# Möbius Transformations of Complex Continued Fractions

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## Voorwoord

Het papierwerk wat u nu in handen heeft is mijn masterscriptie. Het resultaat van maanden hard werken, en de bekroning van acht jaar wiskundestudie. Deze scriptie is in zeven maanden gegroeid van vaag idee naar concreet uitgewerkt, werkend algorithme. Het doel dat in het begin gesteld is, is gehaald. Een resultaat dat ik zelf tot twee maanden voor het einde onwaarschijnlijk achtte. Gelukkig had mijn begeleider Wieb Bosma daar andere gedachten over, en zijn sturing is dan ook onmisbaar geweest in dit proces, en daar wil ik hem voor bedanken. De, in het begin, wekelijkse gesprekken leverden veel ideeën op en zorgden voor een continu leerproces waarin steeds nieuwe, kleine stapjes gezet moesten worden. Het uitvogelen van de karakteristieken van de complexe kettingbreuk was een hele uitdaging! Bedankt dus voor alle steun!

Omdat deze scriptie ook een afsluiting is van mijn studententijd wil ik graag mijn ouders bedanken voor hun steun in de gehele studieperiode. In de tweede helft van mijn studententijd kwam daar ook die van mijn vriendin Rianne bij. Nooit heb je mij veroordeeld, en altijd bleef je in me geloven, ook als daar weinig reden voor was.

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Veel plezier met lezen!

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# 0 Introduction

Continued fractions have been in use for ages now. Some even say thousands of years. Christiaan Huygens used continued fractions for his planetarium. He tried to approximate the ratios of the different orbital periods by rational numbers with smallest denominators possible, because he needed to use as little cogs on the gears as possible. For this he used continued fractions.

The odd thing about continued fractions is that although they give good approximations for real numbers, they are very hard to do arithmetic with. Adding, subtracting and multiplying continued fractions is not an easy task. Even adding a constant to a continued fraction is not always trivial. In the mid and late 20th century algorithms were devised to apply Möbius transformations on continued fractions. There even came an algorithm to add and multiply them with each other! However, these algorithms are not as easy as the ones for adding two decimal numbers. Also they only seem to work for real continued fractions. The goal of this thesis is to construct an algorithm for the complex numbers.

At first I tried what would happen in a specific case: the multiplication by 2. In the real case there already was a specific solution. This is described in chapter two. Although it was intended for the regular continued fraction I try to apply it on the nearest integer continued fraction. Allowing negative numbers as partial quotients causes the uniqueness of a continued fraction to disappear. No longer is there a bijection between the real numbers and the continued fractions. Problems that might occur and possible solutions to those are described in this chapter. The next chapter does essentially the same thing for the complex case.

In chapter five I begin to describe a different viewpoint. Continued fractions can also be represented by matrices, and so can Möbius transformations. Raney [9] makes use of this fact to produce an algorithm which does for the regular case exactly what I want to achieve for the complex case. This algorithm is described, as well as a brief description of Hall's algorithm [3], which has the same outcome as Raney's algorithm, but works a bit differently.

The algorithm by Raney was a new starting point for research on the complex case. The matrix-representation can be extended to include complex continued fractions. With these tools I try to extend Raney's algorithm. I partially succeed. An algorithm is described that generates a continued fraction that converges to the right complex number, but unfortunately the complex continued fraction is not in the right form. This gives rise to the problems described in chapter four.

Eventually I derive an algorithm which, given Hurwitz continued fraction $x$ and Möbius transformation $M$, returns the Hurwitz continued fraction of $M(x)$. This solves or circumvents all problems described in the previous chapters. The first proper chapter gives an introduction into the different continued fractions used in this thesis.

# 1 Continued Fraction Algorithms

A general continued fraction representation of a complex number is an expansion of the following form:

$$a_0 + \cfrac{e_0}{a_1 + \cfrac{e_1}{a_2 + \cfrac{e_2}{\ddots}}}$$

where $a_i$ and $e_i$ are $\in \mathbb{Z}[i]$ for all $i \in \mathbb{N}$. In this thesis all $e_i$ will be 1. When all $a_i$ are positive integers, we say it is the regular continued fraction. RCF$(x)$ is the regular continued fraction of $x$. When all $a_i$ are integers, we say it is a real continued fraction. We can calculate the regular continued fraction in the following way. Let $x \in \mathbb{R}$. Then:

$$\begin{aligned}
a_0 &= \lfloor x \rfloor \\
x_0 &= x && && (1.1) \\
x_{n+1} &= x_n - a_n && \text{for } n \geq 0 && (1.2) \\
a_{n+1} &= \lfloor x_{n+1} \rfloor && \text{for } n \geq 0
\end{aligned}$$

A shorter way to write down a continued fraction is the following:

$$a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{\ddots}}} = [a_0, a_1, a_2, \ldots]$$

Given an infinite (or finite) continued fraction $x = [a_0, a_1, \ldots ...]$ we can define $p_n$ and $q_n$ such that:

$$\frac{p_n}{q_n} = [a_0, a_1, \ldots, a_n]$$

$p_n/q_n$ will be called the $n$-th convergent. The sequence $p_n/q_n$ will converge to $x$.

