is finite.

(iii) ⇒ (iv) This follows by the pigeonhole principle.

(iv) ⇒ (i) This follows directly by Lemma 3.49.

□

The following result is a partial converse of Theorem 3.48. The proof partially follows the proof of the main theorem of [10].

Theorem 3.52. *Let* $x \in \mathbb{C}$ *be quadratic irrational. Let* $(\theta_n)_{n\in\mathbb{N}}$ *be the relative errors of* x *under* CF*. If there exists* $\delta > 0$ *such that* $|\theta_{n-1}| < \delta$ *and* $|\theta_n| < \delta$ *for infinitely many indices* n*, then* $\mathsf{CF}(x)$ *is periodic.*

Proof. Let $\mathsf{CF}(x) = [a_0, e_1/a_1, e_2/a_2, \dots]$ and let $(x_n)_{n\in\mathbb{N}}$ be the complete quotients of x under CF. As x is quadratic irrational, let A, B, $C \in \mathbb{Z}[i]$ such that $A \neq 0$ and $Ax^2 + Bx + C = 0$. Define $D := B^2 - 4AC$.

Claim 1: For every $n \in \mathbb{N}$ we can construct a polynomial $f_n(z) := A_n z^2 + B_n z + C_n$ with coefficients A_n, B_n, $C_n \in \mathbb{Z}[i]$, $A_n \neq 0$, $|B_n^2 - 4A_nC_n| = |B_0^2 - 4A_0C_0|$, such that $f_n(x_n) = 0$.

Proof of claim 1: We prove this by induction: as $x = x_0$, we define $A_0 := A$, $B_0 := B$, $C_0 := C$ and $f_0(z) := A_0z^2 + B_0z + C_0$. Then: $f_0(x_0) = 0$. Now suppose f_n is defined, with $A_n \neq 0$ and $|B_n^2 - 4A_nC_n| = |B_0^2 - 4A_0C_0|$. Then:

$$\begin{aligned} 0 = f_n(x_n)x_{n+1}^2 &= \big(A_n(a_n + \tfrac{e_{n+1}}{x_{n+1}})^2 + B_n(a_n + \tfrac{e_{n+1}}{x_{n+1}}) + C_n\big)x_{n+1}^2 \\ &= (A_na_n^2 + B_na_n + C_n)x_{n+1}^2 + e_{n+1}(2A_na_n + B_n)x_{n+1} + e_{n+1}^2A_n \\ &= f_n(a_n)x_{n+1}^2 + e_{n+1}f_n'(a_n)x_{n+1} + e_{n+1}^2\tfrac{1}{2}f''(a_n). \end{aligned}$$

Therefore we define $A_{n+1} := f_n(a_n)$, $B_{n+1} := e_{n+1}f_n'(a_n)$, $C_{n+1} := e_{n+1}^2\frac{1}{2}f_n''(a_n)$ and $f_{n+1}(z) := A_{n+1}z^2 + B_{n+1}z + C_{n+1}$. It is clear that A_{n+1}, B_{n+1}, $C_{n+1} \in \mathbb{Z}[i]$. By the above equation we immediately see that $f_{n+1}(x_{n+1}) = 0$. It holds that

$$B_{n+1}^2 - 4A_{n+1}C_{n+1} = e_{n+1}^2(2A_na_n + B_n)^2 - 4(A_na_n^2 + B_na_n + C_n)A_ne_{n+1}^2 = e_{n+1}^2(B_n^2 - 4A_nC_n),$$

so $|B_{n+1}^2 - 4A_{n+1}C_{n+1}| = |B_n^2 - 4A_nC_n| = |B_0^2 - 4A_0C_0|$. We have by definition: $A_0 \neq 0$. Suppose $A_n \neq 0$. As x_n is irrational, and $f_n(x_n) = 0$ it follows that also the other root of f_n is irrational. As a_n is rational, we have $f_n(a_n) \neq 0$ and therefore $A_{n+1} \neq 0$. This proves the claim. ■

Define the discriminant $D_n := B_n^2 - 4A_nC_n$ for every $n \in \mathbb{N}$. We see that $|D_n| = |D_0| = |D|$ for every $n \in \mathbb{N}$. Let $y \in \mathbb{C}$ be the other root of $Az^2 + Bz + C$. Define $y_0 := y$ and for every $n \in \mathbb{N}$: $y_{n+1} := \frac{e_{n+1}}{y_n - a_n}$.

Claim 2: The following hold for every $n \in \mathbb{N}$:

i.

$$f_n(y_n) = 0,$$

ii.

$$x_n \neq y_n,$$

iii.

$$y_0 = \frac{y_{n+1}p_n + e_{n+1}p_{n-1}}{y_{n+1}q_n + e_{n+1}q_{n-1}},$$

iv.

$$y_{n+1} = -e_{n+1}\frac{y_0 q_{n-1} - p_{n-1}}{y_0 q_n - p_n}.$$

Proof of claim 2: By definition we have: $f_0(y_0) = 0$. Suppose that $f_n(y_n) = 0$, then: $0 = f_n(y_n)y_{n+1}^2 = A_{n+1}y_{n+1}^2 + B_{n+1}y_{n+1} + C_{n+1} = f_{n+1}(y_{n+1})$. This is obtained in a similar way as in Claim 1. Further, we have $x_0 \neq y_0$. Suppose $x_n \neq y_n$, then $x_{n+1} = \frac{e_{n+1}}{x_n - a_n} \neq \frac{e_{n+1}}{y_n - a_n} = y_{n+1}$. The proofs of the last two assertions are very similar to the proofs of Lemma 3.31 and Lemma 3.32 and are therefore left to the reader. ■

Now we will calculate $x_{n+1} - y_{n+1}$ and $\frac{e_{n+1}}{x_{n+1}} - \frac{e_{n+1}}{y_{n+1}}$, for $n \geq 1$:

$$\begin{aligned}
x_{n+1} - y_{n+1} &= e_{n+1}\frac{y_0 q_{n-1} - p_{n-1}}{y_0 q_n - p_n} - e_{n+1}\frac{x_0 q_{n-1} - p_{n-1}}{x_0 q_n - p_n} \\
&= e_{n+1}\frac{(y_0 q_{n-1} - p_{n-1})(x_0 q_n - p_n) - (x_0 q_{n-1} - p_{n-1})(y_0 q_n - p_n)}{(x_0 q_n - p_n)(y_0 q_n - p_n)} \\
&= e_{n+1}\frac{(x_0 - y_0)(p_n q_{n-1} - p_{n-1}q_n)}{(x_0 q_n - p_n)q_n(y_0 - \frac{p_n}{q_n})} \\
&= \frac{(x_0 - y_0)(-1)^{n-1} \cdot \prod_{k=1}^{n+1} e_k}{\theta_n(y_0 - \frac{p_n}{q_n})},
\end{aligned}$$

and:

$$\frac{e_{n+1}}{x_{n+1}} - \frac{e_{n+1}}{y_{n+1}} = (x_n - a_n) - (y_n - a_n) = x_n - y_n = \frac{(x_0 - y_0)(-1)^n \cdot \prod_{k=1}^{n} e_k}{\theta_{n-1}(y_0 - \frac{p_{n-1}}{q_{n-1}})}.$$

As $\lim_{n\to\infty}(x_0 - \frac{p_n}{q_n}) = 0$ it follows that there exists $\epsilon > 0$ and $N \in \mathbb{N}$ such that $|y_0 - \frac{p_n}{q_n}| < \epsilon$ for every $n \geq N$. Now define $\rho := \frac{|x_0 - y_0|}{\delta\epsilon}$. Note that by definition: $\rho > 0$. As $|\theta_n| < \delta$ and $|\theta_{n-1}| < \delta$ for infinitely many n, we obtain:

$$\begin{aligned}
|x_{n+1} - y_{n+1}| &= \frac{|x_0 - y_0|}{|\theta_n||y_0 - \frac{p_n}{q_n}|} > \frac{|x_0 - y_0|}{\delta\epsilon} = \rho, \\
\Big|\frac{e_{n+1}}{x_{n+1}} - \frac{e_{n+1}}{y_{n+1}}\Big| &= \frac{|x_0 - y_0|}{|\theta_{n-1}||y_0 - \frac{p_{n-1}}{q_{n-1}}|} > \frac{|x_0 - y_0|}{\delta\epsilon} = \rho.
\end{aligned}$$

for infinitely many n. By the quadratic formula we obtain:

$$|x_{n+1} - y_{n+1}| = \Big|\frac{-B_{n+1} + \sqrt{D_{n+1}}}{2A_{n+1}} - \frac{-B_{n+1} - \sqrt{D_{n+1}}}{2A_{n+1}}\Big| = \Big|\frac{\sqrt{D_{n+1}}}{A_{n+1}}\Big|,$$

and

$$\Big|\frac{e_{n+1}}{x_{n+1}} - \frac{e_{n+1}}{y_{n+1}}\Big| = \Big|\frac{2A_{n+1}}{-B_{n+1}+\sqrt{D_{n+1}}} - \frac{2A_{n+1}}{-B_{n+1}-\sqrt{D_{n+1}}}\Big| = \Big|\frac{\sqrt{D_{n+1}}}{C_{n+1}}\Big|.$$

It follows that for infinitely many indices n: $\big|\frac{\sqrt{D_{n+1}}}{A_{n+1}}\big| > \rho$ and $\big|\frac{\sqrt{D_{n+1}}}{C_{n+1}}\big| > \rho$. Note that $|\sqrt{D_{n+1}}| = |\sqrt{D}|$, as $|D_{n+1}| = |D|$. Consequently: $|A_{n+1}| < \frac{|\sqrt{D}|}{\rho}$ and $|C_{n+1}| < \frac{|\sqrt{D}|}{\rho}$ for infinitely many indices n. Also $|B_{n+1}| \leq |B_{n+1}^2| \leq |D| + |4A_{n+1}C_{n+1}| < |D| + \frac{4|D|}{\rho^2}$. So for an infinite number of indices there are only a finite number of possibilities for A_{n+1}, B_{n+1} and C_{n+1}. Therefore there exist k, l, $m \in \mathbb{N}$, $k < l < m$ such that $f_k = f_l = f_m$. Consequently there exist r, $s \in \{k, l, m\}$, $r < s$ such that $x_r = x_s$ and by Lemma 3.49 we conclude: $\mathsf{CF}(x)$ is periodic. □

The next theorem gives us a criterion for a continued fraction of a quadratic irrational number to be periodic.

Theorem 3.53. *Let $x \in \mathbb{C}$ be quadratic irrational and let $\big(\frac{p_n}{q_n}\big)_{n\in\mathbb{N}}$ be the convergents and $(x_n)_{n\in\mathbb{N}}$ be the complete quotients of x under CF. Let $\varepsilon > 0$ and $\eta \geq 1$. If there are infinitely many indices n such that both $\big|x_{n+1} + e_{n+1}\frac{q_{n-1}}{q_n}\big| > \varepsilon$ and $\big|\frac{q_{n-1}}{q_n}\big| < \eta$, then $\mathsf{CF}(x)$ is periodic.*

Proof. According to Theorem 3.52 all we have to show is there exists $\delta > 0$ such that $|\theta_{n-1}| < \delta$ and $|\theta_n| < \delta$ for infinitely many indices n. Let M be an infinite subset of $\mathbb{N}$ such that both $\big|x_{m+1} + e_{m+1}\frac{q_{m-1}}{q_m}\big| > \varepsilon$ and $\big|\frac{q_{m-1}}{q_m}\big| < \eta$ for every $m \in M$. By applying Lemma 3.42 we obtain:

$$\frac{1}{|\theta_m|} = \big|x_{m+1} + e_{m+1}\frac{q_{m-1}}{q_m}\big| > \varepsilon$$

for every $m \in M$. Now define $\delta := \frac{1}{\varepsilon}\eta^2 + \eta$, then for every $m \in M$ we have that $|\theta_m| < \frac{1}{\varepsilon} < \delta$.

<u>Claim:</u> For every $m \in M$: $|\theta_{m-1}| < \delta$.

<u>Proof of claim:</u> Let $m \in M$ and consider the following equations: $x_0 - \frac{p_{m-1}}{q_{m-1}} = \frac{\theta_{m-1}}{q_{m-1}^2}$ and $x_0 - \frac{p_m}{q_m} = \frac{\theta_m}{q_m^2}$. Combining these gives:

$$\frac{\theta_{m-1}}{q_{m-1}^2} = \frac{\theta_m}{q_m^2} + \frac{p_m}{q_m} - \frac{p_{m-1}}{q_{m-1}} = \frac{\theta_m}{q_m^2} + \frac{(-1)^{m-1}\cdot\prod_{j=1}^m e_j}{q_{m-1}q_m}.$$

Therefore:

$$\theta_{m-1} = \theta_m\Big(\frac{q_{m-1}}{q_m}\Big)^2 + \frac{q_{m-1}}{q_m}(-1)^{m-1}\cdot\prod_{j=1}^m e_j.$$

Consequently:

$$|\theta_{m-1}| \leq \Big|\theta_m\Big(\frac{q_{m-1}}{q_m}\Big)^2\Big| + \Big|\frac{q_{m-1}}{q_m}\Big|.$$

As $|\frac{q_{m-1}}{q_m}| < \eta$ we have:

$$|\theta_{m-1}| < |\theta_m|\eta^2 + \eta < \frac{1}{\varepsilon}\eta^2 + \eta = \delta.$$

■

So for every $m \in M$ we have that both $|\theta_{m-1}| < \delta$ and $|\theta_m| < \delta$. From Theorem 3.52 now immediately follows that $\mathsf{CF}(x)$ is periodic. □

This last theorem has three corollaries, which are more convenient criteria for a continued fraction of a quadratic irrational number to be periodic.

Corollary 3.54. *Let $x \in \mathbb{C}$ be quadratic irrational and let $\left(\frac{p_n}{q_n}\right)_{n\in\mathbb{N}}$ be the convergents of x under CF. If $\left|\frac{q_{n-1}}{q_n}\right| < 1$ for every $n \in \mathbb{N}$, then $\mathsf{CF}(x)$ is periodic.*

Proof. According to Proposition 3.29: not $\lim_{n\to\infty} |x_{n+1}| = 1$. As $|x_{n+1}| > 1$ for every $n \in \mathbb{N}$, it follows that

$$\neg\forall\varepsilon > 0\ \exists N \in \mathbb{N}\ \forall n \geq N\ \big[|x_{n+1}| < 1 + \varepsilon\big],$$

or, equivalently:

$$\exists\varepsilon > 0\ \forall N \in \mathbb{N}\ \exists n \geq N\ \big[|x_{n+1}| \geq 1 + \varepsilon\big].$$

Therefore there exist infinitely many indices n such that $|x_{n+1}| \geq 1+\varepsilon_0$, for some $\varepsilon_0 > 0$. As $\left|\frac{q_{n-1}}{q_n}\right| < 1$ for every $n \in \mathbb{N}$, there exist infinitely many indices n such that both $\left|\frac{q_{n-1}}{q_n}\right| < 1$ and $|x_{n+1}| - \left|\frac{q_{n-1}}{q_n}\right| > \varepsilon_0$. From Theorem 3.53 now immediately follows that $\mathsf{CF}(x)$ is periodic. □

Corollary 3.55. *Let $x \in \mathbb{C}$ be quadratic irrational and let $(x_n)_{n\in\mathbb{N}}$ be the complete quotients of x under CF. If there exists $\rho > 1$ such that $|x_{n+1}| \geq \rho$ for every $n \in \mathbb{N}$, then $\mathsf{CF}(x)$ is periodic.*

Proof. As $\lim_{n\to\infty} q_n = \infty$, we have that for an infinite number of indices n that $\left|\frac{q_{n-1}}{q_n}\right| < 1$. Therefore there exist infinitely many indices n such that both $|x_{n+1}| - \left|\frac{q_{n-1}}{q_n}\right| > \rho - 1$ and $\left|\frac{q_{n-1}}{q_n}\right| < 1$. From Theorem 3.53 now immediately follows that $\mathsf{CF}(x)$ is periodic. □

Corollary 3.56. *Let $x \in \mathbb{C}$ be quadratic irrational and let $\left(\frac{p_n}{q_n}\right)_{n\in\mathbb{N}}$ be the convergents of x under CF. Let $0 < \rho < 1$. If there are infinitely many indices n such that $\left|\frac{q_{n-1}}{q_n}\right| \leq \rho$, then $\mathsf{CF}(x)$ is periodic.*

Proof. As $|x_{n+1}| > 1$ for every $n \in \mathbb{N}$, we have that there exist infinitely many indices n such that both $|x_{n+1}| - \left|\frac{q_{n-1}}{q_n}\right| > 1 - \rho$ and $\left|\frac{q_{n-1}}{q_n}\right| < 1$. From Theorem 3.53 now immediately follows that $\mathsf{CF}(x)$ is periodic. □

We end this section with an interesting result about the roots that occur in the proof of Theorem 3.52.

Proposition 3.57. *Let $x \in \mathbb{C}$ be quadratic irrational and let A, B, $C \in \mathbb{Z}[i]$ such that $A \neq 0$ and $Ax^2 + Bx + C = 0$. Let $y \in \mathbb{C}$, $y \neq x$ such that $Ay^2 + By + C = 0$. Let $\mathsf{CF}(x) = [a_0, e_1/a_1, e_2/a_2, \ldots]$, let $y_0 := y$ and let $y_{n+1} := \frac{e_{n+1}}{y_n - a_n}$ for every $n \in \mathbb{N}$ (just as in Claim 2 of Theorem 3.52). Then:*

$$\lim_{n\to\infty} \left(y_{n+1} + e_{n+1}\frac{q_{n-1}}{q_n}\right) = 0.$$

Proof. By Proposition 2.10 we have:

$$\frac{p_n}{q_n} - \frac{p_{n-1}}{q_{n-1}} = \frac{p_n q_{n-1} - p_{n-1} q_n}{q_{n-1} q_n} = \frac{(-1)^{n-1} \cdot \prod_{k=1}^{n} e_k}{q_{n-1} q_n}.$$

By Claim 2 of Theorem 3.52 we obtain:

$$y_{n+1} = -e_{n+1} \cdot \frac{q_{n-1}}{q_n} \cdot \frac{y_0 - \frac{p_{n-1}}{q_{n-1}}}{y_0 - \frac{p_n}{q_n}}.$$

Combining this leads to:

$$y_{n+1} = -e_{n+1} \cdot \frac{q_{n-1}}{q_n} \cdot \frac{y_0 - \frac{p_n}{q_n} + \frac{(-1)^{n-1} \cdot \prod_{k=1}^{n} e_k}{q_{n-1} q_n}}{y_0 - \frac{p_n}{q_n}} = -e_{n+1} \cdot \frac{q_{n-1}}{q_n} \Big(1 + \frac{(-1)^{n-1} \cdot \prod_{k=1}^{n} e_k}{(y_0 - \frac{p_n}{q_n}) q_{n-1} q_n} \Big).$$

A little more mathematical gymnastics gives us:

$$y_{n+1} + e_{n+1} \cdot \frac{q_{n-1}}{q_n} = -e_{n+1} \cdot \frac{q_{n-1}}{q_n} \Big(\frac{(-1)^{n-1} \cdot \prod_{k=1}^{n} e_k}{(y_0 - \frac{p_n}{q_n}) q_{n-1} q_n} \Big) = \frac{(-1)^{n} \cdot \prod_{k=1}^{n+1} e_k}{(y_0 - \frac{p_n}{q_n}) q_n^2}.$$

We have $|(-1)^n \cdot \prod_{k=1}^{n+1} e_k| = 1$. By Proposition 3.36 it follows that $\lim_{n\to\infty} q_n^2 = \infty$. Moreover, as $\lim_{n\to\infty} \frac{p_n}{q_n} = x_0$ and $y_0 \neq x_0$, there exist $\varepsilon > 0$, $N \in \mathbb{N}$ such that $|y_0 - \frac{p_n}{q_n}| > \varepsilon$ for every $n \geq N$. Therefore:

$$\lim_{n\to\infty} \big(y_{n+1} + e_{n+1} \frac{q_{n-1}}{q_n} \big) = \lim_{n\to\infty} \frac{(-1)^n \cdot \prod_{k=1}^{n+1} e_k}{(y_0 - \frac{p_n}{q_n}) q_n^2} = 0$$

and this ends the proof. □

4 A complex continued fraction algorithm by A. Hurwitz

In this chapter we will study one particular complex continued fraction algorithm, which was found by A. Hurwitz in 1887 [3]. We will define the algorithm and we will look at some special properties of this algorithm. First we define the function fl_{AH}.