## 1.1 Nearest Integer Continued Fraction

The nearest integer continued fraction (NICF) is a real continued fraction which allows negative integers as partial quotients. Instead of rounding down, the algorithm to calculate the NICF-expansion of a real number $x$ rounds to the nearest integer. In the case of a tie, it rounds to the smallest integer. Thus $\lfloor 2.5 \rceil = 2$ and $\lfloor -2.5 \rceil = -3$. So, the algorithm to find the nearest integer continued fraction of $x$ includes (1.1) and (1.2), and uses $a_0 = \lfloor x \rceil$ and $a_{n+1} = \lfloor x_{n+1} \rceil$.

In this thesis the nearest integer continued fraction might be denoted a little different from what the reader is used to. In here the nearest integer continued fraction is denoted:

$$[a_0, -a_1, a_2, -a_3] = a_0 + \cfrac{1}{-a_1 + \cfrac{1}{a_2 + \cfrac{1}{-a_3}}}$$

The following notation is more common, and for example used in Smeets [11] and Iofescu and Kraaikamp [7]:

$$\begin{aligned}[a_0, -1/a_1, -1/a_2, -1/a_3] &= a_0 + \cfrac{-1}{a_1 + \cfrac{-1}{a_2 + \cfrac{-1}{a_3}}} \\ &= a_0 + \cfrac{1}{-a_1 + \cfrac{1}{a_2 + \cfrac{1}{-a_3}}}\end{aligned}$$

The following theorem says something about what partial quotients can occur in an NICF-expansion.

**Theorem 1.3.** *A continued fraction expansion of $x$ is an NCIF-expansion if and only if for all partial quotients $a_i$ with $i > 0$, $|a_i| \geq 2$, if $a_i = 2$ then $a_{i+1} > 1$ and if $a_i = -2$, then $a_{i+1} < -1$.*

## 1.2 The Complex Continued Fraction

Complex continued fractions can be defined in various ways. One of the ways known is due to Asmus Schmidt [10]. It gives the best approximations by ratios of Gaussian integers [4, p.67], but the link with the real continued fractions is not immediately clear. An algorithm which still gives good approximations, and is a direct extension of the nearest integer continued fraction is the Hurwitz Continued Fraction [5]. This complex continued fraction is studied in this thesis. To get the Hurwitz continued fraction(HCF$(x)$) of a complex number $x$, again rules (1.1) and (1.2) apply. Again we take the nearest integer, only now, it is a Gaussian integer. In case of a tie the same rules as for the NICF-expansion are extended to the complex plane. $\lfloor -2.5 \rceil = -3$, $\lfloor 2.5 - 3.5i \rceil = 2 - 4i$. The complex plane can then be divided in squares (Figure (1)) that show which complex numbers to round to a specific Gaussian integer.

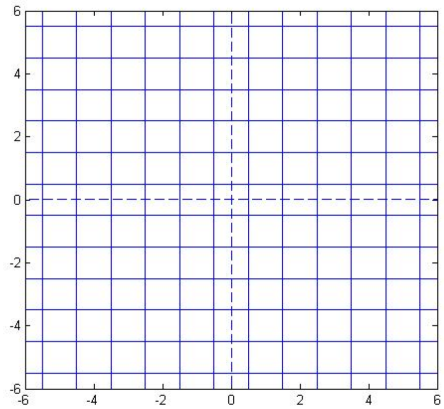

Figure 1: *Squares dividing the complex plane*

## 1.3 Some General Theorems

The following theorems hold for the general, nearest integer and Hurwitz continued fraction. Replace $x \in \mathbb{Q}$ by $x \in \mathbb{Q}[i]$ for the Hurwitz continued fraction.

**Theorem 1.4.** *For every $x \in \mathbb{Q}$ there is an $n \in \mathbb{N}$ and a regular continued fraction $[a_0, \ldots, a_n]$ such that $x = [a_0, \ldots, a_n]$.*

In the real case, if all integers are positive, there are exactly two: $x = [a_0, \ldots, a_n]$ and $x = [a_0, \ldots, a_n - 1, 1]$. If $a_n = 1$ they would be $x = [a_0, \ldots, a_{n-1} + 1]$ and $x = [a_0, \ldots, a_n]$.

For a continued fraction $[a_0, a_1, \ldots]$, its convergents can be calculated in the following way:

$$\begin{aligned} p_{-2} &= q_{-1} = 0 \\ p_{-1} &= q_{-2} = 1 \\ p_{n+1} &= a_{n+1} p_n + p_{n-1} \\ q_{n+1} &= a_{n+1} q_n + q_{n-1}. \end{aligned} \tag{1.5}$$

A few more theorems concerning the convergents:

**Theorem 1.6.** $|q_n| > |q_{n-1}|$.

**Theorem 1.7.** $\left| x - \dfrac{p_n}{q_n} \right| < \left| \dfrac{1}{q_n^2} \right|$.

We say $x = [a_0, a_1, a_2, \ldots]$ if the convergents stemming from $a_i$ converge to $x$. We can also say $[a_0, a_1, a_2, \ldots]$ converges to $x$. For each $x$ there may be more than one continued fraction that converges to it, but there is only one regular, nearest integer or Hurwitz continued fraction that converges to that $x$.

If $[a_0, a_1, a_2, \ldots]$ is the continued fraction of $x$, we define the $n$-th complete quotient of $x$ as $x_n = [a_n, a_{n+1}, a_{n+2}, \ldots]$. It is clear that $x = [a_0, a_1, \ldots, a_{n-1}, x_n]$ for every $n$. The following relation holds:

$$x = \frac{p_n x_{n+1} + p_{n-1}}{q_n x_{n+1} + q_{n-1}}$$

A Möbius transformation of a complex number $z$ is a function of the form $y = \frac{az+b}{cz+d}$ with $a, b, c, d$ complex integers, and $ad - bc \neq 0$. A few simple Möbius transformations of some expansion can easily be deduced by hand, for example multiplication by -1 or i:

$$-1 \cdot x = -a_0 + \frac{-1}{x_1} = -a_0 + \frac{1}{-x_1}$$

$$i \cdot x = ia_0 + \frac{i}{a_1 + \frac{1}{x_2}} = ia_0 + \frac{1}{-ia_1 + \frac{-i}{x_2}} = ia_0 + \frac{1}{-ia_1 + \frac{1}{ix_2}}$$

Obviously, multiplication by $-1$ is different for the regular continued fraction, because only the first partial quotient can be negative. If $\mathrm{RCF}(x) = [a_0, a_1, x_2]$ then $\mathrm{RCF}(-x) = [-a_0 - 1, 1, a_1 - 1, x_2]$. If $x = [0, a_1, a_2, a_3, \ldots]$ then $1/x = [a_1, a_2, a_3, \ldots]$. If $x = [a_1, a_2, a_3, \ldots]$ then $1/x = [0, a_1, a_2, a_3, \ldots]$. Another simple one is addition by an integer. If $k$ is an integer, then $[a_0, a_0, a_2, \ldots] + k = [a_0 + k, a_1, a_2, \ldots]$.

Although these rules might look simple, some complications can be expected. The Hurwitz continued fraction of $i \cdot [a_0, a_1, x_2]$ might be something different then $[ia_0, -ia_1, ix_2]$. If for example $x = \frac{i}{2} + \alpha$ with $0 < \alpha \in \mathbb{R}/\mathbb{Q}$, then $x = [i + \lfloor \alpha \rceil, \ldots]$. Now $ix = i\alpha - \frac{1}{2} = [\lfloor \alpha \rceil i - 0, \ldots]$ instead of $[\lfloor \alpha \rceil i - 1, \ldots]$. both continued fractions can converge to $ix$, but only the first one can be the Hurwitz continued fraction of $ix$. More on this in chapter (3).

# 2 The Nearest Integer Continued Fraction

In order to understand the Hurwitz continued fraction, it might be a good idea to start with the nearest integer continued fraction. The complex one is a direct generalisation. As an example of an easy Möbius transformation I describe in this chapter how to multiply a continued fraction by 2, and what problems might arise when diverging from the NICF-expansion.

## 2.1 Multiplication by 2

More then a hundred years ago, Hurwitz [6] provided us with a way to multiply a regular continued fraction by 2. He came up with the following two rules: Let $a$ and $b$ be any positive integer, and let $\gamma$ be any real number. Then:

$$2[0, 2a, b, \gamma] = [0, a, 2b, \frac{\gamma}{2}]$$
$$2[0, 2a+1, \gamma] = [0, a, 1, 1, \frac{\gamma-1}{2}]$$

If you write that in 'fraction-form' it looks like this:

$$\frac{2}{2a + \cfrac{1}{b + \cfrac{1}{\gamma}}} = \frac{1}{a + \cfrac{1}{2b + \cfrac{2}{\gamma}}} \tag{2.1}$$

$$\frac{2}{2a + 1 + \cfrac{1}{\gamma}} = \frac{1}{a + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{2}{\gamma - 1}}}} \tag{2.2}$$

If we now have $x = [a_0, a_1, a_2, \ldots]$, and we want to know what the continued fraction of $2x$ is, we can repeatedly apply one of these two rules to get it. We do have to pose some restrictions on $\gamma$. If $[0, 2a, b, \gamma]$ or $[0, 2a+1, \gamma]$ is a proper continued fraction, then $\gamma$ is larger then 1, and thus the expansion of $1/\gamma$ starts with 0. However, if $1 < \gamma \leq 2$, then $\gamma - 1 < 1$, and thus $\frac{1}{\gamma-1} > 1$, and the expansion of $\frac{1}{\gamma-1}$ won't start with 0. Suppose $1 < \gamma \leq 2$. Then $\gamma = [1, d, \gamma_1]$

for some $d \in \mathbb{N}$ and $\gamma \in \mathbb{R}$.

$$[1, \frac{\gamma - 1}{2}] =$$
$$= 1 + \frac{2}{\gamma - 1}$$
$$= 1 + \cfrac{2}{0 + \cfrac{1}{d + \cfrac{1}{\gamma_1}}}$$
$$= 1 + \frac{2(d + \frac{1}{\gamma_1})}{1}$$
$$= 1 + 2d + \frac{2}{\gamma_1}$$
$$= [1 + 2d, \frac{\gamma_1}{2}]$$

This also gives us a way to remove any zero from an expansion with only ones as numerators;

$$[a, b, 0, c, \gamma] = [a, b + c, \gamma]$$

Now let's try and calculate an example. The Nearest integer continued fraction of $\pi$ starts with $[3, 7, 16, -294, 3, -4, 5, \ldots]$. We know that $2\pi = [6, 4, -2, -7, -2, -146, -3, -7, 2, 2, \ldots]$. Let's try and get that result by applying the rules we just learned on the continued fraction of $\pi$. Let $x_i$ be the $i$-th complete quotient of $\pi$ for every $i \in \mathbb{N}$.