Definition 4.1. Define $fl_{\mathsf{AH}} : \mathbb{C} \to \mathbb{Z}[i]$ as follows: $fl_{\mathsf{AH}}(z) := \lfloor \mathrm{Re}(z) + \frac{1}{2} \rfloor + \lfloor \mathrm{Im}(z) + \frac{1}{2} \rfloor i$ for every $z \in \mathbb{C}$.

The following two propositions show that fl_{AH} has some specific properties.

Proposition 4.2. *For every $z \in \mathbb{C}$: $|fl_{\mathsf{AH}}(z) - z| \leq \frac{1}{\sqrt{2}}$.*

Proof. Let $z \in \mathbb{C}$. Let $x := \mathrm{Re}(z)$, $y := \mathrm{Im}(z)$, $a := \lfloor x + \frac{1}{2} \rfloor$ and $b := \lfloor y + \frac{1}{2} \rfloor$. Then: $fl_{\mathsf{AH}}(z) = a + bi$, with $a, b \in \mathbb{Z}$. Furthermore: $|x - a| \leq \frac{1}{2}$ and $|y - b| \leq \frac{1}{2}$. Therefore:

$$\begin{aligned}
|z - fl_{\mathsf{AH}}(z)| &= |(x + yi) - (a + bi)| \\
&= |(x - a) + (y - b)i| \\
&= \sqrt{(x-a)^2 + (y-b)^2} \\
&\leq \sqrt{(\tfrac{1}{2})^2 + (\tfrac{1}{2})^2} \\
&= \sqrt{\tfrac{1}{2}}
\end{aligned}$$

and we conclude: $|fl_{\mathsf{AH}}(z) - z| \leq \frac{1}{\sqrt{2}}$. □

Proposition 4.3. *The function fl_{AH} is a shift floor function.*

Proof. From Proposition 4.2 follows: $|fl_{\mathsf{AH}}(z) - z| \leq \frac{1}{\sqrt{2}} < 1$ for every $z \in \mathbb{C}$, so fl_{AH} is a floor function. Now, let $z \in \mathbb{C}$ and $\alpha \in \mathbb{Z}[i]$. Let $x := \mathrm{Re}(z)$, $y := \mathrm{Im}(z)$, $a := \mathrm{Re}(\alpha)$ and $b := \mathrm{Im}(\alpha)$. Then $a, b \in \mathbb{Z}$ and therefore:

$$\begin{aligned}
fl_{\mathsf{AH}}(z + \alpha) &= \lfloor \mathrm{Re}(z + \alpha) + \tfrac{1}{2} \rfloor + \lfloor \mathrm{Im}(z + \alpha) + \tfrac{1}{2} \rfloor i \\
&= \lfloor x + a + \tfrac{1}{2} \rfloor + \lfloor y + b + \tfrac{1}{2} \rfloor i \\
&= \lfloor x + \tfrac{1}{2} \rfloor + a + \lfloor y + \tfrac{1}{2} \rfloor i + bi \\
&= \lfloor \mathrm{Re}(z) + \tfrac{1}{2} \rfloor + \lfloor \mathrm{Im}(z) + \tfrac{1}{2} \rfloor i + a + bi \\
&= fl_{\mathsf{AH}}(z) + \alpha.
\end{aligned}$$

We conclude: fl_{AH} is a shift floor function. □

Definition 4.4. Define $sg_{\mathsf{AH}} : \mathbb{C} \to \{-1, 1, -i, i\}$ as follows: $sg_{\mathsf{AH}}(z) := 1$ for every $z \in \mathbb{C}$.

We have that sg_{AH} is a sign function. Let us investigate $\Delta_{fl_{\mathsf{AH}}}$ and $\Gamma_{fl_{\mathsf{AH}}, sg_{\mathsf{AH}}}$.

Proposition 4.5. $\Delta_{fl_{\mathsf{AH}}} = \{z \in \mathbb{C} \mid -\frac{1}{2} \leq \mathrm{Re}(z) < \frac{1}{2}\ ,\ -\frac{1}{2} \leq \mathrm{Im}(z) < \frac{1}{2}\}$.

Proof. As fl_{AH} is a shift floor function, we have by Proposition 3.6:

$$\begin{aligned}\Delta_{fl_{\mathsf{AH}}} &= \{z \in \mathbb{C} \mid fl_{\mathsf{AH}}(z) = 0\} \\ &= \{z \in \mathbb{C} \mid \lfloor \mathrm{Re}(z) + \tfrac{1}{2} \rfloor + \lfloor \mathrm{Im}(z) + \tfrac{1}{2} \rfloor i = 0\} \\ &= \{z \in \mathbb{C} \mid -\tfrac{1}{2} \leq \mathrm{Re}(z) < \tfrac{1}{2}\ ,\ -\tfrac{1}{2} \leq \mathrm{Im}(z) < \tfrac{1}{2}\}\end{aligned}$$

and this completes the proof. □

Proposition 4.6. $\Gamma_{fl_{\mathsf{AH}}, sg_{\mathsf{AH}}} = \{\frac{1}{z} \mid z \in \Delta_{fl_{\mathsf{AH}}}, z \neq 0\}$.

Proof. As $sg_{\mathsf{AH}}(z) = 1$ for every $z \in \mathbb{C}$, this follows directly from Proposition 3.9. □

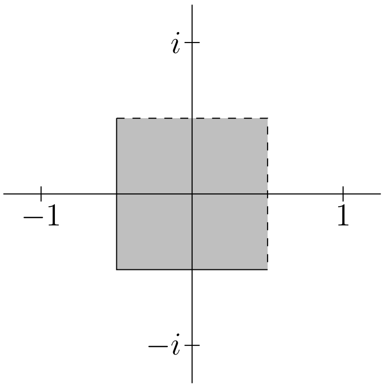

Figure 4: *Fundamental domain* $\Delta_{fl_{\mathsf{AH}}}$.

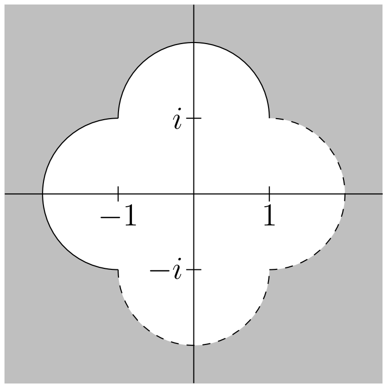

Figure 5: *Fundamental codomain* $\Gamma_{fl_{\mathsf{AH}}, sg_{\mathsf{AH}}}$.

We have defined a floor function and a sign function, therefore we can now define the algorithm.

Algorithm 4.7 (A. Hurwitz, 1887)**.**
input: $x \in \mathbb{C}$,
output: a finite or infinite complex continued fraction, which is generated by

$$\begin{cases} x_0 & := & x \\ a_n & := & fl_{\mathsf{AH}}(x_n) \\ e_{n+1} & := & sg_{\mathsf{AH}}(x_n) \\ x_{n+1} & := & \frac{e_{n+1}}{x_n - a_n} \end{cases}$$

We will refer to this algorithm as $\mathsf{CF}_{\mathsf{AH}}$, and denote the result of the algorithm on $x \in \mathbb{C}$ by $\mathsf{CF}_{\mathsf{AH}}(x)$.

Proposition 4.8. $\mathsf{CF}_{\mathsf{AH}}$ *is a complex continued fraction algorithm.*

Proof. By Proposition 4.3 we have that fl_{AH} is a floor function. We can easily verify that sg_{AH} is a sign function. From this it follows that $\mathsf{CF}_{\mathsf{AH}}$ is a complex continued fraction algorithm. □

According to Proposition 4.8 we have that $\mathsf{CF}_{\mathsf{AH}}$ is a complex continued fraction algorithm. Therefore all the results of Chapter 3 apply to $\mathsf{CF}_{\mathsf{AH}}$. In the remaining part of this chapter we will look at specific properties of $\mathsf{CF}_{\mathsf{AH}}$.

Proposition 4.9. *Let $x \in \mathbb{C}$. Let x_{n+1} be the $(n+1)$-th complete quotient of x under $\mathsf{CF}_{\mathsf{AH}}(x)$. Then: $|x_{n+1}| \geq \sqrt{2}$.*

Proof. According to Proposition 4.2 we have that $|z - \mathit{fl}_{\mathsf{AH}}(z)| \leq \frac{1}{\sqrt{2}}$ for every $z \in \mathbb{C}$. Therefore: $|x_{n+1}| = \left|\frac{1}{x_n - a_n}\right| = \left|\frac{1}{x_n - \mathit{fl}_{\mathsf{AH}}(x_n)}\right| \geq \sqrt{2}$. □

Proposition 4.10. *Let $x \in \mathbb{C}$ and let a_n be the n-th partial quotient of x under $\mathsf{CF}_{\mathsf{AH}}$. If $n \geq 1$ then $a_n \notin \{-1, 1, -i, i\}$.*

Proof. Let $n \geq 1$. Let $T_\alpha^{\mathit{fl}_{\mathsf{AH}}}$ be the tile of $\alpha \in \mathbb{Z}[i]$ under $\mathit{fl}_{\mathsf{AH}}$. Then we have that $T_\alpha^{\mathit{fl}_{\mathsf{AH}}} \cap \Gamma_{\mathit{fl}_{\mathsf{AH}}, sg_{\mathsf{AH}}} = \emptyset$ for $\alpha \in \{-1, 1, -i, i\}$. As $x_n \in \Gamma_{\mathit{fl}_{\mathsf{AH}}, sg_{\mathsf{AH}}}$ we conclude: $x_n \notin T_\alpha^{\mathit{fl}_{\mathsf{AH}}}$ for $\alpha \in \{-1, 1, -i, i\}$. Therefore $a_n \notin \{-1, 1, -i, i\}$. □

Theorem 4.11. *Let $x \in \mathbb{C}$, and let $\frac{p_k}{q_k}$ be the k-th convergent of x under $\mathsf{CF}_{\mathsf{AH}}$. Then for every $0 \leq n \leq |\mathsf{CF}_{\mathsf{AH}}(x)| - 1$:*

i. $|\frac{q_{n-1}}{q_n}| < 1$,

ii. if $n \geq 1$: *either* $\left|\frac{q_{n-2}}{q_{n-1}}\right| \leq \frac{\sqrt{5}-1}{2}$ *or* $\left|\frac{q_{n-1}}{q_n}\right| \leq \frac{\sqrt{5}-1}{2}$.

Proof. The first statement is proven by induction. We have that $|\frac{q_{-1}}{q_0}| < 1$. Now suppose $|\frac{q_{-1}}{q_0}| < 1, \dots, |\frac{q_{n-2}}{q_{n-1}}| < 1$. Suppose for the sake of contradiction that $|\frac{q_n}{q_{n-1}}| \leq 1$. As $\frac{q_n}{q_{n-1}} = a_n + \frac{q_{n-2}}{q_{n-1}}$, we have that $\frac{q_n}{q_{n-1}} \in B_1(a_n)$. As $|\frac{q_n}{q_{n-1}}| \leq 1$ and $a_n \notin \{0, -1, 1, -i, i\}$ it follows that $a_n \in \{1+i, 1-i, -1+i, -1-i\}$. Now one can show that this leads to a contradiction by examining possible orders of the partial quotients. For details of the proof, see [3, pp. 195–196].

For the proof of the second statement, see [2, p. 3566]. □

Now we prove an interesting result concerning the relative errors which occur under the complex continued fraction algorithm $\mathsf{CF}_{\mathsf{AH}}$.

Proposition 4.12. *Let $x \in \mathbb{C}$. Let θ_n be the n-th relative error of x under $\mathsf{CF}_{\mathsf{AH}}$. Then: $|\theta_n| < 1 + \sqrt{2}$.*

Proof. If $\mathsf{CF}_{\mathsf{AH}}(x)$ is finite and $n = |\mathsf{CF}_{\mathsf{AH}}(x)| - 1$, then $\theta_n = 0 < 1 + \sqrt{2}$. Now suppose $n < |\mathsf{CF}_{\mathsf{AH}}(x)| - 1$. Combining Lemma 3.42, Proposition 4.9 and Theorem 4.11 gives:

$$\frac{1}{|\theta_n|} = \left|x_{n+1} + e_{n+1}\frac{q_{n-1}}{q_n}\right| \geq |x_{n+1}| - \left|\frac{q_{n-1}}{q_n}\right| > \sqrt{2} - 1.$$

Therefore:

$$|\theta_n| < \frac{1}{\sqrt{2}-1} = 1 + \sqrt{2}.$$

□

Remark 4.13. Proposition 4.12 can be strengthened by replacing the constant $1 + \sqrt{2}$ with the constant 1. We do not prove this here, as for now, we are only interested in existence of an upper bound for $|\theta_n|$. For the proof, see [5].

Theorem 4.14. *Let $x \in \mathbb{C}$ be quadratic irrational. Then:* $\mathsf{CF}_{\mathsf{AH}}(x)$ *is periodic.*

Proof. By Proposition 4.12 we have that $|\theta_n| < 1 + \sqrt{2}$ for every $n \in \mathbb{N}$. By Theorem 3.52 we conclude: $\mathsf{CF}_{\mathsf{AH}}(x)$ is periodic. $\square$

Let $x \in \mathbb{C}$, let x_{n+1} be the $(n+1)$-th complete quotient and $\frac{p_n}{q_n}$ be the n-th convergent of x under $\mathsf{CF}_{\mathsf{AH}}$. Note that Theorem 4.14 is a consequence of Corollary 3.54, as we have that $|\frac{q_{n-1}}{q_n}| < 1$ for every $n \in \mathbb{N}$. It also follows from Corollary 3.55, as we have that $|x_{n+1}| \geq \sqrt{2}$ for every $n \in \mathbb{N}$. Finally, also Corollary 3.56 implies Theorem 4.14, as we have that $|\frac{q_{n-1}}{q_n}| \leq \frac{\sqrt{5}-1}{2}$ for infinitely indices n.

5 Two complex continued fraction algorithms by J. Hurwitz

In this chapter we will study two complex continued fraction algorithms, which were found by J. Hurwitz in 1902 [4]. We will define the algorithms and we will look at some special properties of the algorithms. We will also find a specific relation between the two algorithms.

5.1 Complex continued fraction algorithm of the first kind

Consider the complex plane, where we identify every complex number $z = x+yi$, where x, $y \in \mathbb{R}$, with the point (x, y). We define the lines $l_k := \{z \,|\, \mathrm{Re}(z) = \mathrm{Im}(z) + k\}$ and $m_k := \{z \,|\, \mathrm{Re}(z) = -\mathrm{Im}(z) + k\}$, for every $k \in \mathbb{Z}$. Define $L := \{l_{2k+1} \,|\, k \in \mathbb{Z}\} \cup \{m_{2k+1} \,|\, k \in \mathbb{Z}\}$. Now L divides the complex plane in an infinite number of squares, and the centre points of these squares are precisely the numbers in $(1+i)\mathbb{Z}[i]$. We define S_α to be the square with centre point α, for every $\alpha \in (1+i)\mathbb{Z}[i]$. That is: S_α contains all the points in the interior of the square, and we will later determine which points on the border of the square belong to S_α. Now we will give each square a type, except for S_0. Consider S_α, with $\alpha \in (1+i)\mathbb{Z}[i]$. Then:

$$
\begin{array}{lll}
\text{if } \mathrm{Re}(\alpha) = \mathrm{Im}(\alpha) > 0 & \text{then the type of } S_\alpha \text{ is} & 1+i, \\
\text{if } \mathrm{Re}(\alpha) = -\mathrm{Im}(\alpha) > 0 & \text{then the type of } S_\alpha \text{ is} & 1-i, \\
\text{if } \mathrm{Re}(\alpha) = -\mathrm{Im}(\alpha) < 0 & \text{then the type of } S_\alpha \text{ is} & -1+i, \\
\text{if } \mathrm{Re}(\alpha) = \mathrm{Im}(\alpha) < 0 & \text{then the type of } S_\alpha \text{ is} & -1-i, \\
\text{if } \mathrm{Re}(\alpha) > \mathrm{Im}(\alpha) \text{ and } -\mathrm{Re}(\alpha) < \mathrm{Im}(\alpha) & \text{then the type of } S_\alpha \text{ is} & 2, \\
\text{if } \mathrm{Re}(\alpha) < \mathrm{Im}(\alpha) \text{ and } -\mathrm{Re}(\alpha) < \mathrm{Im}(\alpha) & \text{then the type of } S_\alpha \text{ is} & 2i, \\
\text{if } \mathrm{Re}(\alpha) < \mathrm{Im}(\alpha) \text{ and } -\mathrm{Re}(\alpha) > \mathrm{Im}(\alpha) & \text{then the type of } S_\alpha \text{ is} & -2, \\
\text{if } \mathrm{Re}(\alpha) > \mathrm{Im}(\alpha) \text{ and } -\mathrm{Re}(\alpha) > \mathrm{Im}(\alpha) & \text{then the type of } S_\alpha \text{ is} & -2i.
\end{array}
$$

Now every square has a type, except for S_0. We did not yet decide which points on the border of the square belong to S_α. For S_0: every edge belongs to S_0. To the squares with type $1+i$, $1-i$, $-1+i$ and $-1-i$ belong three edges, and the one edge that does not belong to S_α is the edge which is enclosed by the two vertices of S_α which have the least distance to 0. To the squares with type 2, -2, $2i$ and $-2i$ belong two edges, those who intersect in the vertex that has the greatest distance to 0. A vertex belongs to S_α if the two intersecting edges of that vertex belong to S_α. We will regard S_α as a subset of $\mathbb{C}$, for every $\alpha \in (1+i)\mathbb{Z}[i]$. Then we have: for every $z \in \mathbb{C}$ there exists precisely one $\alpha \in (1+i)\mathbb{Z}[i]$ such that $z \in S_\alpha$.