$$\begin{aligned}
&2 \cdot [3, 7, 16, -294, 3, -4, 5, x_6] \\
=&6 + 2 \cdot [0, 7, 16, -294, 3, -4, 5, x_6] \\
=&6 + 2 \cdot [0, 7, x_2] \\
=&6 + [0, 3, 1, 1, \frac{x_2 - 1}{2}] \\
x_2 - 1 =&[15, x_3] = [15, -294, 3, -4, 5, x_6] \\
\frac{2}{x_2 - 1} =&2 \cdot [0, 15, -294, 3, -4, 5, x_6] \\
=&[0, 7, 1, 1, \frac{x_3 - 1}{2}]
\end{aligned}$$

We now applied 2.2 twice. Notice that I continue with $\frac{2}{x_2-1}$, and not with $\frac{x_2-1}{2}$. This is more clear when you write it down in fractionform:

$$[0,3,1,1,\frac{x_2-1}{2}] = \cfrac{1}{3+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{\cfrac{x_2-1}{2}}}}} = \cfrac{1}{3+\cfrac{1}{1+\cfrac{1}{1+\cfrac{2}{x_2-1}}}}$$

A few more calculations yield:

$$\begin{aligned}
\frac{2}{x_3-1} &= 2\cdot[0,-295,3,-4,5,x_6] \\
&= [0,-148,1,1,\frac{x_4-1}{2}] \\
\frac{2}{x_4-1} &= 2\cdot[0,2,-4,5,x_6] \\
&= [0,1,-8,\frac{x_5}{2}] \\
\frac{2}{x_5} &= 2\cdot[0,5,x_6] \\
&= [0,2,1,1,\frac{x_6-1}{2}]
\end{aligned}$$

Now we have to construct the continued fraction of $2\pi$ out the small bits of information we have gathered.

$$\begin{aligned}
&2\cdot[3,7,16,-294,3,-4,5,x_6] \\
&= 6+[0,3,1,1,\frac{x_2-1}{2}] \\
&= [6,3,1,1,7,1,1,\frac{x_3-1}{2}] \\
&= [6,3,1,1,7,1,1,-148,1,1,\frac{x_4-1}{2}] \\
&= [6,3,1,1,7,1,1,-148,1,1,1,-8,\frac{x_5}{2}] \\
&= [6,3,1,1,7,1,1,-148,1,1,1,-8,2,1,1,\frac{x_6-1}{2}]
\end{aligned}$$

This is different from what we expected it to be! It isn't completely wrong though, this continued fraction still converges to $2\pi$. It is just not an NICF-expansion.

## 2.2 Repairing an NICF-expansion

The result of the example of the last paragraph isn't very satisfactory. If we apply the rules for multiplication by 2 on a nearest integer continued fraction, we might get some odd results. We might get a negative number after a 2 in the expansion, which is not allowed, and we can even get a 1 or a $-1$. This can be really bad, because for example theorems (1.6) and (1.7) might not be true anymore. More bad things can happen as this paragraph will show.

The following rules can be applied to remove unwanted combinations in the coefficients. The first one is also called singularization. For all $a \in \mathbb{Z}, \gamma \in \mathbb{R}$. The following holds:

$$[0, a, 1, \gamma] = \cfrac{1}{a + \cfrac{1}{1 + \cfrac{1}{\gamma}}} = \cfrac{1}{a + 1 + \cfrac{1}{-\gamma - 1}} = [0, a+1, -\gamma - 1]$$

$$[0, a, -1, \gamma] = \cfrac{1}{a + \cfrac{1}{-1 + \cfrac{1}{\gamma}}} = \cfrac{1}{a - 1 + \cfrac{1}{1 - \gamma}} = [0, a-1, 1 - \gamma]$$

There is also an equation connecting 2 and $-2$

$$[0, a, -2, \gamma] = \cfrac{1}{a + \cfrac{1}{-2 + \cfrac{1}{\gamma}}} = \cfrac{1}{a - 1 + \cfrac{1}{2 + \cfrac{1}{\gamma - 1}}} = [0, a-1, 2, \gamma - 1]$$

Summarising what we know of transforming continued fractions:

$$\begin{aligned}
[a, 0, b, \gamma] &= [a + b, \gamma] \\
[a, 1, \gamma] &= [a + 1, -\gamma - 1] \\
[a, -1, \gamma] &= [a - 1, 1 - \gamma] \\
[a, -2, \gamma] &= [a - 1, 2, \gamma - 1] \\
[a, 2, \gamma] &= [a + 1, -2, \gamma + 1]
\end{aligned} \tag{2.3}$$

With these tools, we can hope to repair any irregularities, and we can, in a finite case.

**Theorem 2.4.** *Let $x = [a_0, a_1, \ldots, a_n]$ be a general continued fraction with integer partial quotients. we can construct NICF($x$) $= [b_0, b_1, \ldots, b_m]$ without calculating $x$.*

Theorem (1.3) states that a continued fraction expansion is an NICF-expansion if we remove any 1 or $-1$, and have no $(-)2$ followed by a (positive)negative number. The equations (2.3) have a solution for each of the properties in (1.3). If you look closely, you see that upon applying one of the rules in (2.3) both the partial quotient to the left and to the right change. And if one of those numbers is changed, a new $1, -1$ or unwanted 2 or $-2$ might be created. If it is on the left side of the current number, you can hop back and replace that number as well. In the finite case this won't give any problems, as I will prove. In the infinite case it might.

Assume we are in the following situation:

$$x = [b_1, \ldots, b_k, 2, -l, a_j, \ldots, a_n]$$

with $l \in \mathbb{N}^*$, and $[b_1, \ldots, b_k, 2]$ a valid NICF-expansion. Now we apply the fourth rule, and get:

$$x = [b_1, \ldots, b_k, 2, -l, a_j, \ldots, a_n] = [b_1, \ldots, b_k + 1, -2, 1 - l, a_j, \ldots, a_n]$$

Now a problem can arise if $b_k + 1$ equals $-1, 0, 1$ or $2$. Since we are working from left to right we don't mind problems arising with $1 - l$. If $b_k + 1$ equals one of $0, 1, 2$, then $b_k$ must have been one of $-1, 0, 1$. All three are in violation with the fact that $[b_1, \ldots, b_k, 2]$ is an NICF-expansion. That leaves us with $b_k + 1 = -1$. Then $b_k = -2$. This is also not possible, because we would have had $-2$ followed by a positive number (namely 2). We now know that transforming 2 into $-2$ at index $k$ doesn't create a new problem at index $k-1$. Going from $-2$ to 2 will yield the same result in a similar proof.

Applying one of the first three rules of (2.3) however, might cause some problems. The good thing is, in this case one partial quotient is eliminated. In the finite case it means that those rules can only be applied finitely many times. Thus the algorithm works in the finite case.

The following example illustrates that it can also work in the infinite case. Let $\phi$ be the golden ratio. It's regular continued fraction is $[1, 1, 1, 1, \ldots]$. Now let's try and convert it to an NICF-expansion:

$$\begin{aligned}
\phi &= [1, 1, 1, 1, 1, 1, \ldots] \\
&= [2, -2, -1, -1, -1, -1, \ldots] \\
&= [2, -3, 2, 1, 1, 1, 1, \ldots] \\
&= [2, -3, 3, -2, -1, -1, -1, -1, \ldots] \\
&\vdots \\
&= [2, -3, 3, -3, 3, -3, 3, \ldots]
\end{aligned}$$

Also, the continued fraction of $2\pi$ of the last paragraph can be converted:

$$\begin{aligned}
&[6, 3, 1, 1, 7, 1, 1, -148, 1, 1, 1, -8, 2, 1, 1] = \\
&[6, 4, -2, -7, -1, -1, 148, -1, -1, -1, 8, -2, -1, -1] = \\
&[6, 4, -2, -8, 2, -148, 1, 1, -8, 2, 1, 1] = \\
&[6, 4, -2, -7, -2, -147, 1, 1, 1, -8, 2, 1, 1] = \\
&[6, 4, -2, -7, -2, -146, -2, -1, 8, -2, -1, -1] = \\
&[6, 4, -2, -7, -2, -146, -3, -7, 2, 1, 1] = \\
&[6, 4, -2, -7, -2, -146, -3, -7, 3, -2]
\end{aligned}$$

The last two partial quotients are still wrong, but that is because I took only the first few partial quotients in my calculations, while $\pi$ and $2\pi$ have infinitely many. It does not always work in the infinite case, as the following example illustrates. Let $x$ be $[\overline{2,-2}]$. Then $x = 2 - \frac{1}{x}$. Thus $x^2 - 2x + 1 = 0$, which means $x = 1$. It is clear that there is a much shorter way to write down $x$ . If we try and repair $x$, step by step, the following happens:

$$
\begin{aligned}
x &= [2,-2,2,-2,2\ldots] \\
&= [1,2,1,-2,2,\ldots] \\
&= [1,3,1,-2,2,\ldots] \\
&= [1,4,1,-2,2,\ldots] \\
&\vdots \\
&= [1,\infty,\ldots]
\end{aligned}
$$

Of course, the last continued fraction isn't well defined. The sequence of continued fractions show that it converges to 1, but it never really get's there.

An even more intruiging example is the following. Let $[a_0, a_1, \ldots, a_n]$ be a valid continued fraction. Now let's try and extend it with $[2, -1, 2, -a_n - 1, -a_{n-1}, \ldots, -a_0]$, and see what happens:

$$
\begin{aligned}
x &= [a_0, a_1, \ldots, a_n, 2, -1, 2, -a_n - 1, -a_{n-1}, \ldots, -a_0] \\
&= [a_0, a_1, \ldots, a_n + 1, -2, 0, 2, -a_n - 1, -a_{n-1}, \ldots, -a_0] \\
&= [a_0, a_1, \ldots, a_n + 1, 0, -a_n - 1, -a_{n-1}, \ldots, -a_0] \\
&= [a_0, a_1, \ldots, a_{n-1}, 0, -a_{n-1}, \ldots, -a_0] \\
&\vdots \\
&= [a_0, 0, -a_0] \\
&= [0]
\end{aligned}
\tag{2.5}
$$

This shows that the convergents $p_n/q_n$ of $x$ eventually end up at 0, while they were not during the process. If we copy and paste this continued fraction infinitely many times, we see that the convergents go up and down, but never converge to anything. Thus, a random continued fraction does not need to converge. Another reason to try and stick to the nearest integer, or any other well-defined expansion.