Definition 5.1. Define $\mathit{fl}_{\mathsf{JH1}} : \mathbb{C} \to \mathbb{Z}[i]$ as follows:

$$\mathit{fl}_{\mathsf{JH1}}(z) := \begin{cases} z & \text{if } z \in \mathbb{Z}[i], \\ \alpha & \text{otherwise, where } \alpha \in (1+i)\mathbb{Z}[i] \text{ such that } z \in S_\alpha. \end{cases}$$

Proposition 5.2. *The function $\mathit{fl}_{\mathsf{JH1}}$ is a floor function.*

Proof. Let $z \in \mathbb{C}$. First suppose $z \in \mathbb{Z}[i]$, then $\mathit{fl}_{\mathsf{JH1}}(z) = z$, so $|\mathit{fl}_{\mathsf{JH1}}(z) - z| < 1$. Now suppose $z \notin \mathbb{Z}[i]$ and let α such that $\mathit{fl}_{\mathsf{JH1}}(z) = \alpha$. Then $z \in S_\alpha$, and as $S_\alpha \subseteq \{z \,|\, |z-\alpha| \leq 1\}$ and $S_\alpha \cap \{z \,|\, |z-\alpha| = 1\} \subseteq \mathbb{Z}[i]$ we conclude: $|\mathit{fl}_{\mathsf{JH1}}(z) - z| < 1$. So in either case $|\mathit{fl}_{\mathsf{JH1}}(z) - z| < 1$, therefore: $\mathit{fl}_{\mathsf{JH1}}$ is a floor function. □

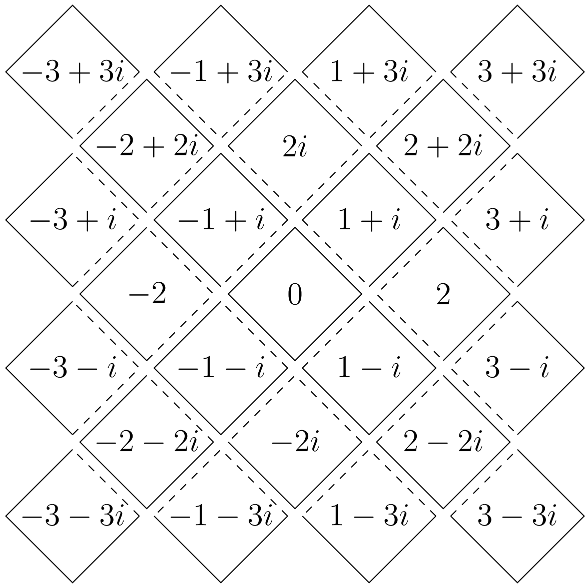

Figure 6: *The squares S_α in $\mathbb{C}$, for every edge is indicated to which square S_α it belongs.*

Note that $\mathit{fl}_{\mathsf{JH1}}$ is not a shift floor function, as $\mathit{fl}_{\mathsf{JH1}}(\frac{1}{2}+1) = 2 \neq 1 = \mathit{fl}_{\mathsf{JH1}}(\frac{1}{2}) + 1$. It is even not the case that $\mathit{fl}_{\mathsf{JH1}}(z+\alpha) = \mathit{fl}_{\mathsf{JH1}}(z) + \alpha$, for every $z \in \mathbb{C}$, $\alpha \in (1+i)\mathbb{Z}[i]$. This is illustrated by $\mathit{fl}_{\mathsf{JH1}}(\frac{1+3i}{2}+2) = 2+2i \neq 3+i = \mathit{fl}_{\mathsf{JH1}}(\frac{1+3i}{2}) + 2$.

Definition 5.3. Define $\mathit{sg}_{\mathsf{JH1}} : \mathbb{C} \to \{-1, 1, -i, i\}$ as follows: $\mathit{sg}_{\mathsf{JH1}}(z) := -1$ for every $z \in \mathbb{C}$.

Proposition 5.4. $\Delta_{\mathit{fl}_{\mathsf{JH1}}} = \{z \in \mathbb{C} \mid \mathit{fl}_{\mathsf{JH1}}(z) = 0\}$.

Proof. Let $T_\alpha^{\mathit{fl}_{\mathsf{JH1}}}$ the tile and $U_\alpha^{\mathit{fl}_{\mathsf{JH1}}}$ the corresponding set of $\alpha \in \mathbb{Z}[i]$ under $\mathit{fl}_{\mathsf{JH1}}$. Now let $\alpha \in \mathbb{Z}[i]$. If $\alpha \notin (1+i)\mathbb{Z}[i]$ then $T_\alpha^{\mathit{fl}_{\mathsf{JH1}}} = \{\alpha\}$, so $U_\alpha^{\mathit{fl}_{\mathsf{JH1}}} = \{\ 0\}$. If $\alpha \in (1+i)\mathbb{Z}[i]$ then we have that $U_\alpha^{\mathit{fl}_{\mathsf{JH1}}} \subseteq U_0^{\mathit{fl}_{\mathsf{JH1}}}$, as $\{z - \alpha \mid z \in S_\alpha\} \subseteq S_0$. Therefore

$$\Delta_{\mathit{fl}_{\mathsf{JH1}}} = \bigcup_{\alpha \in (1+i)\mathbb{Z}[i]} U_\alpha^{\mathit{fl}_{\mathsf{JH1}}} \cup \bigcup_{\alpha \in 1+(1+i)\mathbb{Z}[i]} U_\alpha^{\mathit{fl}_{\mathsf{JH1}}} = U_0^{\mathit{fl}_{\mathsf{JH1}}} \cup \{0\} = T_0^{\mathit{fl}_{\mathsf{JH1}}}$$

and this completes the proof. □

Proposition 5.5. $\Gamma_{\mathit{fl}_{\mathsf{JH1}}, \mathit{sg}_{\mathsf{JH1}}} = \{\frac{-1}{z} \mid z \in \Delta_{\mathit{fl}_{\mathsf{JH1}}}, z \neq 0\}$.

Proof. As $\mathit{sg}_{\mathsf{JH1}}(z) = -1$ for every $z \in \mathbb{C}$, this follows directly from Proposition 3.9. □

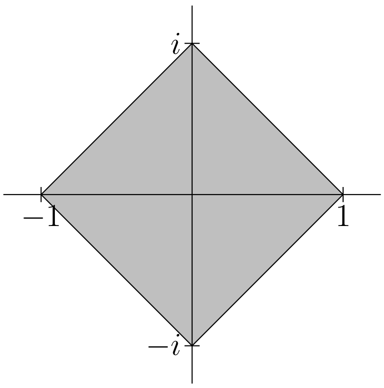

Figure 7: *Fundamental domain $\Delta_{\mathit{fl}_{\mathsf{JH1}}}$.*

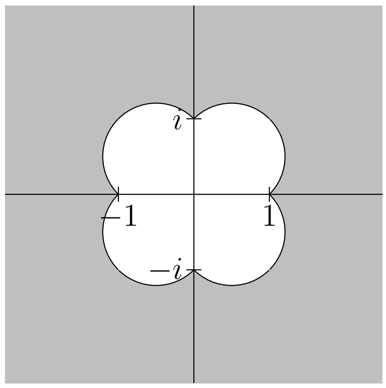

Figure 8: *Fundamental codomain $\Gamma_{\mathit{fl}_{\mathsf{JH1}}, \mathit{sg}_{\mathsf{JH1}}}$.*

As we have a floor function $\mathit{fl}_{\mathsf{JH1}}$ and a sign function sg_{JH1} we can define the algorithm.

Algorithm 5.6 (J. Hurwitz, 1902)**.**
input: $x \in \mathbb{C}$,
output: a finite or infinite complex continued fraction, which is generated by

$$\begin{cases} x_0 & := & x \\ a_n & := & \mathit{fl}_{\mathsf{JH1}}(x_n) \\ e_{n+1} & := & sg_{\mathsf{JH1}}(x_n) \\ x_{n+1} & := & \frac{e_{n+1}}{x_n - a_n} \end{cases}$$

We will refer to this algorithm as $\mathsf{CF}_{\mathsf{JH1}}$, and denote the result of the algorithm on $x \in \mathbb{C}$ by $\mathsf{CF}_{\mathsf{JH1}}(x)$.

Proposition 5.7. $\mathsf{CF}_{\mathsf{JH1}}$ *is a complex continued fraction algorithm.*

Proof. By Proposition 5.2 we have that $\mathit{fl}_{\mathsf{JH1}}$ is a floor function. We see that sg_{JH1} is a sign function. From this it follows that $\mathsf{CF}_{\mathsf{JH1}}$ is a complex continued fraction algorithm. □

As $\mathsf{CF}_{\mathsf{JH1}}$ is a complex continued fraction algorithm, all the results of Chapter 3 apply to this algorithm. In a continued fraction obtained by $\mathsf{CF}_{\mathsf{JH1}}$ we have that every partial numerator is equal to -1. In this section we will omit the partial numerators from $[a_0, e_1/a_1, \ldots, e_n/a_n]$ and $[a_0, e_1/a_1, e_2/a_2, \ldots]$ if this fraction is a result of $\mathsf{CF}_{\mathsf{JH1}}(x)$. We will write $[a_0, a_1, \ldots, a_n]$ and $[a_0, a_1, a_2, \ldots]$ respectively.

Proposition 5.8. *Let $z \in \mathbb{C}$ and let $\mathsf{CF}_{\mathsf{JH1}}(z) = [a_0, a_1, a_2, a_3, \ldots, a_n, \mathsf{CF}_{\mathsf{JH1}}(z_{n+1})]$. Then:*

$$\begin{aligned} \mathsf{CF}_{\mathsf{JH1}}(iz) &= [ia_0, -ia_1, ia_2, -ia_3, \ldots, (-1)^n ia_n, \mathsf{CF}_{\mathsf{JH1}}((-1)^{n+1} iz_{n+1})], \\ \mathsf{CF}_{\mathsf{JH1}}(-z) &= [-a_0, -a_1, -a_2, -a_3, \ldots, -a_n, \mathsf{CF}_{\mathsf{JH1}}(-z_{n+1})], \\ \mathsf{CF}_{\mathsf{JH1}}(-iz) &= [-ia_0, ia_1, -ia_2, ia_3, \ldots, (-1)^{n-1} ia_n, \mathsf{CF}_{\mathsf{JH1}}((-1)^n iz_{n+1})], \\ \mathsf{CF}_{\mathsf{JH1}}(\overline{z}) &= [\overline{a_0}, \overline{a_1}, \overline{a_2}, \overline{a_3}, \ldots, \overline{a_n}, \mathsf{CF}_{\mathsf{JH1}}(\overline{z_{n+1}})]. \end{aligned}$$

Proof. As $\mathsf{CF}_{\mathsf{JH1}}(z) = [a_0, a_1, a_2, a_3, \ldots, a_n, \mathsf{CF}_{\mathsf{JH1}}(z_{n+1})]$, we have for every $k \leq n$ that $z_k = a_k - \frac{1}{z_{k+1}}$. Therefore we have also the following equations: $iz_k = ia_k - \frac{1}{-iz_{k+1}}$, $-z_k = -a_k - \frac{1}{-z_{k+1}}$ and $-iz_k = -ia_k - \frac{1}{iz_{k+1}}$. This proves the validity of the first three statements. For the last statement, observe that if $\mathit{fl}_{\mathsf{JH1}}(z_k) = a_k$, then $\mathit{fl}_{\mathsf{JH1}}(\overline{z_k}) = \overline{a_k}$. We have $\frac{-1}{\overline{z_k} - \overline{a_k}} = \overline{(\frac{-1}{z_k - a_k})} = \overline{z_{k+1}}$, and the statement follows. □

The following are two results about the partial quotients under the algorithm $\mathsf{CF}_{\mathsf{JH1}}$.

Proposition 5.9. *Let $z \in \mathbb{C}$ be irrational, let $\mathsf{CF}_{\mathsf{JH1}}(z) = [a_0, a_1, a_2, \ldots]$. Then: $a_k \in (1+i)\mathbb{Z}[i]$ for every $k \in \mathbb{N}$.*

Proof. Let $k \in \mathbb{N}$. As z is irrational, we have that z_k is irrational. Accordingly: $z_k \notin \mathbb{Z}[i]$ and therefore $a_k = \mathit{fl}_{\mathsf{JH1}}(z_k) = \alpha$, where $\alpha \in (1+i)\mathbb{Z}[i]$ such that $z \in S_\alpha$. □

Proposition 5.10. *Let $z \in \mathbb{C}$ be rational, let $\mathsf{CF}_{\mathsf{JH1}}(z) = [a_0, a_1, \ldots, a_n]$. Let r, $s \in \mathbb{Z}[i]$ such that $z = \frac{r}{s}$ and* $\gcd(r, s) = 1$*. Then:*

i. $a_k \in (1+i)\mathbb{Z}[i]$ *for every* $k < n$,

ii. $a_n \notin (1+i)\mathbb{Z}[i]$ *iff* $1+i \nmid r$ *and* $1+i \nmid s$.

Proof. Let $k < n$. Regarding the algorithm, it is not the case that $x_k - \mathit{fl}_{\mathsf{JS}}(x_k) = 0$ (otherwise the algorithm would halt at this point). So: $x_k \notin \mathbb{Z}[i]$ and considering the floor function $\mathit{fl}_{\mathsf{JS}}$ we see that $a_k = \mathit{fl}_{\mathsf{JH1}}(z_k) \in (1+i)\mathbb{Z}[i]$.

For the second statement, define the following sequences.

$$\begin{aligned} r_0 &:= r, & r_{k+1} &:= s_k e_{k+1}, \\ s_0 &:= s, & s_{k+1} &:= r_k - s_k a_k. \end{aligned}$$

From the proof of Proposition 3.27 we know: $x_k = \frac{r_k}{s_k}$ and $\gcd(r_k, s_k) = \gcd(r, s) = 1$ for every $k \le n$. We also have $|s_n| = 1$, as $s_n = \gcd(r, s)$.
First suppose that $1+i \nmid r$ and $1+i \nmid s$. We will now prove: $1+i \nmid r_k$ and $1+i \nmid s_k$ for every $k \le n$. We have $1+i \nmid r_0$ and $1+i \nmid s_0$ by assumption. Now suppose $1+i \nmid r_k$ and $1+i \nmid s_k$ Then $r_{k+1} = s_k e_{k+1}$, so $1+i \nmid r_{k+1}$. Also: $s_{k+1} = r_k - s_k a_k$, where $a_k \in (1+i)\mathbb{Z}[i]$, so $1+i \nmid s_{k+1}$. As $a_n = x_n = \frac{r_n}{s_n}$, and $|s_n| = 1$, we conclude: $1+i \nmid a_n$. Therefore: $a_n \notin (1+i)\mathbb{Z}[i]$.
Now suppose $1+i \mid r$ or $1+i \mid s$. Note that not both $1+i \mid r$ and $1+i \mid s$, otherwise $1+i \mid \gcd(r,s) = 1$. We will prove: $1+i \nmid r_k + s_k$ for every $k \le n$. By assumption we have $1+i \nmid r_0 + s_0$. Now: $r_{k+1} + s_{k+1} = s_k e_{k+1} + r_k - s_k a_k$, where $a_k \in \mathbb{Z}[i]$, so $1+i \nmid r_{k+1} + s_{k+1}$. As $|s_n| = 1$, we have $1+i \mid r_n$ and therefore $1+i \mid a_n$. We conclude: $a_n \in (1+i)\mathbb{Z}[i]$.
By combining this we obtain: $a_n \notin (1+i)\mathbb{Z}[i]$ iff $1+i \nmid r$ and $1+i \nmid s$. □

The following result will be useful in proving periodicity for quadratic irrationals for $\mathsf{CF}_{\mathsf{JH1}}$.

Proposition 5.11. *Let* $x \in \mathbb{C}$ *be irrational, and let* $\frac{p_k}{q_k}$ *be the* k*-th convergent of* x *under* $\mathsf{CF}_{\mathsf{JH1}}$. *Then for every* $n \in \mathbb{N}$: $|\frac{q_{n-1}}{q_n}| < 1$.

Proof. This follows by proving that $\frac{q_n}{q_{n-1}}$ is in the interior of W_{a_n} for every $n \in \mathbb{N}$. (W_α is defined in Section 5.2). This is achieved by examining possible orders of the partial quotients. As $a_n \neq 0$ for every $n \in \mathbb{N}_{>0}$, we have that $\left|\frac{q_n}{q_{n-1}}\right| > 1$ for every $n \in \mathbb{N}_{>0}$. For details of the proof, see [4, pp. 246–248]. □

Theorem 5.12. *Let* $x \in \mathbb{C}$ *be quadratic irrational. Then:* $\mathsf{CF}_{\mathsf{JH1}}(x)$ *is periodic.*

Proof. By Proposition 5.11 we have that $|\frac{q_{n-1}}{q_n}| < 1$ for every $n \in \mathbb{N}$. By Corollary 3.54 we conclude: $\mathsf{CF}_{\mathsf{JH1}}(x)$ is periodic. □

5.2 Complex continued fraction algorithm of the second kind

Consider the set $(1+i)\mathbb{Z}[i]$, and let $B_\alpha := \{z \in \mathbb{C} \mid |z - \alpha| < 1\}$ for every $\alpha \in (1+i)\mathbb{Z}[i]$. We will use the same distribution of types for B_α as in Section 5.1. For example: B_{3+i} has type 2, B_{2-2i} has type $1-i$, and so on. Now we will define $W_\alpha \subseteq \mathbb{C}$ for every $\alpha \in (1+i)\mathbb{Z}[i]$:

$$
\begin{array}{lll}
W_0 := B_0 & & \\
W_\alpha := B_\alpha \setminus B_{\alpha-1-i} & \text{for every } \alpha \text{ of type} & 1+i, \\
W_\alpha := B_\alpha \setminus B_{\alpha-1+i} & \text{for every } \alpha \text{ of type} & 1-i, \\
W_\alpha := B_\alpha \setminus B_{\alpha+1-i} & \text{for every } \alpha \text{ of type} & -1+i, \\
W_\alpha := B_\alpha \setminus B_{\alpha+1+i} & \text{for every } \alpha \text{ of type} & -1-i, \\
W_\alpha := (B_\alpha \cup \{\alpha-1\}) \setminus (B_{\alpha-1+i} \cup B_{\alpha-1-i}) & \text{for every } \alpha \text{ of type} & 2, \\
W_\alpha := (B_\alpha \cup \{\alpha-i\}) \setminus (B_{\alpha-1-i} \cup B_{\alpha+1-i}) & \text{for every } \alpha \text{ of type} & 2i, \\
W_\alpha := (B_\alpha \cup \{\alpha+1\}) \setminus (B_{\alpha+1+i} \cup B_{\alpha+1-i}) & \text{for every } \alpha \text{ of type} & -2, \\
W_\alpha := (B_\alpha \cup \{\alpha+i\}) \setminus (B_{\alpha-1+i} \cup B_{\alpha+1+i}) & \text{for every } \alpha \text{ of type} & -2i.
\end{array}
$$

Now we have that for every $z \in \mathbb{C}$ there exists precisely one $\alpha \in (1+i)\mathbb{Z}[i]$ such that $z \in W_\alpha$.

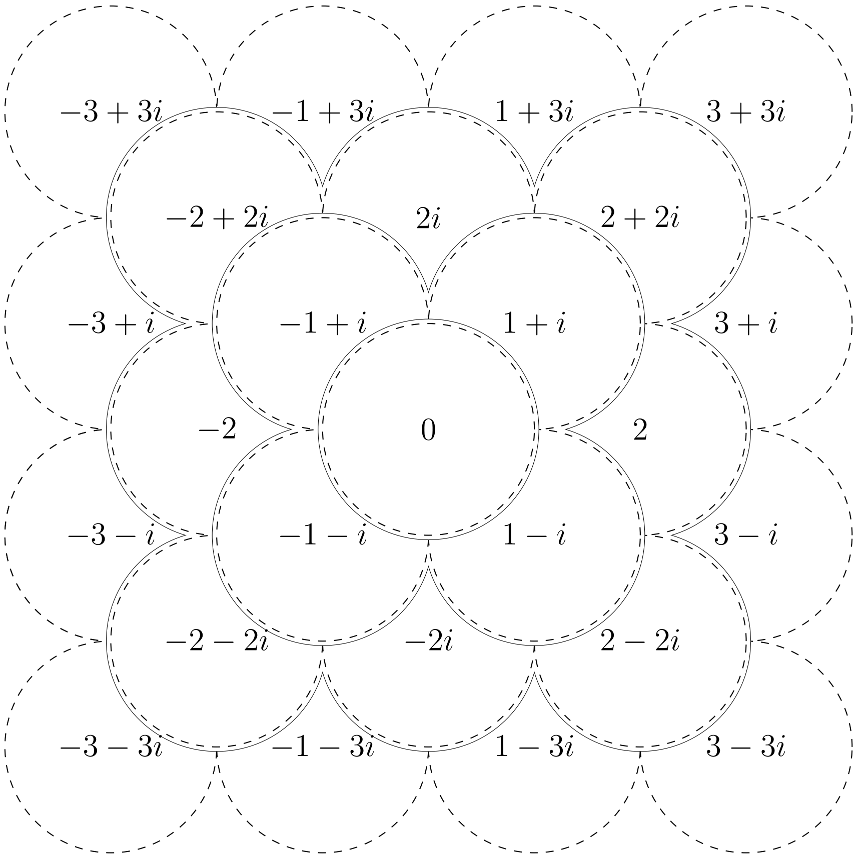

Figure 9: *The areas W_α in $\mathbb{C}$.*

We now define the following function.

Definition 5.13. Define $\mathit{fl}_{\mathsf{JH2}} : \mathbb{C} \to \mathbb{Z}[i]$ as follows:

$$
\mathit{fl}_{\mathsf{JH2}}(z) := \begin{cases} z & \text{if } z \in \mathbb{Z}[i], \\ \alpha & \text{otherwise, where } \alpha \in (1+i)\mathbb{Z}[i] \text{ such that } z \in W_\alpha. \end{cases}
$$

Proposition 5.14. *The function $\mathit{fl}_{\mathsf{JH2}}$ is a floor function.*

Proof. Let $z \in \mathbb{C}$. First suppose $z \in \mathbb{Z}[i]$, then $\mathit{fl}_{\mathsf{JH2}}(z) = z$, so $|\mathit{fl}_{\mathsf{JH2}}(z) - z| < 1$. Now suppose $z \notin \mathbb{Z}[i]$ and let α such that $\mathit{fl}_{\mathsf{JH2}}(z) = \alpha$. Then $z \in W_\alpha$, and as $W_\alpha \subseteq$

$\{z \mid |z-\alpha| \leq 1\}$ and $W_\alpha \cap \{z \mid |z-\alpha| = 1\} \subseteq \mathbb{Z}[i]$ we conclude: $|\mathit{fl}_{\mathsf{JH2}}(z) - z| < 1$. So in either case $|\mathit{fl}_{\mathsf{JH2}}(z) - z| < 1$, therefore: $\mathit{fl}_{\mathsf{JH2}}$ is a floor function. $\square$

Note that $\mathit{fl}_{\mathsf{JH2}}$ is not a shift floor function, as again, $\mathit{fl}_{\mathsf{JH2}}(\frac{1}{2}+1) = 2 \neq 1 = \mathit{fl}_{\mathsf{JH2}}(\frac{1}{2}) + 1$. Also, it should be clear that in general it is not the case that $\mathit{fl}_{\mathsf{JH2}}(z+\alpha) = \mathit{fl}_{\mathsf{JH2}}(z) + \alpha$ for $z \in \mathbb{C}$, $\alpha \in (1+i)\mathbb{Z}[i]$.

Definition 5.15. Define $sg_{\mathsf{JH2}} : \mathbb{C} \to \{-1, 1, -i, i\}$ as follows: $sg_{\mathsf{JH2}}(z) := -1$ for every $z \in \mathbb{C}$.

Let us investigate $\Delta_{\mathit{fl}_{\mathsf{JH2}}}$ and $\Gamma_{\mathit{fl}_{\mathsf{JH2}}, sg_{\mathsf{JH2}}}$.

Proposition 5.16. $\Delta_{\mathit{fl}_{\mathsf{JH2}}} = \{z \in \mathbb{C} \mid \mathit{fl}_{\mathsf{JH2}}(z) = 0\}$.

Proof. This proof is similar to the proof of Proposition 5.4 and is therefore left to the reader. $\square$

Proposition 5.17. $\Gamma_{\mathit{fl}_{\mathsf{JH2}}, sg_{\mathsf{JH2}}} = \{\frac{-1}{z} \mid z \in \Delta_{\mathit{fl}_{\mathsf{JH2}}}, z \neq 0\}$.

Proof. As $sg_{\mathsf{JH2}}(z) = -1$ for every $z \in \mathbb{C}$, this follows directly from Proposition 3.9. $\square$

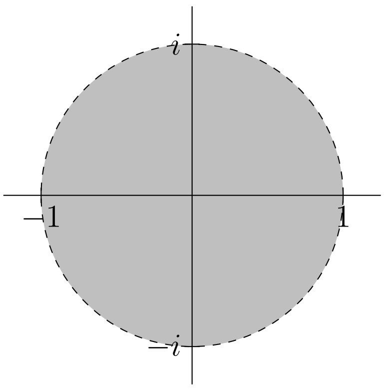

Figure 10: *Fundamental domain* $\Delta_{\mathit{fl}_{\mathsf{JH2}}}$.

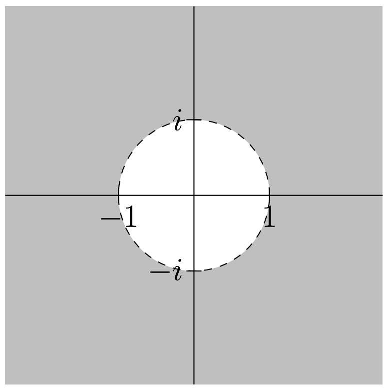

Figure 11: *Fundamental codomain* $\Gamma_{\mathit{fl}_{\mathsf{JH2}}, sg_{\mathsf{JH2}}}$.

We have defined a floor function and a sign function, therefore we can now define the algorithm.

Algorithm 5.18 (J. Hurwitz, 1902)**.**
input: $x \in \mathbb{C}$,
output: a finite or infinite complex continued fraction, which is generated by

$$\begin{cases} x_0 & := & x \\ a_n & := & \mathit{fl}_{\mathsf{JH2}}(x_n) \\ e_{n+1} & := & sg_{\mathsf{JH2}}(x_n) \\ x_{n+1} & := & \frac{e_{n+1}}{x_n - a_n} \end{cases}$$

We will refer to this algorithm as $\mathsf{CF}_{\mathsf{JH2}}$, and denote the result of the algorithm on $x \in \mathbb{C}$ by $\mathsf{CF}_{\mathsf{JH2}}(x)$.

Proposition 5.19. $\mathsf{CF}_{\mathsf{JH2}}$ *is a complex continued fraction algorithm.*

Proof. By Proposition 5.2 we have that fl_{JH2} is a floor function. Of course, sg_{JH2} is a sign function. From this it follows that $\mathsf{CF}_{\mathsf{JH2}}$ is a complex continued fraction algorithm. □

As $\mathsf{CF}_{\mathsf{JH2}}$ is a complex continued fraction algorithm, all the results of Chapter 3 apply to $\mathsf{CF}_{\mathsf{JH2}}$. Let $z \in \mathbb{C}$, we have that all the partial numerators of $\mathsf{CF}_{\mathsf{JH2}}(z)$ are equal to -1, and therefore we will omit these partial numerators from $\mathsf{CF}_{\mathsf{JH2}}(z)$. The following results show that $\mathsf{CF}_{\mathsf{JH2}}$ and $\mathsf{CF}_{\mathsf{JH1}}$ have some similar properties.

Proposition 5.20. *Let $z \in \mathbb{C}$ and let $\mathsf{CF}_{\mathsf{JH2}}(z) = [a_0, a_1, a_2, a_3, \ldots, a_n, \mathsf{CF}_{\mathsf{JH2}}(z_{n+1})]$. Then:*

$$\begin{aligned}
\mathsf{CF}_{\mathsf{JH2}}(iz) &= [ia_0, -ia_1, ia_2, -ia_3, \ldots, (-1)^n ia_n, \mathsf{CF}_{\mathsf{JH2}}((-1)^{n+1} iz_{n+1})], \\
\mathsf{CF}_{\mathsf{JH2}}(-z) &= [-a_0, -a_1, -a_2, -a_3, \ldots, -a_n, \mathsf{CF}_{\mathsf{JH2}}(-z_{n+1})], \\
\mathsf{CF}_{\mathsf{JH2}}(-iz) &= [-ia_0, ia_1, -ia_2, ia_3, \ldots, (-1)^{n-1} ia_n, \mathsf{CF}_{\mathsf{JH2}}((-1)^n iz_{n+1})], \\
\mathsf{CF}_{\mathsf{JH2}}(\overline{z}) &= [\overline{a_0}, \overline{a_1}, \overline{a_2}, \overline{a_3}, \ldots, \overline{a_n}, \mathsf{CF}_{\mathsf{JH2}}(\overline{z_{n+1}})].
\end{aligned}$$

Proof. This proof is similar to the proof of Proposition 5.8. □

Proposition 5.21. *Let $z \in \mathbb{C}$ be irrational, let $\mathsf{CF}_{\mathsf{JH2}}(z) = [a_0, a_1, a_2, \ldots]$. Then: $a_k \in (1+i)\mathbb{Z}[i]$ for every $k \in \mathbb{N}$.*

Proof. This proof is similar to the proof of Proposition 5.9. □

Proposition 5.22. *Let $z \in \mathbb{C}$ be rational, let $\mathsf{CF}_{\mathsf{JH2}}(z) = [a_0, a_1, \ldots, a_n]$. Let r, $s \in \mathbb{Z}[i]$ such that $z = \frac{r}{s}$ and $\gcd(r, s) = 1$. Then:*

i. $a_k \in (1+i)\mathbb{Z}[i]$ *for every* $k < n$,

ii. $a_n \notin (1+i)\mathbb{Z}[i]$ *iff* $1+i \nmid r$ *and* $1+i \nmid s$.

Proof. This proof is similar to the proof of Proposition 5.10. □

Proposition 5.23. *Let $x \in \mathbb{C}$ be irrational, and let $\frac{p_k}{q_k}$ be the k-th convergent of x under $\mathsf{CF}_{\mathsf{JH2}}$. Then for every $n \in \mathbb{N}$: $|\frac{q_{n-1}}{q_n}| < 1$.*

Proof. This follows by proving that $\frac{q_n}{q_{n-1}} \in \Gamma_{fl_{\mathsf{JH2}}, sg_{\mathsf{JH2}}}$ and $\frac{q_n}{q_{n-1}} \in S_{a_n}$ for every $n \in \mathbb{N}$, where S_{a_n} as defined in Section 5.1. This is again achieved by examining possible orders of the partial quotients. If $\frac{q_n}{q_{n-1}} \in \Gamma_{fl_{\mathsf{JH2}}, sg_{\mathsf{JH2}}}$, then $\left|\frac{q_{n-1}}{q_n}\right| < 1$. For details of the proof, see [4, pp. 260–261]. □

Theorem 5.24. *Let $x \in \mathbb{C}$ be quadratic irrational. Then: $\mathsf{CF}_{\mathsf{JH2}}(x)$ is periodic.*

Proof. By Proposition 5.23 we have that $|\frac{q_{n-1}}{q_n}| < 1$ for every $n \in \mathbb{N}$. By Corollary 3.54 we conclude: $\mathsf{CF}_{\mathsf{JH2}}(x)$ is periodic. □

Remark 5.25. Theorem 5.24 cannot be found in [4], as J. Hurwitz does not prove that $\mathsf{CF}_{\mathsf{JH2}}(x)$ is periodic for $x \in \mathbb{C}$ quadratic irrational. This theorem is partly a consequence of Proposition 3.29, which may have been unknown in 1902.

5.3 Correspondence between the two algorithms

In this section we will see that there is an interesting correspondence between the algorithms $\mathsf{CF}_{\mathsf{JH1}}$ and $\mathsf{CF}_{\mathsf{JH2}}$.

Theorem 5.26. *Let $x \in \mathbb{C}$ be irrational and let $n \in \mathbb{N}$. Let $\mathsf{CF}_{\mathsf{JH1}}(x) = [a_0, a_1, a_2, \ldots]$. Let $\frac{p_n}{q_n}$ be the n-th convergent of $\mathsf{CF}_{\mathsf{JH1}}(x)$. Then: $\mathsf{CF}_{\mathsf{JH2}}(\frac{q_n}{q_{n-1}}) = [a_n, a_{n-1}, \ldots, a_1]$.*

Proof. This follows as we can show that $\frac{q_n}{q_{n-1}} \in W_{a_n}$ for every $n \in \mathbb{N}$. For details of the proof, see [4, pp. 260–262]. $\square$

Theorem 5.27. *Let $x \in \mathbb{C}$ be irrational and let $n \in \mathbb{N}$. Let $\mathsf{CF}_{\mathsf{JH2}}(x) = [a_0, a_1, a_2, \ldots]$. Let $\frac{p_n}{q_n}$ be the n-th convergent of $\mathsf{CF}_{\mathsf{JH2}}(x)$. If $a_1 \notin \{1+i, 1-i, -1+i, -1-i\}$ then $\mathsf{CF}_{\mathsf{JH1}}(\frac{q_n}{q_{n-1}}) = [a_n, a_{n-1}, \ldots, a_1]$.*

Proof. This follows because $\frac{q_n}{q_{n-1}} \in S_{a_n}$ for every $n \in \mathbb{N}$. For details of the proof, see [4, pp. 260–262]. $\square$

Note that Theorem 5.26 and Theorem 5.27 are false when we assume x to be rational. Consider $\frac{7}{5}$, we have $\mathsf{CF}_{\mathsf{JH1}}(\frac{7}{5}) = \mathsf{CF}_{\mathsf{JH2}}(\frac{7}{5}) = [2, 2, 3]$. Then: $q_1 = 2$ and $q_2 = 5$, so $\frac{q_2}{q_1} = \frac{5}{2}$. Now we obtain: $\mathsf{CF}_{\mathsf{JH2}}(\frac{5}{2}) = \mathsf{CF}_{\mathsf{JH1}}(\frac{5}{2}) = [2, -2] \neq [3, 2]$.

5.4 Description of the output of the algorithms

In this section we try to convince the reader that there exists a finite automaton which accepts the output of $\mathsf{CF}_{\mathsf{JH1}}$, for rational input. We will also examine this property for other complex continued fraction algorithms.

Let $z \in \mathbb{Q}[i]$ and let $\mathit{fl}_{\mathsf{JH1}}(z) = [a_0, a_1, \ldots, a_n]$. We consider $[a_0, a_1, \ldots, a_n]$ as a word over the alphabet $\mathbb{Z}[i]$. It seems that there exists a finite automaton which precisely accepts the set $\{\mathsf{CF}_{\mathsf{JH1}}(z) \mid z \in \mathbb{Q}[i]\} \subseteq \mathbb{Z}[i]^*$. Such an automaton could be found by considering the tiles obtained by $\mathit{fl}_{\mathsf{JH1}}$ and the intersection of these tiles with the fundamental codomain $\Gamma_{\mathit{fl}_{\mathsf{JH1}}, sg_{\mathsf{JH1}}}$. This gives subsets of tiles, and therefore subsets of the fundamental domain $\Delta_{\mathit{fl}_{\mathsf{JH1}}}$ and subsequently subsets of $\Gamma_{\mathit{fl}_{\mathsf{JH1}}, sg_{\mathsf{JH1}}}$. Iterating this process gives again new subsets of tiles. It looks like there are only finitely many subsets of $\Delta_{\mathit{fl}_{\mathsf{JH1}}}$ which appear in this manner. The states of the automaton should correspond with the subsets of $\Delta_{\mathit{fl}_{\mathsf{JH1}}}$ that are found. It seems that the corresponding automaton for $\mathsf{CF}_{\mathsf{JH1}}$ has 27 states.

In the same fashion it is likely that this is also possible for $\mathsf{CF}_{\mathsf{JH2}}$, with an automaton that has only 11 states. For $\mathsf{CF}_{\mathsf{AH}}$ it seems that such an automaton exists as well and the number of states of this automaton is presumably 60. In Chapter 6 we will define the complex continued fraction algorithm $\mathsf{CF}_{\mathsf{JS}}$. In [9] is shown that there exists a finite automaton with 25 states which precisely accepts the set $\{\mathsf{CF}_{\mathsf{JS}}(z) \mid z \in \mathbb{Q}[i]\}$.

This topic asks for a rigorous approach, which unfortunately goes beyond the scope of this thesis. We refer the reader who is interested in this subject to [1].

6 A complex continued fraction algorithm by J. O. Shallit

In this chapter we will look at a complex continued fraction algorithm devised by J. O. Shallit [9]. We will define the algorithm, try to investigate whether continued fractions obtained by this algorithm are periodic for quadratic irrational numbers, and look at some specific properties of the algorithm.

6.1 The algorithm

In this section we will define the algorithm. The floor function of this algorithm is due to E. E. McDonnell [6].