# 3 The Complex Continued Fraction

Just as with the nearest integer continued fraction, a complex number is represented by a unique Hurwitz continued fraction, but several general complex continued fractions. Actually, there are infinitely many. To recognise what is and what isn't a Hurwitz continued fraction is more difficult however.

## 3.1 Multiplication by 2

We go on with investigating the multiplication by 2. If we look closely to equalities (2.1) and (2.2) we see that $a$, $b$ and $\gamma$ don't have to be positive integers, or even real numbers. The same rules apply when $a$, $b$ and $\gamma$ are negative, or complex numbers. That gives us a foothold in trying to find a complex variant of multiplying with 2. But, since the Gauss integers can't be divided in odd and even numbers, we would need an extra set of rules. We need to know what happens when the first partial quotient is equal to $2a+i$ or $2a+i+1$ for some $a \in \mathbb{Z}[i]$. And indeed, the equalities (2.1) and (2.2) can be extended with two new ones.

Suppose that $x = [0, 2a+i, \gamma]$, with $a$, $b \in \mathbb{Z}[i]$ and $\gamma \in \mathbb{C}$. Then:

$$[0, a, 1-i, -bi-1, i+1, (i-1-\gamma)/2] =$$

$$\cfrac{1}{a+\cfrac{1}{1-i+\cfrac{1}{-bi-1+\cfrac{1}{i+1+\cfrac{2}{i-1-\gamma}}}}} = \cfrac{1}{a+\cfrac{1}{1-i+\cfrac{1}{-bi-1+\cfrac{i-1-\gamma}{-\gamma i-\gamma}}}} =$$

$$\cfrac{1}{a+\cfrac{1}{1-i+\cfrac{-\gamma i-\gamma}{i-1-\gamma-b\gamma+\gamma i+b\gamma i+c}}} = \cfrac{1}{a+\cfrac{i-1-\gamma-b\gamma+\gamma i+b\gamma i+\gamma}{2i+2b\gamma i}} =$$

$$\frac{2i+2b\gamma i}{2ai+2ab\gamma i+i-1-b\gamma+\gamma i+b\gamma i} = \frac{2+2b\gamma}{2a+2ab\gamma+b\gamma i+i+b\gamma+1+\gamma} =$$

$$\cfrac{2}{2a+i+1+\cfrac{\gamma}{b\gamma+1}} = \cfrac{2}{2a+i+1+\cfrac{1}{b+\cfrac{1}{\gamma}}} =$$

$$= 2[0, 2a+i+1, b, \gamma]$$

Similarly, we have:

$$2x = 2[0, 2a+i, \gamma] = \frac{2}{2a+i+\frac{1}{\gamma}} = \frac{2\gamma}{2a\gamma+\gamma i+1} = \frac{-2\gamma i}{-2a\gamma i+\gamma-i}$$

$$= \frac{1}{a+\frac{\gamma-i}{-2\gamma i}} = \frac{1}{a+\frac{1}{\frac{-2\gamma i}{\gamma-i}}} = \frac{1}{a+\frac{1}{-2i+\frac{2}{\gamma-i}}} = [0, a, -2i, \frac{\gamma-i}{2}]$$

So, the four rules to convert the complex continued fraction of $z$ into the continued fraction of $2z$ are:

$$\begin{aligned}
2[0, 2a, b, \gamma] &= [0, a, 2b, \frac{\gamma}{2}] \\
2[0, 2a+1, \gamma] &= [0, a, 1, 1, \frac{\gamma-1}{2}] \\
2[0, 2a+i, \gamma] &= [0, a, -2i, \frac{\gamma-i}{2}] \\
2[0, 2a+i+1, b, \gamma] &= [0, a, 1-i, -bi-1, i+1, (i-1-\gamma)/2]
\end{aligned}$$

The way these rules have to be applied is as in paragraph (2.1). In the complex case the same things can go wrong as the examples show there. Therefore studying what partial quotients can and what partial quotients can't occur in the Hurwitz continued fraction is subject of the next paragraph.

## 3.2 Secure Partial Quotients

Just like in the real case, you can try to convert an arbitrary complex continued fraction into a Hurwitz continued fraction. The following rules aid in that process. For any $a$ and $b \in \mathbb{C}$:

$$\begin{aligned}
i[a, i, b] &= [ia, 1, ib] = [ai+1, -ib-1] = i[a-i, b-i] \\
-[a, -i, b] &= [-a, i, -b] = [-a-i, -b-i] = -[a+i, b+i]
\end{aligned}$$

And, like in the real case, $2i$ can not be succeeded by $-ki$, $k \in \mathbb{N}$. The conversion here can be deduced from the real case. Let $k \in \mathbb{N}, a \in \mathbb{Z}[i], z \in \mathbb{C}$.

$$i[a, 2i, ki, z] = [ai, 2, -k, -zi] = [ai+1, -2, -k+1, -zi] = i[a-i, -2i, ki-i, z]$$

The few rules that we have are thus:

$$\begin{aligned}
[a, i, b] &= [a-i, b-i] \\
[a, -i, b] &= [a+i, b+i] \\
[a, 2i, -ki] &= [a-i, -2i, -ki-i] \\
[a, -2i, ki] &= = [a+i, 2i, ki+i]
\end{aligned}$$