Definition 6.1. Define $fl_{\mathsf{JS}} : \mathbb{C} \to \mathbb{Z}[i]$ as follows: Let $z \in \mathbb{C}$, and let $x := \mathrm{Re}(z)$ and $y := \mathrm{Im}(z)$, and let $a := \lfloor x \rfloor$ and $b := \lfloor y \rfloor$. Then:

$$fl_{\mathsf{JS}}(z) := \begin{cases} a + bi & \text{if } (x-a)+(y-b) < 1 \\ a+1+bi & \text{if } (x-a)+(y-b) \geq 1 \text{ and } x-a \geq y-b \\ a+(b+1)i & \text{if } (x-a)+(y-b) \geq 1 \text{ and } x-a < y-b. \end{cases}$$

Proposition 6.2. *The function fl_{JS} is a shift floor function.*

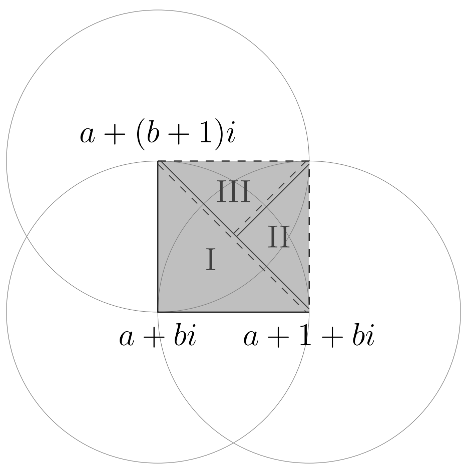

Figure 12: *The set $G_{a,b}$, which is partitioned in the subsets $\mathrm{I}_{a,b}$, $\mathrm{II}_{a,b}$ and $\mathrm{III}_{a,b}$. The circles have radius 1.*

Proof. Let $z \in \mathbb{C}$, let $x := \mathrm{Re}(z)$, $y := \mathrm{Im}(z)$, $a := \lfloor x \rfloor$ and $b := \lfloor y \rfloor$. We define $G_{a,b} := \{w \in \mathbb{C} \mid a \leq \mathrm{Re}(w) < a+1,\ b \leq \mathrm{Im}(w) < b+1\}$. Then we have: $z \in G_{a,b}$. We define three subsets of $G_{a,b}$:

$$\begin{aligned} \mathrm{I}_{a,b} &:= \{z \in G_{a,b} \mid (x-a)+(y-b) < 1\}, \\ \mathrm{II}_{a,b} &:= \{z \in G_{a,b} \mid (x-a)+(y-b) \geq 1 \text{ and } x-a \geq y-b\}, \\ \mathrm{III}_{a,b} &:= \{z \in G_{a,b} \mid (x-a)+(y-b) \geq 1 \text{ and } x-a < y-b\}. \end{aligned}$$

We see that $\mathrm{I}_{a,b}$, $\mathrm{II}_{a,b}$, $\mathrm{III}_{a,b}$ is a partitioning of $G_{a,b}$, see also Figure 12.
Suppose $(x-a)+(y-b)<1$ (the first case in the function definition of fl_{JS}). Then we have that $z \in \mathrm{I}_{a,b}$ and $fl_{\mathsf{JS}}(z) = a+bi$, and it follows that $|z - fl_{\mathsf{JS}}(z)| < 1$.
Suppose $(x-a)+(y-b) \geq 1$ and $x-a \geq y-b$ (the second case in the function definition of fl_{JS}). Then we have that $z \in \mathrm{II}_{a,b}$ and $fl_{\mathsf{JS}}(z) = a+1+bi$, and it follows that $|z - fl_{\mathsf{JS}}(z)| < 1$.
Finally, suppose $(x-a)+(y-b) \geq 1$ and $x-a < y-b$ (the third case in the function definition of fl_{JS}). Then we have that $z \in \mathrm{III}_{a,b}$ and $fl_{\mathsf{JS}}(z) = a+(b+1)i$, and it follows that $|z - fl_{\mathsf{JS}}(z)| < 1$.
In every case we have $|z - fl_{\mathsf{JS}}(z)| < 1$ and we conclude: fl_{JS} is a floor function.

For the second part of the proof, let $z \in \mathbb{C}$, $\alpha \in \mathbb{Z}[i]$ and $\alpha = p+qi$, where p, $q \in \mathbb{Z}$. Now, let $x = \mathrm{Re}(z+\alpha)$, $y = \mathrm{Im}(z+\alpha)$, $a = \lfloor x \rfloor$ and $b = \lfloor y \rfloor$. Let $x' = \mathrm{Re}(z)$, $y' = \mathrm{Im}(z)$, $a' = \lfloor x' \rfloor$ and $b' = \lfloor y' \rfloor$. Then we have $a = a'+p$, $b = b'+q$, $x-a = x'-a'$ and $y-b = y'-b'$. Consequently, following the three different cases: $fl_{\mathsf{JS}}(z+\alpha) = fl_{\mathsf{JS}}(z)+\alpha$. We conclude: fl_{JS} is a shift floor function. $\square$

Definition 6.3. Define $sg_{\mathsf{JS}} : \mathbb{C} \to \{-1, 1, -i, i\}$ as follows: $sg_{\mathsf{JS}}(z) := 1$ for every $z \in \mathbb{C}$.

We have that fl_{JS} is a shift floor function and sg_{JS} is a sign function. Therefore we can define a complex continued fraction algorithm, but first we investigate the fundamental domain $\Delta_{fl_{\mathsf{JS}}}$ and the fundamental codomain $\Gamma_{fl_{\mathsf{JS}}, sg_{\mathsf{JS}}}$.

Proposition 6.4. $\Delta_{fl_{\mathsf{JS}}} = \{z \in \mathbb{C} \mid \mathrm{Re}(z) \geq -\mathrm{Im}(z), \mathrm{Re}(z) < -\mathrm{Im}(z)+1, \mathrm{Re}(z) \geq \mathrm{Im}(z)+1, \mathrm{Re}(z) < \mathrm{Im}(z)-1\}$.

Proof. As fl_{JS} is a shift floor function, we have by Proposition 3.6:

$$\Delta_{fl_{\mathsf{JS}}} = \{z \in \mathbb{C} \mid fl_{\mathsf{JS}}(z) = 0\}$$

Considering the definition of fl_{JS} and the subsets $\mathrm{I}_{a,b}$, $\mathrm{II}_{a,b}$, $\mathrm{III}_{a,b}$ of $G_{a,b}$ from the proof of Proposition 6.2 we obtain the following. If $z \in \mathrm{I}_{0,0}$, then $fl_{\mathsf{JS}}(z) = 0$. If $z \in \mathrm{II}_{-1,0}$, then $fl_{\mathsf{JS}}(z) = 0$. If $z \in \mathrm{III}_{0,-1}$, then $fl_{\mathsf{JS}}(z) = 0$. Also: if $z \notin \mathrm{I}_{0,0} \cup \mathrm{II}_{-1,0} \cup \mathrm{III}_{0,-1}$, then $fl_{\mathsf{JS}}(z) \neq 0$. Therefore: $\Delta_{fl_{\mathsf{JS}}} = \mathrm{I}_{0,0} \cup \mathrm{II}_{-1,0} \cup \mathrm{III}_{0,-1}$, and this set is equal to the set $\{z \in \mathbb{C} \mid \mathrm{Re}(z) \geq -\mathrm{Im}(z), \mathrm{Re}(z) < -\mathrm{Im}(z)+1, \mathrm{Re}(z) \geq \mathrm{Im}(z)+1, \mathrm{Re}(z) < \mathrm{Im}(z)-1\}$, which completes the proof. $\square$

Proposition 6.5. $\Gamma_{fl_{\mathsf{JS}}, sg_{\mathsf{JS}}} = \{\frac{1}{z} \mid z \in \Delta_{fl_{\mathsf{JS}}}, z \neq 0\}$.

Proof. As $sg_{\mathsf{JS}}(z) = 1$ for every $z \in \mathbb{C}$, this follows directly from Proposition 3.9. $\square$

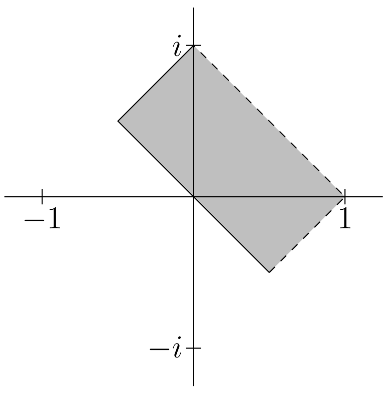

Figure 13: *Fundamental domain* $\Delta_{fl_{\mathsf{JS}}}$.

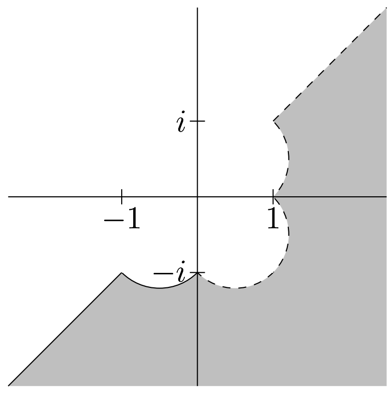

Figure 14: *Fundamental codomain* $\Gamma_{fl_{\mathsf{JS}}, sg_{\mathsf{JS}}}$.

Algorithm 6.6 (J. O. Shallit, 1979)**.**
input: $x \in \mathbb{C}$,
output: a finite or infinite complex continued fraction, which is generated by

$$\begin{cases} x_0 & := & x \\ a_n & := & \mathit{fl}_{\mathsf{JS}}(x_n) \\ e_{n+1} & := & sg_{\mathsf{JS}}(x_n) \\ x_{n+1} & := & \frac{e_{n+1}}{x_n - a_n} \end{cases}$$

We will refer to this algorithm as $\mathsf{CF}_{\mathsf{JS}}$, and denote the result of the algorithm on $x \in \mathbb{C}$ by $\mathsf{CF}_{\mathsf{JS}}(x)$.

Remark 6.7. During the writing of my thesis, I implemented this algorithm in Magma. For rational or quadratic irrational input, the output of the algorithm is exact. The examples in Section 6.3 were generated with this implementation of the algorithm.

Proposition 6.8. $\mathsf{CF}_{\mathsf{JS}}$ *is a complex continued fraction algorithm.*

Proof. By Proposition 6.2 we have that $\mathit{fl}_{\mathsf{JS}}$ is a floor function. It is obvious that sg_{JS} is a sign function. Now it follows that $\mathsf{CF}_{\mathsf{JS}}$ is a complex continued fraction algorithm. □

As $\mathsf{CF}_{\mathsf{JS}}$ is a complex continued fraction algorithm, all the results of Chapter 3 apply to this algorithm.

6.2 Convergence

In this section we look at some aspects of convergence for the algorithm $\mathsf{CF}_{\mathsf{JS}}$.

Proposition 6.9. *Let $x \in \mathbb{C}$ be irrational and let $(x_{n+1})_{n\in\mathbb{N}}$ be the complete quotients of x under $\mathsf{CF}_{\mathsf{JS}}$. Then not:* $\lim_{n\to\infty} |x_{n+1}| = 1$.

Note that Proposition 6.9 is a special case of Proposition 3.29. Therefore, we already have a proof of Proposition 6.9. However, we will give another proof which provides a bit more insight.

Proof. Suppose for the sake of contradiction: $\lim_{n\to\infty} |x_{n+1}| = 1$. As $x_{n+1} \in \Gamma_{\mathit{fl}_{\mathsf{JS}}, sg_{\mathsf{JS}}}$ for every $n \in \mathbb{N}$, this is only possible if $\lim_{n\to\infty} d(x_{n+1}, \{1, -i\}) = 0$. Now consider $D_1 := B_{\frac{1}{3}}(1)$ and $D_{-i} := B_{\frac{1}{3}}(-i)$. Accordingly, there exists $N \in \mathbb{N}$ such that $x_{n+1} \in D_1 \cup D_{-i}$ for every $n > N$. Let T_α be the tile of $\alpha \in \mathbb{Z}[i]$ under $\mathit{fl}_{\mathsf{JS}}$. Note that, if $x_{n+1} \in D_1$ then $x_{n+1} \in T_1 \cup T_{1-i}$. Likewise, if $x_{n+1} \in D_{-i}$ then $x_{n+1} \in T_{-1-i} \cup T_{-2i} \cup T_{-i}$. See also Figure 15. Consider $n > N$; we distinguish five cases.

I. If $x_{n+1} \in D_1 \cap T_1$, then $|x_{n+2}| = \frac{1}{|x_{n+1} - a_{n+1}|} = \frac{1}{|x_{n+1} - 1|} > \frac{1}{1/3} = 3$.

II. If $x_{n+1} \in D_1 \cap T_{1-i}$, then $x_{n+2} = \frac{1}{x_{n+1} - a_{n+1}} = \frac{1}{x_{n+1} - (1-i)}$. As $x_{n+1} \approx 1$, thus $x_{n+2} = \frac{1}{x_{n+1} - (1-i)} \approx \frac{1}{i} = -i$, we conclude: $x_{n+2} \in D_{-i}$. Now:

$$\begin{aligned} x_{n+2} + i &= \frac{1}{x_{n+1} - 1 + i} + i \cdot \frac{x_{n+1} - 1 + i}{x_{n+1} - 1 + i} \\ &= \frac{1 + i x_{n+1} - i - 1}{x_{n+1} - 1 + i} \\ &= \frac{i(x_{n+1} - 1)}{x_{n+1} - 1 + i}. \end{aligned}$$

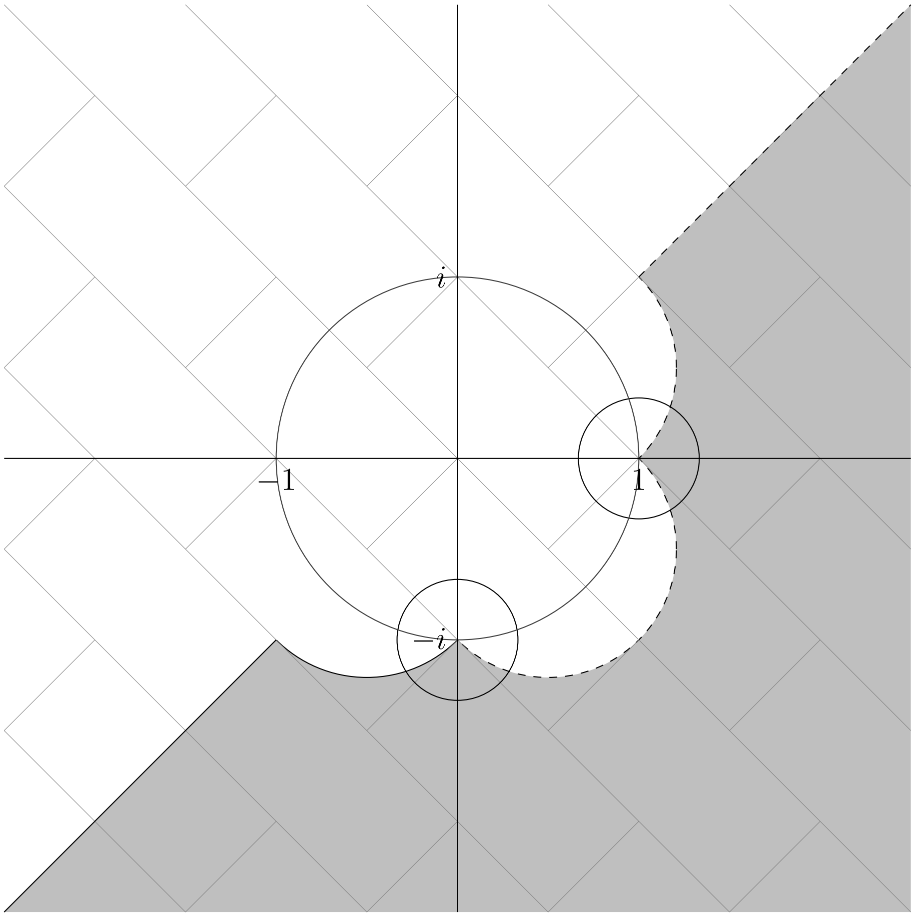

Figure 15: *The complex plane with the tiling obtained by fl_{JS}. The fundamental codomain* $\Gamma_{fl_{\mathsf{JS}},sg_{\mathsf{JS}}}$ *is indicated in grey. Also the unit circle and the areas D_1 and D_{-i} are included.*

So: $|x_{n+2}+i| = \frac{|x_{n+1}-1|}{|x_{n+1}-1+i|}$, and because $x_{n+1} \in T_{1-i}$ we have $|x_{n+1}-1+i| < 1$. Therefore: $|x_{n+2}+i| > |x_{n+1}-1|$.

III. If $x_{n+1} \in D_{-i} \cap T_{-1-i}$ then $x_{n+2} = \frac{1}{x_{n+1}-a_{n+1}} = \frac{1}{x_{n+1}-(-1-i)}$. As $x_{n+1} \approx -i$, thus $x_{n+2} = \frac{1}{x_{n+1}-(-1-i)} \approx \frac{1}{1} = 1$, we conclude: $x_{n+2} \in D_1$. Now:

$$\begin{aligned} x_{n+2}-1 &= \frac{1}{x_{n+1}+1+i} - \frac{x_{n+1}+1+i}{x_{n+1}+1+i} \\ &= \frac{1-x_{n+1}-1-i}{x_{n+1}+1+i} \\ &= \frac{-x_{n+1}-i}{x_{n+1}+1+i}. \end{aligned}$$

So: $|x_{n+2}-1| = \frac{|x_{n+1}+i|}{|x_{n+1}+1+i|}$, and because $x_{n+1} \in T_{-1-i}$ we have $|x_{n+1}+1+i| < 1$. Therefore: $|x_{n+2}-1| > |x_{n+1}+i|$.

IV. If $x_{n+1} \in D_{-i} \cap T_{-2i}$, then $x_{n+2} = \frac{1}{x_{n+1}-a_{n+1}} = \frac{1}{x_{n+1}-(-2i)}$. As $x_{n+1} \approx -i$, thus

$x_{n+2} = \frac{1}{x_{n+1}-(-2i)} \approx \frac{1}{i} = -i$, we conclude: $x_{n+2} \in D_{-i}$. Now:

$$\begin{aligned} x_{n+2} + i &= \frac{1}{x_{n+1}+2i} + i \cdot \frac{x_{n+1}+2i}{x_{n+1}+2i} \\ &= \frac{1 + ix_{n+1} - 2}{x_{n+1}+2i} \\ &= \frac{ix_{n+1} - 1}{x_{n+1}+2i}. \end{aligned}$$

So: $|x_{n+2}+i| = \frac{|x_{n+1}+i|}{|x_{n+1}+2i|}$, and because $x_{n+1} \in T_{-2i}$ we have $|x_{n+1}+2i| < 1$. Therefore: $|x_{n+2}+i| > |x_{n+1}+i|$.

V. If $x_{n+1} \in D_{-i} \cap T_{-i}$, then $|x_{n+2}| = \frac{1}{|x_{n+1}-a_{n+1}|} = \frac{1}{|x_{n+1}-(-i)|} > \frac{1}{1/3} = 3$.

Considering these cases, we see that the cases I and V will not occur, as $x_{n+2} \in D_1 \cup D_{-i}$. Therefore, for $n = N, N+1, N+2, \ldots$ we will have a sequence of the cases II, III and IV. However, this contradicts that $\lim_{n\to\infty} d(x_{n+1}, \{1, -i\}) = 0$, as can be verified from the last line of the cases II, III and IV. Therefore we conclude: not $\lim_{n\to\infty} |x_{n+1}| = 1$. $\square$

We will later see that in general, for $x \in \mathbb{C}$ irrational, it is not the case that $\left|\frac{q_{n-1}}{q_n}\right| < 1$ for every $n \in \mathbb{N}$, where $\left(\frac{p_n}{q_n}\right)_{n\in\mathbb{N}}$ are the convergents of $\mathsf{CF}_{\mathsf{JS}}(x)$. However, we have the following two results. In this chapter we will omit the partial numerators from $\mathsf{CF}_{\mathsf{JS}}(x) = [a_0, e_1/a_1, e_2/a_2, \ldots]$, as they are all equal to 1, and write $\mathsf{CF}_{\mathsf{JS}}(x) = [a_0, a_1, a_2, \ldots]$.