However, a theorem like (1.3) is not easily obtained, and the algorithm that proves theorem (2.4) can't be extended with just these rules. Much more is needed! As you divide 1 by a real number in $[-\frac{1}{2}, \frac{1}{2})$, the outcome will be in $(-\infty, -2] \cup (2, \infty)$, explaining theorem (1.3). If you divide 1 by a complex number with real and complex values in $[-\frac{1}{2}, \frac{1}{2})$ (region $F$ in Figure (2)), the outcome (region $1/F$) is a more difficult region. The borders consist of four semi-circles with radius 1 around the points $1, -1, i, -i$. The right and lower semicircles are part of $1/F$, the other two are not. If you calculate the Hurwitz continued fraction of $x$, in any step $x_i$ will end up in $F$, and $1/x_i$ will end up in $1/F$. This means that except for the first partial quotient, you will never end up in the region between $F$ and $1/F$. That tells us something about what partial quotients can occur, and what not.

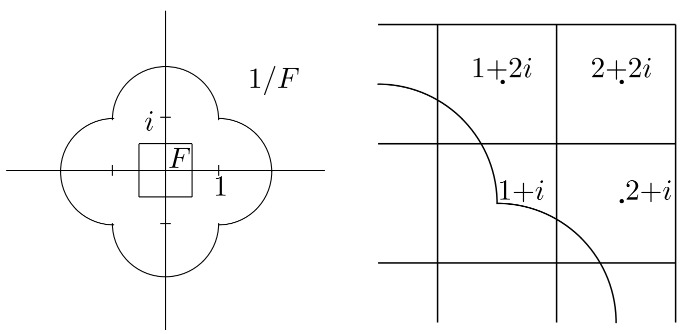


Figure 2: *The regions $F$ and $1/F$ , and a small zoom in on $1+i$*

**Definition 3.1.** *Let $[a_0, \ldots, a_{j-1}]$ be a Hurwitz continued fraction. We call the partial quotient $a_j$ secure if for every complete quotient $x_{j+1} \in 1/F$ the first $j+1$ partial quotients of the Hurwitz continued fraction of $Z_{x_{j+1}} = [a_0, \ldots, a_j, x_{j+1}]$ are $[a_0, \ldots, a_j]$.*

If for every $x_{j+1}$ the $j$-th partial quotient of $Z_{x_{j+1}}$ is not $a_j$ we say $a_j$ is not admissable. If it can go either way, we say it is insecure.

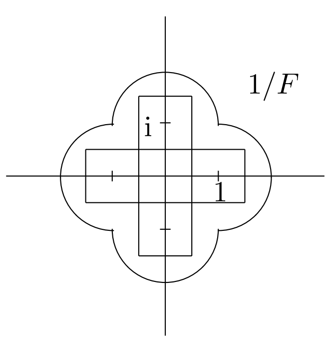


Figure 3: *Squares around $1, -1, i, -i$ and $0$ not in $1/F$*

**Theorem 3.2.** *If $a_j \in \{1, -1, i, -i, 0\}$, and $j > 0$, then $a_j$ is not secure.*

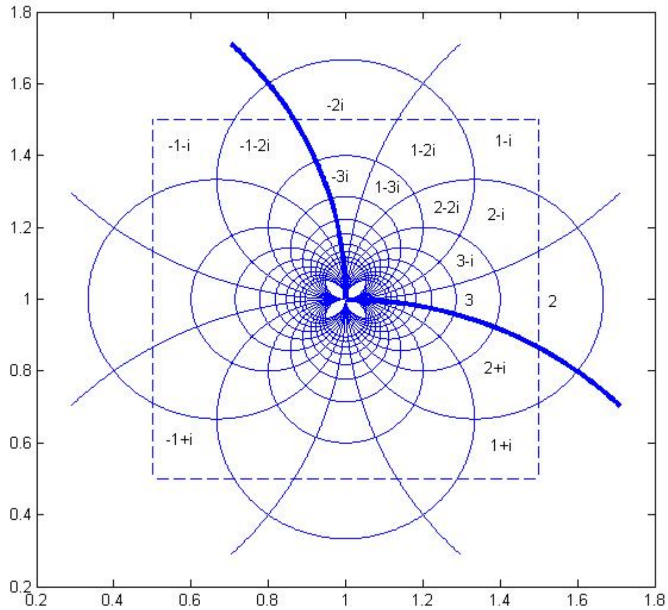


Figure 4: *Showing* $1+i+1/z$ *for* $z$ *on the lines of Figure* (1)*. The numbers show where the square around* $z$ *goes for some* $z$

*Proof.* If $j > 0$, then $x_j$ lies in the area $1/F$. The number $x_j$ can only be rounded to $a_j$ if $x_j$ lies inside one of the 5 squares around $1, -1, i, -i$ or 0. All those squares lie outside the region $1/F$. □

It gets a little more complicated with the values $2i, 1+i, 2+i, 1+2i$ and their rotations by a power of $i$. For example, it is clear that a 2 can't be succeeded by a negative real partial quotient (just as in the real case). As $\frac{1}{a+bi} = \frac{a-bi}{a^2+b^2}$, $1+i$ can be succeeded by any complex number $z$ with $\mathrm{Re}(z) \geq 1$ and $\mathrm{Im}(z) \leq 1$. It can't be succeeded by any complex number $z$ with $\mathrm{Re}(z) \leq -1$ and $\mathrm{Im}(z) \geq -1$. $2+i$ can't be succeeded by $-1+i$, while $-2+2i$ is uncertain.