Proposition 6.10. *Let $x \in \mathbb{R}$ be irrational. Let $\left(\frac{p_n}{q_n}\right)_{n\in\mathbb{N}}$ be the convergents of $\mathsf{CF}_{\mathsf{JS}}(x)$. Then: $\left|\frac{q_{n-1}}{q_n}\right| < 1$ for every $n \in \mathbb{N}$.*

Proof. Let $x \in \mathbb{R}$ and let $\mathsf{CF}_{\mathsf{JS}}(x) = [a_0, a_1, a_2, \ldots]$. We have that $a_n \in \mathbb{N}_{>0}$ as $x_n \in \{x \,|\, x \in \mathbb{R}, x > 1\}$ for every $n \in \mathbb{N}_{>0}$. We prove the statement by induction, first: $0 = \frac{q_{-1}}{q_0} < 1$. Now suppose $0 \leq \frac{q_{n-1}}{q_n} < 1$. Then: $\frac{q_{n+1}}{q_n} = a_{n+1} + \frac{q_{n-1}}{q_n} > 1$, consequently: $0 \leq \frac{q_n}{q_{n+1}} < 1$. This completes the proof. $\square$

Proposition 6.11. *Let $x \in \mathbb{R}$ be irrational. Let $\left(\frac{p_n}{q_n}\right)_{n\in\mathbb{N}}$ be the convergents of $\mathsf{CF}_{\mathsf{JS}}(ix)$. Then: $\left|\frac{q_{n-1}}{q_n}\right| < 1$ for every $n \in \mathbb{N}$.*

Proof. Let $x \in \mathbb{R}$ and let $\mathsf{CF}_{\mathsf{JS}}(ix) = [a_0, a_1, a_2, \ldots]$. We have that $a_n \in \{in \,|\, n \in \mathbb{Z}, n \leq 2\}$ as $x_n \in \{ix \,|\, x \in \mathbb{R}, x < -1\}$ for every $n \in \mathbb{N}_{>0}$. We prove the statement by induction, first: $\left|\frac{q_{-1}}{q_0}\right| < 1$. Then: $\left|\frac{q_{n+1}}{q_n}\right| = \left|a_{n+1} + \frac{q_{n-1}}{q_n}\right| \geq |a_{n+1}| - \left|\frac{q_{n-1}}{q_n}\right| > 1$, as $|a_{n+1}| \geq 2$. Consequently: $\left|\frac{q_n}{q_{n+1}}\right| < 1$ and this completes the proof. $\square$

6.3 Periodicity?

In this section we try to investigate whether $\mathsf{CF}_{\mathsf{JS}}(x)$ is periodic for x quadratic irrational. Let $z \in \mathbb{C}$ be irrational. By Theorem 3.48 we have that if $\mathsf{CF}_{\mathsf{JS}}(z)$ is periodic, then z is quadratic irrational. We formulate the converse of this result.

Conjecture 6.12. *If $z \in \mathbb{C}$ is quadratic irrational, then $\mathsf{CF}_{\mathsf{JS}}(z)$ is periodic.*

The following examples seem to suggest that Conjecture 6.12 is true.

Example 6.13. Here we compute $\mathsf{CF}_{\mathsf{JS}}(z)$ for some values $z \in \mathbb{C}$, where z is quadratic irrational. The partial numerators are omitted, as they are all equal to 1.

$$\mathsf{CF}_{\mathsf{JS}}(3+i+\sqrt{9+6i}) = [\overline{6+2i}]$$
$$\mathsf{CF}_{\mathsf{JS}}(\sqrt{1+11i}) = [2+2i, \overline{1-i, 1, 2+i, 4, 2+i, 1, 1-i, 4+4i}]$$
$$\mathsf{CF}_{\mathsf{JS}}(\tfrac{1+7i+\sqrt{4+2i}}{6+4i}) = [1, -2i, -1-2i, 6-i, \overline{-1-2i, -i, 1+i, 1+i, 5}]$$

Unfortunately, I did not succeed in proving Conjecture 6.12. However, we have the following result.

Proposition 6.14. *Let $z \in \mathbb{C}$ be quadratic irrational. Let z_n be the n-th complete quotient and let $\frac{p_n}{q_n}$ be the n-th convergent of z under $\mathsf{CF}_{\mathsf{JS}}$. If either*

i. $\left|\frac{q_{n-1}}{q_n}\right| < 1$ *for every $n \in \mathbb{N}$, or*

ii. $|z_{n+1}| \geq \rho$ *for every $n \in \mathbb{N}$, for some $\rho > 1$, or*

iii. $\left|\frac{q_{n-1}}{q_n}\right| \leq \sigma$ *for infinitely many indices n, for some $\sigma < 1$,*

then $\mathsf{CF}_{\mathsf{JS}}(z)$ is periodic.

Proof. This is just Corollary 3.54, Corollary 3.55 and Corollary 3.56. □

Let us investigate the assumptions in Proposition 6.14. In Proposition 6.23 we will see that in general, the first assumption is false. For the second assumption: note that $|z|-1$ can be arbitrarily small for $z \in \Gamma_{fl_{\mathsf{JS}}, sg_{\mathsf{JS}}}$. Therefore there is no obvious reason to expect that there exists $\rho > 1$, such that $|z_{n+1}| \geq \rho$ for every $n \in \mathbb{N}$. The third assumption seems reasonable, as $\lim_{n\to\infty} q_n = \infty$. However, we did not prove anything about the growth of $|q_0|, |q_1|, |q_2|, \ldots$ and it is not clear why this assumption would be true in general.

Although I cannot prove Conjecture 6.12, the statement is true if we assume some more about the input z.

Proposition 6.15. *Let $z \in \mathbb{C}$ be quadratic irrational. If z is either real or purely imaginary, then $\mathsf{CF}_{\mathsf{JS}}(z)$ is periodic.*

Proof. Suppose z is either real or purely imaginary. Let $\left(\frac{p_n}{q_n}\right)_{n\in\mathbb{N}}$ be the convergents of z under $\mathsf{CF}_{\mathsf{JS}}$. By Proposition 6.10 and Proposition 6.11 we have that $\left|\frac{q_{n-1}}{q_n}\right| < 1$ for every $n \in \mathbb{N}$. Therefore by Corollary 3.54: $\mathsf{CF}_{\mathsf{JS}}(z)$ is periodic. □

Now I will explain how I tried to prove Conjecture 6.12 and why I did not succeed in this. Theorem 3.52 seems a good starting point for proving this conjecture; in fact, this is how A. Hurwitz and J. Hurwitz prove periodicity for their algorithms. So, I reduced proving periodicity to proving a property about the sequence $(\theta_n)_{n\in\mathbb{N}}$. Now we prove two more results.

Proposition 6.16. *Let $x \in \mathbb{C}$ be irrational. Let θ_n be the n-th relative error of x under $\mathsf{CF}_{\mathsf{JS}}$. If $\lim_{n\to\infty} \theta_n = \infty$, then $\mathsf{CF}_{\mathsf{JS}}(x)$ is not periodic.*

Proof. Suppose $\lim_{n\to\infty} \theta_n = \infty$, then by Lemma 3.42 we have: $\lim_{n\to\infty} |x_{n+1} + \frac{q_{n-1}}{q_n}| = 0$. As $|x_{n+1}| > 1$ for every $n \in \mathbb{N}$ and $|\frac{q_{n-1}}{q_n}| \leq 1$ for infinitely many indices n, we have that there exists a subsequence $(k_n)_{n\in\mathbb{N}}$ of $\mathbb{N}$ such that $\lim_{n\to\infty} |x_{k_n+1}| = 1$. Therefore: as $|x_{k_n+1}| > 1$ it follows that the set $\{x_{k_n+1} \bigm| n \in \mathbb{N}\}$ is infinite. Consequently: the set $\{x_n \bigm| n \in \mathbb{N}\}$ is infinite. By Proposition 3.51 we conclude: $\mathsf{CF}_{\mathsf{JS}}(x)$ is not periodic. □

Corollary 6.17. *Let $x \in \mathbb{C}$ be irrational. Let θ_n be the n-th relative error of x under $\mathsf{CF}_{\mathsf{JS}}$. If $\mathsf{CF}_{\mathsf{JS}}(x)$ is periodic, then there exists $\delta > 0$ such that $|\theta_n| < \delta$ for infinitely many indices $n \in \mathbb{N}$.*

Proof. Suppose $\mathsf{CF}_{\mathsf{JS}}(x)$ is periodic. By Proposition 6.16 follows not: $\lim_{n\to\infty} \theta_n = \infty$. This implies there exists $\delta > 0$ such that $|\theta_n| < \delta$ for infinitely many indices $n \in \mathbb{N}$. □

Let $x \in \mathbb{C}$ be quadratic irrational and let θ_n be the n-th relative error of x under $\mathsf{CF}_{\mathsf{JS}}$. By Corollary 6.17 we see that it is necessary that there exists $\delta > 0$ such that $|\theta_n| < \delta$ for an infinite number of indices n for $\mathsf{CF}_{\mathsf{JS}}(x)$ to be periodic. By Theorem 3.52 it follows that it is sufficient that there exists $\delta > 0$ such that both $|\theta_{n-1}| < \delta$ and $|\theta_n| < \delta$ for an infinite number of indices n for $\mathsf{CF}_{\mathsf{JS}}(x)$ to be periodic.

By Proposition 6.16 we have: $\mathsf{CF}_{\mathsf{JS}}(x)$ is periodic implies that not: $\lim_{n\to\infty} \theta_n = \infty$, for x quadratic irrational. We now formulate a weakening of Conjecture 6.12.

Conjecture 6.18. *Let $x \in \mathbb{C}$ be quadratic irrational. Let θ_n be the n-th relative error of x under $\mathsf{CF}_{\mathsf{JS}}$. Then not: $\lim_{n\to\infty} \theta_n = \infty$.*

However, also this conjecture seems hard to prove, and I did not succeed in this. I tried to prove Conjecture 6.18 by contradiction: I supposed both $x \in \mathbb{C}$ quadratic irrational and $\lim_{n\to\infty} \theta_n = \infty$. When we do this, we get the following.

Proposition 6.19. *Let $x \in \mathbb{C}$ be quadratic irrational. Let θ_n be the n-th relative error of x under $\mathsf{CF}_{\mathsf{JS}}$. Suppose $\lim_{n\to\infty} \theta_n = \infty$. Then:*

i. $\lim_{n\to\infty} \left(x_{n+1} + \frac{q_{n-1}}{q_n}\right) = 0$,

ii. $\lim_{n\to\infty} \left(x_{n+1} \frac{q_n}{q_{n-1}}\right) = -1$,

iii. $\lim_{n\to\infty} \frac{\prod_{k=0}^{n} x_{k+1}}{q_n} = 0$,

iv. $\lim_{n\to\infty} (x_{n+1} - y_{n+1}) = 0$,

v. $\lim_{n\to\infty} \prod_{k=0}^{n} x_{k+1} y_{k+1} = 0$,

where $\frac{p_n}{q_n}$ is the n-th convergent and x_{n+1} is the $(n+1)$-th complete quotient of x under $\mathsf{CF}_{\mathsf{JS}}$. We set y_0 to be the other root of the polynomial that x satisfies, and $y_{n+1} := \frac{1}{y_n - a_n}$, for $\mathsf{CF}_{\mathsf{JS}}(x) = [a_0, a_1, a_2, \ldots]$ (just as in Claim 2 of the proof of Theorem 3.52).

Proof. i. This directly follows from Lemma 3.42 and the assumption $\lim_{n\to\infty} \theta_n = \infty$.

ii. We have $\left|1 + \frac{1}{x_{n+1}} \frac{q_{n-1}}{q_n}\right| = \left|\frac{x_{n+1} + \frac{q_{n-1}}{q_n}}{x_{n+1}}\right| < \left|x_{n+1} + \frac{q_{n-1}}{q_n}\right|$, so: $\lim_{n\to\infty} \left(1 + \frac{1}{x_{n+1}} \frac{q_{n-1}}{q_n}\right) = 0$. Therefore: $\lim_{n\to\infty} \left(\frac{1}{x_{n+1}} \frac{q_{n-1}}{q_n}\right) = -1$ and we conclude: $\lim_{n\to\infty} \left(x_{n+1} \frac{q_n}{q_{n-1}}\right) = -1$.

iii. By Lemma 3.33 we have: $\theta_n = \frac{q_n(-1)^n}{\prod_{k=0}^n x_{k+1}}$, and the statement follows.

iv. By Proposition 3.57 we have: $\lim_{n\to\infty}(y_{n+1} + \frac{q_{n-1}}{q_n}) = 0$. Then it follows by (i) that $\lim_{n\to\infty}(x_{n+1} - y_{n+1}) = 0$.

v. We have: $x_n - y_n = (x_n - a_n) - (y_n - a_n) = \frac{1}{x_{n+1}} - \frac{1}{y_{n+1}} = \frac{y_{n+1}-x_{n+1}}{x_{n+1}y_{n+1}}$, and therefore: $x_{n+1}y_{n+1} = -\frac{x_{n+1}-y_{n+1}}{x_n-y_n}$. Consequently: $\prod_{k=0}^n x_{k+1}y_{k+1} = (-1)^{n-1} \cdot \frac{x_{n+1}-y_{n+1}}{x_0-y_0}$. According to (iv) we have: $\lim_{n\to\infty} \prod_{k=0}^n x_{k+1}y_{k+1} = 0$.

$\square$

While some of the statements above may seem curious, I cannot show that any of them leads to a contradiction. For example, by (ii) it follows that $\lim_{n\to\infty} \left|x_{n+1}\frac{q_n}{q_{n-1}}\right| = 1$, while $|x_{n+1}| > 1$ for every $n \in \mathbb{N}$, and $\left|\frac{q_n}{q_{n-1}}\right| > 1$ for an infinite number of indices n. By (iv) it follows that $\lim_{n\to\infty}(x_{n+1} - y_{n+1}) = 0$. We have that $|x_{n+1}| > 1$ for every $n \in \mathbb{N}$, and therefore (v) seems peculiar, but again, I could not find a contradiction.

Although it looks like $\mathsf{CF}_{\mathsf{JS}}(x)$ is periodic for every x quadratic irrational, as the many examples I calculated show, I am not able to prove this. We know that $\mathsf{CF}_{\mathsf{AH}}$, $\mathsf{CF}_{\mathsf{JH1}}$ and $\mathsf{CF}_{\mathsf{JH2}}$ do have this property, and this is a hint that it applies to $\mathsf{CF}_{\mathsf{JS}}$ too. However, in proving periodicity for $\mathsf{CF}_{\mathsf{AH}}$, $\mathsf{CF}_{\mathsf{JH1}}$ and $\mathsf{CF}_{\mathsf{JH2}}$ we used the fact that $|\frac{q_{n-1}}{q_n}| < 1$ for every $n \in \mathbb{N}$, which does not apply to $\mathsf{CF}_{\mathsf{JS}}$. I have to conclude: Conjecture 6.12 and Conjecture 6.18 are still open questions.

6.4 Specific properties of the algorithm

In this section we will look at some specific properties of $\mathsf{CF}_{\mathsf{JS}}$ and we will see that $\mathsf{CF}_{\mathsf{JS}}$ lacks a property that is shared by $\mathsf{CF}_{\mathsf{AH}}$, $\mathsf{CF}_{\mathsf{JH1}}$ and $\mathsf{CF}_{\mathsf{JH2}}$.

The main theorem in [9] gives a complete description of the output of $\mathsf{CF}_{\mathsf{JS}}(z)$, where z is rational. More precisely: a context-free grammar G over the alphabet $\mathbb{Z}[i]$ is given. Let $L(G)$ be the language of G and consider $\mathsf{CF}_{\mathsf{JS}}(z) = [a_0, a_1, \ldots, a_n]$ as a word over the alphabet $\mathbb{Z}[i]$. The main theorem states that $L(G) = \{\mathsf{CF}_{\mathsf{JS}}(z) \mid z \in \mathbb{Q}[i]\}$. We will bring this theorem into practice in the following two propositions.

Proposition 6.20. *For every $n \in \mathbb{N}_{>0}$ there exists $z \in \mathbb{Q}[i]$ such that, if $\frac{p_m}{q_m}$ is the m-th convergent of $\mathsf{CF}_{\mathsf{JS}}(z)$, then:*

i. $q_n = (-i)^n$, $\frac{p_n}{q_n} = 0$, $\left|\frac{q_{n-1}}{q_n}\right| = n$, *and*

ii. $\left|z - \frac{p_{n-1}}{q_{n-1}}\right| = \frac{2\sqrt{2}}{n}\left|z - \frac{p_n}{q_n}\right|$.

Proof. Let $n \in \mathbb{N}_{>0}$. Consider $z := \mathrm{Val}([-i, \underbrace{-2i, \ldots, -2i}_{n-1}, -i, 2+2i])$. Then, by Theorem II.3.3 in [9] we have: $\mathsf{CF}_{\mathsf{JS}}(z) = [-i, \underbrace{-2i, \ldots, -2i}_{n-1}, -i, 2+2i]$. We compute the convergents $\left(\frac{p_m}{q_m}\right)_{m \le n+1}$ of $\mathsf{CF}_{\mathsf{JS}}(z)$.

Claim: We have that $p_{n-1} = (-i)^n$ and $q_{n-1} = n(-i)^{n-1}$.

Proof of claim: We prove this by induction. We have $p_0 = a_0 = -i = (-i)^1$ and $q_0 = 1 = 1(-i)^0$. Moreover: $p_1 = a_1 a_0 + 1 = -1 = (-i)^2$ and $q_1 = a_1 = -2i = 2(-i)^1$. Now, suppose $p_{m-2} = (-i)^{m-1}$, $p_{m-1} = (-i)^m$, $q_{m-2} = (m-1)(-i)^{m-2}$ and $q_{m-1} = m(-i)^{m-1}$ Then for $2 \le m \le n-1$:

$$\begin{aligned}
p_m &= a_m p_{m-1} + p_{m-2} = -2i(-i)^m + (-i)^{m-1} = 2(-i)^{m+1} - (-i)^{m+1} \\
&= (-i)^{m+1}, \\
q_m &= a_m q_{m-1} + q_{m-2} = -2im(-i)^{m-1} + (m-1)(-i)^{m-2} = 2m(-i)^m - (m-1)(-i)^m \\
&= (m+1)(-i)^m,
\end{aligned}$$

and the claim follows. ■

For p_n and q_n we have the following:

$$\begin{aligned}
p_n &= a_n p_{n-1} + p_{n-2} = -i(-i)^n + (-i)^{n-1} = (-i)^{n+1} - (-i)^{n+1} = 0, \\
q_n &= a_n q_{n-1} + q_{n-2} = -in(-i)^{n-1} + (n-1)(-i)^{n-2} = n(-i)^n - (n-1)(-i)^n = (-i)^n.
\end{aligned}$$

Therefore: $q_n = (-i)^n$, $\frac{p_n}{q_n} = 0$, $\left|\frac{q_{n-1}}{q_n}\right| = \left|\frac{n(-i)^{n-1}}{(-i)^n}\right| = n$ and this completes the first part of the proof.

Now we compute p_{n+1} and q_{n+1}:

$$\begin{aligned}
p_{n+1} &= a_{n+1} p_n + p_{n-1} = (2+2i) \cdot 0 + (-i)^n = (-i)^n, \\
q_{n+1} &= a_{n+1} q_n + q_{n-1} = (2+2i)(-i)^n + n(-i)^{n-1} = (2 + (2+n)i)(-i)^n.
\end{aligned}$$

As a consequence we obtain:

$$z = \mathrm{Val}([-i, \underbrace{-2i, \ldots, -2i}_{n-1}, -i, 2+2i]) = \frac{p_{n+1}}{q_{n+1}} = \frac{(-i)^n}{(2+(2+n)i)(-i)^n} = \frac{1}{2+(2+n)i}.$$

Therefore:

$$\begin{aligned}
\left|z - \frac{p_n}{q_n}\right| &= \left|\frac{1}{2+(2+n)i} - 0\right| \\
&= \frac{1}{|2+(2+n)i|}, \\
\left|z - \frac{p_{n-1}}{q_{n-1}}\right| &= \left|\frac{1}{2+(2+n)i} - \frac{-i}{n}\right| = \left|\frac{n + i(2+(2+n)i)}{n(2+(2+n)i)}\right| \\
&= \frac{|n+2i-2-n|}{n|2+(2+n)i|} = \frac{|2i-2|}{n} \cdot \frac{1}{|2+(2+n)i|} \\
&= \frac{2\sqrt{2}}{n}\left|z - \frac{p_n}{q_n}\right|,
\end{aligned}$$

and this completes the proof. □

Let $z \in \mathbb{Q}[i]$, and let $\frac{p_n}{q_n}$ be the n-th convergent of $\mathsf{CF}_{\mathsf{JS}}(z)$. According to Proposition 6.20 we see that it is not the case that $|\frac{q_{n-1}}{q_n}| < 1$ for every $n < |\mathsf{CF}_{\mathsf{JS}}(z)|$. This is in contrast to the convergents of $\mathsf{CF}_{\mathsf{AH}}(z)$. It is even the case that $|\frac{q_{n+1}}{q_n}|$ can be arbitrary small. Also, $\frac{p_{n+1}}{q_{n+1}}$ is not always a better approximation to z than $\frac{p_n}{q_n}$, which is again in contrast to the convergents of $\mathsf{CF}_{\mathsf{AH}}(z)$.

Remark 6.21. Note that, in a sense, $\mathsf{CF}_{\mathsf{AH}}(\frac{1}{2+(2+n)i})$ is much simpler than $\mathsf{CF}_{\mathsf{JS}}(\frac{1}{2+(2+n)i})$, for $n \in \mathbb{N}_{>0}$. From Proposition 6.20: $\mathsf{CF}_{\mathsf{JS}}(\frac{1}{2+(2+n)i}) = [-i, \underbrace{-2i, \ldots, -2i}_{n-1}, -i, 2+2i]$.

Now we will compute $\mathsf{CF}_{\mathsf{AH}}(\frac{1}{2+(2+n)i})$. As $|\frac{1}{2+(2+n)i}| \leq |\frac{1}{2+(2+1)i}| = \frac{1}{\sqrt{13}} < \frac{1}{2}$ we have that $a_0 = \mathit{fl}_{\mathsf{AH}}(\frac{1}{2+(2+n)i}) = 0$. Therefore $z_1 = \frac{1}{\frac{1}{2+(2+n)i} - 0} = 2 + (2+n)i$, and $a_1 = \mathit{fl}_{\mathsf{AH}}(2 + (2+n)i) = 2 + (2+n)i$. As $z_1 = a_1$, the algorithm terminates and we obtain: $\mathsf{CF}_{\mathsf{AH}}(\frac{1}{2+(2+n)i}) = [0, 1/(2+(2+n)i)]$. With the same reasoning we find: $\mathsf{CF}_{\mathsf{JH1}}(\frac{1}{2+(2+n)i}) = \mathsf{CF}_{\mathsf{JH2}}(\frac{1}{2+(2+n)i}) = [0, -1/(-2-(2+n)i)]$.
The next result shows that the relative error θ_n can be arbitrary large and arbitrary small for specific $z \in \mathbb{Q}[i]$.

Proposition 6.22. *For every $n \in \mathbb{N} \setminus \{0,1\}$ there exists $z \in \mathbb{Q}[i]$ such that, if θ_m is the m-th relative error of $\mathsf{CF}_{\mathsf{JS}}(z)$, then $|\theta_{n-1}| = \frac{n\sqrt{2}}{\sqrt{5}}$ and $|\theta_n| = \frac{1}{n\sqrt{5}}$.*

Proof. Let $n \in \mathbb{N} \setminus \{0,1\}$ and consider $z := \mathrm{Val}([-i, \underbrace{-2i, \ldots, -2i}_{n-1}, -i, n + ni])$. Then, by Theorem II.3.3 in [9] we have: $\mathsf{CF}_{\mathsf{JS}}(z) = [-i, \underbrace{-2i, \ldots, -2i}_{n-1}, -i, n+ni]$. Note that the first $n+1$ partial quotients coincide with the number constructed in Proposition 6.20, therefore we have:

$$\begin{aligned} p_{n-1} &= (-i)^n, & p_n &= 0, \\ q_{n-1} &= n(-i)^{n-1}, & q_n &= (-i)^n. \end{aligned}$$

Now we can compute p_{n+1} and q_{n+1}:

$$\begin{aligned} p_{n+1} &= a_{n+1}p_n + p_{n-1} = (n+ni) \cdot 0 + (-i)^n = (-i)^n, \\ q_{n+1} &= a_{n+1}q_n + q_{n-1} = (n+ni)(-i)^n + n(-i)^{n-1} = n(1+2i)(-i)^n. \end{aligned}$$

Therefore we obtain:

$$z = \mathrm{Val}([-i, \underbrace{-2i, \ldots, -2i}_{n-1}, -i, n+ni]) = \frac{p_{n+1}}{q_{n+1}} = \frac{(-i)^n}{n(1+2i)(-i)^n} = \frac{1}{n(1+2i)}.$$

Then:

$$\begin{aligned} \theta_{n-1} &= q_{n-1}^2(z - \tfrac{p_{n-1}}{q_{n-1}}) \\ &= n^2((-i)^{n-1})^2\big(\tfrac{1}{n(1+2i)} + \tfrac{i(1+2i)}{n(1+2i)}\big) \\ &= n^2((-i)^2)^{n-1}\tfrac{i-1}{n(1+2i)} \\ &= n(-1)^{n-1}\tfrac{i-1}{1+2i}. \end{aligned}$$

We also have:

$$\theta_n = q_n^2(z - \tfrac{p_n}{q_n}) = ((-i)^n)^2(\tfrac{1}{n(1+2i)} - \tfrac{0}{(-i)^n}) = (-1)^n\tfrac{1}{n(1+2i)}.$$

Therefore:

$$|\theta_{n-1}| = n\tfrac{|i-1|}{|1+2i|} = \tfrac{n\sqrt{2}}{\sqrt{5}} \quad \text{and} \quad |\theta_n| = \tfrac{1}{n|1+2i|} = \tfrac{1}{n\sqrt{5}},$$

and this completes the proof. Note that $|z - \frac{p_{n-1}}{q_{n-1}}| = \frac{|\theta_{n-1}|}{|q_{n-1}^2|} = \frac{n\sqrt{\frac{2}{5}}}{n^2} = \frac{\sqrt{2}}{n\sqrt{5}}$ and on the other hand: $|z - \frac{p_n}{q_n}| = \frac{|\theta_n|}{|q_n^2|} = \frac{\frac{1}{n\sqrt{5}}}{1} = \frac{1}{n\sqrt{5}}$. Therefore $\sqrt{2} \cdot |z - \frac{p_n}{q_n}| = |z - \frac{p_{n-1}}{q_{n-1}}|$. So $\frac{p_n}{q_n}$ is a better approximation than $\frac{p_{n-1}}{q_{n-1}}$. □

Recall that for $\mathsf{CF}_{\mathsf{AH}}(z)$ we have that $|\theta_n| < 1 + \sqrt{2}$ for every $z \in \mathbb{C}$, for every n. For $\mathsf{CF}_{\mathsf{JH1}}(z)$ and $\mathsf{CF}_{\mathsf{JH2}}(z)$ it is not clear whether $|\theta_n|$ can be arbitrary large for given $z \in \mathbb{C}$.

Let $z \in \mathbb{C}$ and let $\left(\frac{p_n}{q_n}\right)_{n\in\mathbb{N}}$ be the convergents of $\mathsf{CF}_{\mathsf{JS}}(z)$. The following result shows that in general it is not the case that $\left|\frac{q_{n-1}}{q_n}\right| < 1$ for every $n \in \mathbb{N}$. We will see that it is even possible that $\left|\frac{q_{n-1}}{q_n}\right| > 1$ for infinitely many indices n. Note that $\mathsf{CF}_{\mathsf{AH}}$, $\mathsf{CF}_{\mathsf{JH1}}$ and $\mathsf{CF}_{\mathsf{JH2}}$ do have the property that $\left|\frac{q_{n-1}}{q_n}\right| < 1$ for every $n \in \mathbb{N}$.

Proposition 6.23. *There exists $z \in \mathbb{C}$ such that, if $\left(\frac{p_n}{q_n}\right)_{n\in\mathbb{N}}$ are the convergents of $\mathsf{CF}_{\mathsf{JS}}(z)$, then there are infinitely many indices n such that $\left|\frac{q_n}{q_{n-1}}\right| < 1$.*

Proof. Consider $z := 1 + \sqrt{4i-2}$. We calculate the continued fraction $\mathsf{CF}_{\mathsf{JS}}(z)$ using Algorithm 6.6. Then: $z_0 = z = 1 + \sqrt{4i-2}$ and $a_0 = \mathit{fl}_{\mathsf{JS}}(z_0) = 2 + i$. Therefore:

$$z_1 = \frac{1}{z_0 - a_0} = \frac{1}{1 + \sqrt{4i-2} - (2+i)} = \frac{-2i - (1+i)\sqrt{4i-2}}{4},$$

and $a_1 = \mathit{fl}_{\mathsf{JS}}(z_1) = -2i$. Next:

$$z_2 = \frac{1}{z_1 - a_1} = \frac{1}{\frac{-2i-(1+i)\sqrt{4i-2}}{4} - (-2i)} = \frac{3 - 21i - (3+4i)\sqrt{4i-2}}{25},$$

and $a_2 = \mathit{fl}_{\mathsf{JS}}(z_2) = -i$. Finally:

$$z_3 = \frac{1}{z_2 - a_2} = \frac{1}{\frac{3-21i-(3+4i)\sqrt{4i-2}}{25} - (-i)} = 1 + \sqrt{4i-2} = z_0.$$

As $z_3 = z_0$, the continued fraction of z is periodic with period length 3, and we obtain $\mathsf{CF}_{\mathsf{JS}}(z) = [\overline{2+i, -2i, -i}]$. Now define the complex valued function $f : \mathbb{C}\backslash\{\frac{i-2}{5}, \frac{6i-8}{25}\} \to \mathbb{C}$ as follows:

$$f(z) := -i + \cfrac{1}{-2i + \cfrac{1}{2 + i + \cfrac{1}{z}}} = a_2 + \cfrac{1}{a_1 + \cfrac{1}{a_0 + \cfrac{1}{z}}}.$$

We exclude $\frac{i-2}{5}$ and $\frac{6i-8}{25}$ from the domain of f because f would not be defined on these numbers. We can convince ourselves that these are the only two numbers where f would be undefined. Define $D := \{z \in \mathbb{C} \,\big|\, |z| < 1, \operatorname{Re}(z) \geq 0, \operatorname{Im}(z) \leq 0\}$. Note that $\frac{i-2}{5} \notin D$ and $\frac{6i-8}{25} \notin D$.

<u>Claim 1:</u> $f(D) := \{f(z) \in \mathbb{C} \;\big|\; z \in D\} \subseteq D$.

Proof of claim 1: Suppose $z \in D$, then:

$$\begin{aligned}
\frac{1}{z} &\in \{z \in \mathbb{C} \,\big|\, |z| > 1,\ \mathrm{Re}(z) \geq 0,\ \mathrm{Im}(z) \geq 0\}, \\
2 + i + \frac{1}{z} &\in \{z \in \mathbb{C} \,\big|\, \mathrm{Re}(z) \geq 2,\ \mathrm{Im}(z) \geq 1\}, \\
\cfrac{1}{2 + i + \cfrac{1}{z}} &\in \{z \in \mathbb{C} \,\big|\, |z| < 1,\ \mathrm{Re}(z) \geq 0,\ \mathrm{Im}(z) \leq 0\} = D, \\
-2i + \cfrac{1}{2 + i + \cfrac{1}{z}} &\in \{z \in \mathbb{C} \,\big|\, 0 \leq \mathrm{Re}(z) \leq 1,\ -3 \leq \mathrm{Im}(z) \leq -2\}, \\
\cfrac{1}{-2i + \cfrac{1}{2 + i + \cfrac{1}{z}}} &\in \{z \in \mathbb{C} \,\big|\, \mathrm{Re}(z) \geq 0,\ z \in Y\}, \\
&\text{where } Y := \{y \in \mathbb{C} \,\big|\, |y - \frac{i}{4}| \leq \frac{1}{4},\ |y - \frac{i}{6}| \geq \frac{1}{6},\ |y - \frac{1}{2}| \geq \frac{1}{2}\}, \\
f(z) = -i + \cfrac{1}{-2i + \cfrac{1}{2 + i + \cfrac{1}{z}}} &\in D.
\end{aligned}$$

We conclude: if $z \in D$, then $f(z) \in D$, therefore: $f(D) \subseteq D$. It may be convenient to make some drawings here. ■

Claim 2: For every $k \in \mathbb{N}$: $\frac{q_{3(k+1)+2}}{q_{3(k+1)+1}} = f\left(\frac{q_{3k+2}}{q_{3k+1}}\right)$.

Proof of claim 2: Let $k \in \mathbb{N}$, then:

$$\frac{q_{3(k+1)+2}}{q_{3(k+1)+1}} = a_{3(k+1)+2} + \cfrac{1}{a_{3(k+1)+1} + \cfrac{1}{a_{3(k+1)} + \cfrac{1}{\frac{q_{3k+2}}{q_{3k+1}}}}} = a_2 + \cfrac{1}{a_1 + \cfrac{1}{a_0 + \cfrac{1}{\frac{q_{3k+2}}{q_{3k+1}}}}} = f\left(\frac{q_{3k+2}}{q_{3k+1}}\right).$$

■

We calculate $(q_n)_{n \in \mathbb{N}}$. For the first three terms this gives: $q_0 = 1$, $q_1 = -2i$ and $q_2 = -1$. Therefore we have $\frac{q_2}{q_1} = \frac{-1}{-2i} = \frac{-i}{2}$, and we get: $\frac{q_2}{q_1} \in D$. By applying the preceding two claims we obtain: $\frac{q_{3k+2}}{q_{3k+1}} \in D$ for every $k \in \mathbb{N}$. As $z \in D$ implies that $|z| < 1$ we conclude: $\left|\frac{q_{3k+2}}{q_{3k+1}}\right| < 1$ for every $k \in \mathbb{N}$. □

Remark 6.24. Let $z \in \mathbb{C}$ be irrational. During the writing of this thesis the question arose if it is possible to rewrite $\mathsf{CF}_{\mathsf{JH1}}(z) = [a_0, -1/a_1, -1/a_2, \ldots]$ to $[b_0, 1/b_1, 1/b_2, \ldots]$, such that $[b_0, 1/b_1, 1/b_2, \ldots] = \mathsf{CF}_{\mathsf{JS}}(z)$. This was motivated by the idea that this could be helpful proving periodicity for $\mathsf{CF}_{\mathsf{JS}}(z)$, for z quadratic irrational. Also, the tiling obtained by $\mathit{fl}_{\mathsf{JS}}$ is strongly related to the tiling obtained by $\mathit{fl}_{\mathsf{JH1}}$. Despite several attempts, I did not succeed in finding a procedure to rewrite $\mathsf{CF}_{\mathsf{JH1}}(z)$ to $\mathsf{CF}_{\mathsf{JS}}(z)$.

7 Real continued fraction algorithms

In this chapter we will consider real continued fractions as a special case of complex continued fractions. We will define what a real continued fraction algorithm is, and we will see that a complex continued fraction algorithm is in fact also a real continued fraction algorithm. Finally we will define some well-known real continued fraction algorithms and we will find relations with the complex continued fraction algorithms of A. Hurwitz, J. Hurwitz and J. O. Shallit.

First we give the definition of a real continued fraction and prove a proposition about finite real continued fractions.

Definition 7.1. A *finite real continued fraction* is a finite complex continued fraction $[a_0, e_1/a_1, \ldots, e_n/a_n]$ where $e_k \in \{-1, 1\}$ for every $k \in \{1, \ldots, n\}$ and $a_k \in \mathbb{Z}$ for every $k \in \{0, \ldots, n\}$.

Definition 7.2. An *infinite real continued fraction* is an infinite complex continued fraction $[a_0, e_1/a_1, e_2/a_2, \ldots]$ where $e_k \in \{-1, 1\}$ for every $k \in \mathbb{N}_{>0}$ and $a_k \in \mathbb{Z}$ for every $k \in \mathbb{N}$.

Proposition 7.3. *Let $[a_0, e_1/a_1, \ldots, e_n/a_n]$ be a proper finite real continued fraction. Then:* $\mathrm{Val}([a_0, e_1/a_1, \ldots, e_n/a_n]) \in \mathbb{Q}$.

Proof. We prove this by induction. We have: $\mathrm{Val}([a_n]) = a_n \in \mathbb{Z}$, so $\mathrm{Val}([a_n]) \in \mathbb{Q}$. Now suppose $\mathrm{Val}([a_{k+1}, e_{k+2}/a_{k+2}, \ldots, e_n/a_n]) \in \mathbb{Q}$. As $[a_0, e_1/a_1, \ldots, e_n/a_n]$ is proper we have:

$$\mathrm{Val}([a_k, e_{k+1}/a_{k+1}, \ldots, e_n/a_n]) = a_k + \frac{e_{k+1}}{\mathrm{Val}([a_{k+1}, e_{k+2}/a_{k+2}, \ldots, e_n/a_n])},$$

with $e_{k+1} \in \{-1, 1\}$ and $a_k \in \mathbb{Z}$. Therefore $\mathrm{Val}([a_k, e_{k+1}/a_{k+1}, \ldots, e_n/a_n]) \in \mathbb{Q}$ and this completes the proof. □

In the remaining part of this chapter, let $\mathsf{CF}_{\mathbb{C}}$ be a complex continued fraction algorithm, let $\mathit{fl}_{\mathbb{C}}$ be the associated floor function and $\mathit{sg}_{\mathbb{C}}$ be the associated sign function.

Lemma 7.4. *If $x \in \mathbb{R}$, then $\mathit{fl}_{\mathbb{C}}(x) \in \mathbb{Z}$.*

Proof. Let $x \in \mathbb{R}$, suppose $\mathit{fl}_{\mathbb{C}}(x) \notin \mathbb{Z}$. As $\mathit{fl}_{\mathbb{C}}(x) \in \mathbb{Z}[i]$, we have $|\,\mathrm{Im}(\mathit{fl}_{\mathbb{C}}(x))| \geq 1$. On the other hand: $\mathrm{Im}(x) = 0$. Therefore: $1 \leq |\,\mathrm{Im}(\mathit{fl}(x)) - \mathrm{Im}(x)| = |\,\mathrm{Im}(\mathit{fl}_{\mathbb{C}}(x) - x)| \leq |\mathit{fl}_{\mathbb{C}}(x) - x| < 1$. So we obtain a contradiction and conclude: $\mathit{fl}_{\mathbb{C}}(x) \in \mathbb{Z}$. □

Proposition 7.5. *Let $x \in \mathbb{R}$. Suppose $\mathit{sg}_{\mathbb{C}}(\mathbb{R}) \subseteq \{-1, 1\}$, then $\mathsf{CF}_{\mathbb{C}}(x)$ is a real continued fraction.*

Proof. We assume that $x \in \mathbb{R}$ is irrational. For rational x the proof is similar. Let $\mathsf{CF}_{\mathbb{C}}(x) = [a_0, e_1/a_1, e_2/a_2, \ldots]$, and let x_k be the k-th complete quotient of x under $\mathsf{CF}_{\mathbb{C}}$.

Claim: $x_k \in \mathbb{R}$, $a_k \in \mathbb{Z}$ and $e_{k+1} \in \{-1, 1\}$ for every $k \in \mathbb{N}$.

Proof of claim: We prove this by induction. By assumption we have $x_0 = x \in \mathbb{R}$. By Lemma 7.4 we have that $\mathit{fl}_{\mathbb{C}}(x_0) \in \mathbb{Z}$, so $a_0 = \mathit{fl}_{\mathbb{C}}(x_0) \in \mathbb{Z}$. By assumption we have

$e_1 = sg_{\mathbb{C}}(x_0) \in \{-1, 1\}$. Now suppose $x_k \in \mathbb{R}$, $a_k \in \mathbb{Z}$ and $e_{k+1} \in \{-1, 1\}$. Then also $x_{k+1} = \frac{e_{k+1}}{x_k - a_k} \in \mathbb{R}$, as e_{k+1}, x_k, $a_k \in \mathbb{R}$. Then, again by Lemma 7.4 we have $a_{k+1} = fl_{\mathbb{C}}(x_{k+1}) \in \mathbb{Z}$. We also have $e_{k+2} = sg_{\mathbb{C}}(x_{k+1}) \in \{-1, 1\}$. ■

By the claim we obtain: $a_n \in \mathbb{Z}$ and $e_{n+1} \in \{-1, 1\}$ for every $n \in \mathbb{N}$. Therefore we conclude: $\mathsf{CF}_{\mathbb{C}}(x)$ is a real infinite continued fraction. □

Suppose $sg_{\mathbb{C}}(\mathbb{R}) \subseteq \{-1, 1\}$, by Proposition 7.5 it follows that $\mathsf{CF}_{\mathbb{C}}(x)$ is a real continued fraction for every $x \in \mathbb{R}$. So, in a sense, $\mathsf{CF}_{\mathbb{C}}$ is also a 'real continued fraction algorithm'. We will soon make this more precise. First we will give a definition of a real floor function and a real sign function.

Definition 7.6. Let $f : \mathbb{R} \to \mathbb{Z}$ be a function. We call f a *real floor function* if $|f(x) - x| < 1$ for every $x \in \mathbb{R}$. We call any function $g : \mathbb{R} \to \{-1, 1\}$ a *real sign function.*

Now we give a definition of a real continued fraction algorithm.

Definition 7.7. A *real continued fraction algorithm* is a real floor function fl and a real sign function sg, together with the following sequence of transformations:

$$\begin{cases} x_0 & := & x \\ a_n & := & fl(x_n) \\ e_{n+1} & := & sg(x_n) \\ x_{n+1} & := & \frac{e_{n+1}}{x_n - a_n}. \end{cases}$$

The input of such an algorithm should be a real number $x \in \mathbb{R}$. The algorithm outputs either a finite list $[a_0, e_1/a_1, \ldots, e_n/a_n]$ if $x_n - a_n = 0$ for some $n \in \mathbb{N}$. Otherwise, it outputs an infinite list $[a_0, e_1/a_1, e_2/a_2, \ldots]$.

In the remaining part of this chapter, let $\mathsf{CF}_{\mathbb{R}}$ be a real continued fraction algorithm, let $fl_{\mathbb{R}}$ be the associated real floor function and $sg_{\mathbb{R}}$ be the associated real sign function. From Definition 7.7 we immediately see that $\mathsf{CF}_{\mathbb{R}}(x)$ is a finite or infinite real continued fraction for every $x \in \mathbb{R}$. The following result gives a condition on $\mathsf{CF}_{\mathbb{C}}$ and $\mathsf{CF}_{\mathbb{R}}$ for $\mathsf{CF}_{\mathbb{C}}(x)$ to be equal to $\mathsf{CF}_{\mathbb{R}}(x)$.

Proposition 7.8. *Let $z \in \mathbb{R}$. If $fl_{\mathbb{C}}(w) = fl_{\mathbb{R}}(w)$ and $sg_{\mathbb{C}}(w) = sg_{\mathbb{R}}(w)$ for every $w \in \mathbb{R}$, then $\mathsf{CF}_{\mathbb{C}}(z) = \mathsf{CF}_{\mathbb{R}}(z)$.*

Proof. We assume $z \in \mathbb{R}$ is irrational, if z is rational, the proof is similar. Let $x := z$ and $y := z$. Let $\mathsf{CF}_{\mathbb{C}}(x) = [a_0, e_1/a_1, e_2/a_2, \ldots]$ and let x_n be the n-th complete quotient of $\mathsf{CF}_{\mathbb{C}}(x)$. Similarly, let $\mathsf{CF}_{\mathbb{R}}(y) = [b_0, f_1/b_1, f_2/b_2, \ldots]$ and let y_n be the n-th complete quotient of $\mathsf{CF}_{\mathbb{R}}(y)$.

Claim: $x_k = y_k$, $a_k = b_k$ and $e_{k+1} = f_{k+1}$ for every $k \in \mathbb{N}$.

Proof of claim: We prove this by induction. By assumption we have $x_0 = x = y = y_0$. Also: $a_0 = fl_{\mathbb{C}}(x_0) = fl_{\mathbb{R}}(y_0) = b_0$ and $e_1 = sg_{\mathbb{C}}(x_0) = sg_{\mathbb{R}}(y_0) = f_1$. Now suppose $x_k = y_k$, $a_k = b_k$ and $e_{k+1} = f_{k+1}$. Then: $x_{k+1} = \frac{e_{k+1}}{x_k - a_k} = \frac{f_{k+1}}{y_k - b_k} = y_{k+1}$. Also: $a_{k+1} = fl_{\mathbb{C}}(x_{k+1}) = fl_{\mathbb{R}}(y_{k+1}) = b_{k+1}$ and $e_{k+2} = sg_{\mathbb{C}}(x_{k+1}) = sg_{\mathbb{R}}(y_{k+1}) = f_{k+2}$. ■

Now it immediately follows from the claim that $\mathsf{CF}_{\mathbb{C}}(z) = \mathsf{CF}_{\mathbb{R}}(z)$. □

We will now see that a complex continued fraction algorithm gives rise to a real continued fraction algorithm.

Proposition 7.9. *Suppose $sg_{\mathbb{C}}(\mathbb{R}) \subseteq \{-1, 1\}$. Define $f : \mathbb{R} \to \mathbb{Z}$ by $f(z) = fl_{\mathbb{C}}(z)$ for every $z \in \mathbb{R}$. Define $g : \mathbb{R} \to \{-1, 1\}$ by $g(z) = sg_{\mathbb{C}}(z)$ for every $z \in \mathbb{R}$. Then: f, g, together with the transformations in Definition 7.7, is a real continued fraction algorithm.*

Proof. All we have to check is that f is a real floor function and g is a real sign function. As we have that $|f(z) - z| = |fl_{\mathbb{C}}(z) - z| < 1$ for every $z \in \mathbb{R}$, we have that f is a real floor function. It is clear that g is a real sign function and this ends the proof. □

We will now define (variants of) three well-known real continued fraction algorithms.

Algorithm 7.10 (Regular continued fraction)**.**
input: $x \in \mathbb{R}$,
output: a finite or infinite real continued fraction, which is generated by

$$\begin{cases} x_0 & := & x & \\ a_n & := & \lfloor x_n \rfloor & (= fl_{\mathsf{RCF}}(x_n)) \\ e_{n+1} & := & 1 & (= sg_{\mathsf{RCF}}(x_n)) \\ x_{n+1} & := & \frac{e_{n+1}}{x_n - a_n} & \end{cases}$$

We will refer to this algorithm as RCF, and denote the result of the algorithm on $x \in \mathbb{R}$ by $\mathsf{RCF}(x)$.

Remark 7.11. Algorithm 7.10 gives the regular continued fraction representation of a real number x. This is the most common continued fraction representation that can be found in the literature.

Proposition 7.12. RCF *is a real continued fraction algorithm.*

Proof. In this algorithm we have $fl_{\mathsf{RCF}}(x) = \lfloor x \rfloor$. As for every $x \in \mathbb{R}$: $|x - \lfloor x \rfloor| < 1$ it follows that fl_{RCF} is a real floor function. We have $sg_{\mathsf{RCF}}(x) = 1$ for every $x \in \mathbb{R}$ and it follows that sg_{RCF} is a real sign function. Consequently: RCF is a real continued fraction algorithm. □

Algorithm 7.13 (Nearest integer continued fraction)**.**
input: $x \in \mathbb{R}$,
output: a finite or infinite real continued fraction, which is generated by

$$\begin{cases} x_0 & := & x & \\ a_n & := & \lfloor x_n + \frac{1}{2} \rfloor & (= fl_{\mathsf{NICF}}(x_n)) \\ e_{n+1} & := & 1 & (= sg_{\mathsf{NICF}}(x_n)) \\ x_{n+1} & := & \frac{e_{n+1}}{x_n - a_n} & \end{cases}$$

We will refer to this algorithm as NICF, and denote the result of the algorithm on $x \in \mathbb{R}$ by $\mathsf{NICF}(x)$.

Remark 7.14. Algorithm 7.13 gives a variant of the nearest integer continued fraction expansion of x. This is a well-known expansion that is common in the literature. In Algorithm 7.13 all the partial numerators are assigned the value 1, where in the nearest integer continued fraction expansion of x, the partial numerators are chosen from $\{-1, 1\}$, such that the partial quotients a_n are positive for every positive index n.

Proposition 7.15. NICF *is a real continued fraction algorithm.*

Proof. Let $x \in \mathbb{R}$. We have $|fl_{\mathsf{NICF}}(x) - x| = |\lfloor x + \frac{1}{2} \rfloor - x| \leq \frac{1}{2} < 1$. Furthermore we have $sg_{\mathsf{NICF}}(x) = 1$. It follows that fl_{NICF} is a real floor function and sg_{NICF} is a real sign function. Therefore: NICF is a real continued fraction algorithm. $\square$

Now we will consider a less well known real continued fraction algorithm. First, we define the following function.

Definition 7.16. Define for every $\beta \in 2\mathbb{Z}$ the interval $I_\beta \subseteq \mathbb{R}$ as follows:

$$\begin{array}{lll} I_\beta & := [-1, 1] & \text{if } \beta = 0, \\ I_\beta & := (\beta - 1, \beta + 1] & \text{if } \beta \geq 2, \\ I_\beta & := [\beta - 1, \beta + 1) & \text{if } \beta \leq -2. \end{array}$$

We define $fl_{\mathsf{ECF}} : \mathbb{R} \to \mathbb{Z}$ as follows:

$$fl_{\mathsf{ECF}}(x) := \begin{cases} x & \text{if } x \in \mathbb{Z}, \\ \beta & \text{otherwise, where } \beta \in 2\mathbb{Z} \text{ such that } x \in I_\beta. \end{cases}$$

Algorithm 7.17 (Even continued fraction)**.**
input: $x \in \mathbb{R}$,
output: a finite or infinite real continued fraction, which is generated by

$$\begin{cases} x_0 & := & x \\ a_n & := & fl_{\mathsf{ECF}}(x_n) \\ e_{n+1} & := & -1 & (= sg_{\mathsf{ECF}}(x_n)) \\ x_{n+1} & := & \frac{e_{n+1}}{x_n - a_n} \end{cases}$$

We will refer to this algorithm as ECF, and denote the result of the algorithm on $x \in \mathbb{R}$ by $\mathsf{ECF}(x)$.

Remark 7.18. Algorithm 7.17 is a variation of the continued fraction expansion with even partial quotients. In Algorithm 7.17 all the partial numerators are assigned the value -1, where in the even continued fraction expansion of x, the partial numerators are chosen from $\{-1, 1\}$, such that the partial quotients a_n are positive for every positive index n.

Proposition 7.19. ECF *is a real continued fraction algorithm.*

Proof. Let $x \in \mathbb{R}$, suppose $x \in \mathbb{Z}$, then $fl_{\mathsf{ECF}}(x) = x$, so $|fl_{\mathsf{ECF}}(x) - x| < 1$. Now suppose $x \notin \mathbb{Z}$, let $\beta \in 2\mathbb{Z}$ such that $x \in [\beta - 1, \beta + 1]$. Then we have $fl_{\mathsf{ECF}}(x) = \beta$ and consequently: $|fl_{\mathsf{ECF}}(x) - x| < 1$. Therefore: fl_{ECF} is a real floor function. It is clean that sg_{ECF} is a real sign function. We conclude: ECF is a real continued fraction algorithm. $\square$

We have just defined three real continued fraction algorithms, namely RCF, NICF and ECF. In the previous chapters we defined the complex continued fraction algorithms $\mathsf{CF}_{\mathsf{AH}}$, $\mathsf{CF}_{\mathsf{JH1}}$, $\mathsf{CF}_{\mathsf{JH2}}$ and $\mathsf{CF}_{\mathsf{JS}}$. Note that the image of the sign functions of every of these four algorithms is either $\{-1\}$ or $\{1\}$. Therefore, by Proposition 7.5 we have that $\mathsf{CF}_{\mathsf{AH}}(x)$, $\mathsf{CF}_{\mathsf{JH1}}(x)$, $\mathsf{CF}_{\mathsf{JH2}}(x)$ and $\mathsf{CF}_{\mathsf{JS}}(x)$ are real continued fractions if $x \in \mathbb{R}$. The following propositions show some interesting relations between the real continued fraction algorithms and the complex continued fraction algorithms we have defined.

Proposition 7.20. *Let $x \in \mathbb{R}$. Then* $\mathsf{RCF}(x) = \mathsf{CF}_{\mathsf{JS}}(x)$.

Proof. According to Proposition 7.8 we only have to show: $fl_{\mathsf{JS}}(z) = fl_{\mathsf{RCF}}(z)$ and $sg_{\mathsf{JS}}(z) = sg_{\mathsf{RCF}}(z)$ for every $z \in \mathbb{R}$. Now, let $z \in \mathbb{R}$, then let $x := \mathrm{Re}(z)$, $y := \mathrm{Im}(z) = 0$. Let $a := \lfloor x \rfloor$, $b := \lfloor y \rfloor = 0$. Then: $(x - a) + (y - b) = x - \lfloor x \rfloor + (0 - 0) = x - \lfloor x \rfloor < 1$. So according to Definition 6.1 we have: $fl_{\mathsf{JS}}(z) = a + bi = \lfloor x \rfloor = \lfloor z \rfloor = fl_{\mathsf{RCF}}(z)$. By definition: $sg_{\mathsf{JS}}(z) = 1 = sg_{\mathsf{RCF}}(z)$ and this completes the proof. □

Proposition 7.21. *Let $x \in \mathbb{R}$. Then* $\mathsf{NICF}(x) = \mathsf{CF}_{\mathsf{AH}}(x)$.

Proof. According to Proposition 7.8 we only have to show: $fl_{\mathsf{AH}}(z) = fl_{\mathsf{NICF}}(z)$ and $sg_{\mathsf{AH}}(z) = sg_{\mathsf{NICF}}(z)$ for every $z \in \mathbb{R}$. Now let $z \in \mathbb{R}$, then we have that $fl_{\mathsf{AH}}(z) = \lfloor \mathrm{Re}(z) + \frac{1}{2} \rfloor + \lfloor \mathrm{Im}(z) + \frac{1}{2} \rfloor i = \lfloor z + \frac{1}{2} \rfloor + \lfloor 0 + \frac{1}{2} \rfloor i = \lfloor z + \frac{1}{2} \rfloor = fl_{\mathsf{NICF}}(z)$. We also have $sg_{\mathsf{AH}}(z) = 1 = sg_{\mathsf{NICF}}(z)$ and this ends the proof. □

Proposition 7.22. *Let $x \in \mathbb{R}$. Then* $\mathsf{ECF}(x) = \mathsf{CF}_{\mathsf{JH1}}(x) = \mathsf{CF}_{\mathsf{JH2}}(x)$.

Proof. We will show: $fl_{\mathsf{JH1}}(z) = fl_{\mathsf{ECF}}(z) = fl_{\mathsf{JH2}}(z)$ and $sg_{\mathsf{JH1}}(z) = sg_{\mathsf{ECF}}(z) = sg_{\mathsf{JH2}}(z)$ for every $z \in \mathbb{R}$. Let $z \in \mathbb{R}$. Then: $sg_{\mathsf{JH1}}(z) = -1 = sg_{\mathsf{ECF}}(z) = -1 = sg_{\mathsf{JH2}}(z)$. Let $\alpha \in 2\mathbb{Z}$ and let S_α be as defined in Section 5.1 and W_α be as defined in Section 5.2. Let I_α be as in Definition 7.16 and let $P := \mathbb{R} \setminus \mathbb{Z}$.

Claim: $S_\alpha \cap P = I_\alpha \cap P = W_\alpha \cap P$ for every $\alpha \in 2\mathbb{Z}$.

Proof of claim: Let $\alpha \in 2\mathbb{Z}$. Then $S_\alpha \cap P = (\alpha - 1, \alpha + 1) \setminus \{\alpha\}$, $I_\alpha \cap P = (\alpha - 1, \alpha + 1) \setminus \{\alpha\}$ and $W_\alpha \cap P = (\alpha - 1, \alpha + 1) \setminus \{\alpha\}$. This proves the claim. ■

Now, let $z \in \mathbb{R}$. Suppose $z \in \mathbb{Z}$, then $fl_{\mathsf{ECF}}(z) = z = fl_{\mathsf{JH1}}(z) = z = fl_{\mathsf{JH2}}(z)$. Now suppose $z \notin \mathbb{Z}$; note that in this case: $z \in P$. By applying the claim and using the fact that if $z \in P$, then $z \in S_\alpha$, for some $\alpha \in 2\mathbb{Z}$, we have:

$$\begin{aligned} fl_{\mathsf{ECF}}(z) &= \alpha, \text{where } \alpha \in 2\mathbb{Z} \text{ such that } z \in I_\alpha \\ &= \alpha, \text{where } \alpha \in (1+i)\mathbb{Z} \text{ such that } z \in S_\alpha \\ &= fl_{\mathsf{JH1}}(z). \end{aligned}$$

Similarly, by applying the claim and using the fact that if $z \in P$, then $z \in W_\alpha$, for some $\alpha \in 2\mathbb{Z}$, we have:

$$\begin{aligned} fl_{\mathsf{ECF}}(z) &= \alpha, \text{where } \alpha \in 2\mathbb{Z} \text{ such that } z \in I_\alpha \\ &= \alpha, \text{where } \alpha \in (1+i)\mathbb{Z} \text{ such that } z \in W_\alpha \\ &= fl_{\mathsf{JH2}}(z). \end{aligned}$$

Therefore: $fl_{\mathsf{JH1}}(z) = fl_{\mathsf{ECF}}(z) = fl_{\mathsf{JH2}}(z)$. By Proposition 7.8 we conclude: $\mathsf{CF}_{\mathsf{ECF}}(x) = \mathsf{CF}_{\mathsf{JH1}}(x) = \mathsf{CF}_{\mathsf{JH2}}(x)$. □

According to the last three results we have that $\mathsf{CF}_{\mathsf{JS}}$ is a generalisation of RCF, $\mathsf{CF}_{\mathsf{AH}}$ generalises NICF and both $\mathsf{CF}_{\mathsf{JH1}}$ and $\mathsf{CF}_{\mathsf{JH2}}$ are generalisations of ECF.

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