Figure (4) gives some more insight. It is zoomed in on $1+i$, and shows where $1+i+1/z$ will end up. All the circles stem from either $1+i+1/(zi+1/2+k)$ or $1+i+1/(z+i/2+ki)$ for some $k \in \mathbb{Z}$ and $z \in \mathbb{R}$. Now all the areas are the images of the square around some Gaussian number. The thick lines are the edges of the clover as shown in Figure (2).

One interesting thing noticed right away is that the thick lines coincide with the rest of the figure. This is easily proven. The image of the right thick line has two preimages: $f(a) = 1+i+\dfrac{1}{a+\frac{i}{2}}$ for $a \in [0, \infty)$, and $g(a) = \dfrac{1}{\frac{1}{2}+ai}$ for

$a \in [-\frac{1}{2}, \frac{1}{2})$. Now

$$\frac{1}{1+i+\cfrac{1}{a+\cfrac{i}{2}}} = \frac{1}{i+1+\cfrac{2}{2a+i}} = \frac{2a+i}{2a+1+2ai+i} =$$

$$\frac{4a^2+4a+1-(4a^2-2a+3)i}{8a2+8a+2} = \frac{1}{2} + yi$$

with $y = \frac{-\left(4a^2+2a-3\right)}{8a^2+8a+2}$. If $a \to \infty$ , $y$ goes to $-\frac{1}{2}$. If $a = 0$, then $y = \frac{3}{2}$. The interval $[-\frac{1}{2}, \frac{3}{2}]$ overlaps $[-\frac{1}{2}, \frac{1}{2}]$, and $y$ is a continuous function of $a$, thus the image of $f(a)$ and $g(a)$ overlap. For the upper thick line a similar thing can be proven. Except for the borders we now know exactly which partial quotients are secure and which quotients are not admissable after $1+i$. For $k, l \in \mathbb{N}^*$:

| | |
|---|---|
| Not secure | $0, 1, i, -1, -i, -k+li, k+li, -k-li, -k, li$ |
| Insecure | $2, -2i, 1-i, 2-i, 1-2i$ |
| Not admissable | $k-li, k+1, -(l+1)i$ |

We are not so lucky if we look at the square around $2+i$ (Figure (5)). We

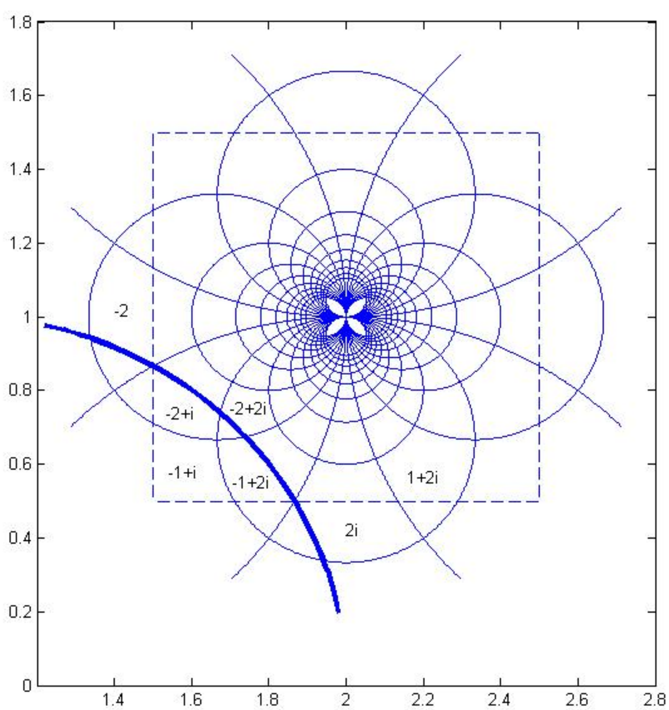


Figure 5: *Showing* $2+i+1/z$ *for* $z$ *on the lines in Figure* (1)

see here that we are not sure whether $-2+2i, -2+i$ and $-i+2i$ are secure as partial quotient, because the thick line intersects the distorted squares. The areas coinciding with the numbers $-2, -2-i, -1-i, -1-2i, -2i, 1-2i, 1-i, 2-i, 2, 2+i, 1+i, 1+2i$ and $2i$ are intersected by the edge of the square, and that is why these are also not decidable without knowing the following partial quotients. $1-i$ is not admissable as partial quotient after $2+i$.

Note that for most Gaussian numbers, the only numbers where you have doubt about whether they can be the next partial quotient are $2i, 2i-1, i-1, i-2, -2, -2-i, -1-i, -1-2i, -2i, 1-2i, 1-i, 2-i, 2, 2+i, 1+i$ and $1+2i$. These are in accordance with the areas overlapping the edge of the square, just as in Figure (5) clockwise, starting from the bottom center one. In these edges it can be very hard to find out exactly what combinations can occur or not. Figure (6) shows that $[1+i, 1-i, 2+2i]$ is a good Hurwitz continued fraction, while we don't know whether $[1+i, 1-i, 1+i]$ is. That depends on what the next partial quotients are. It also means that the $1-i$ is still insecure. This can go on for quite a while, as the next theorem suggests.

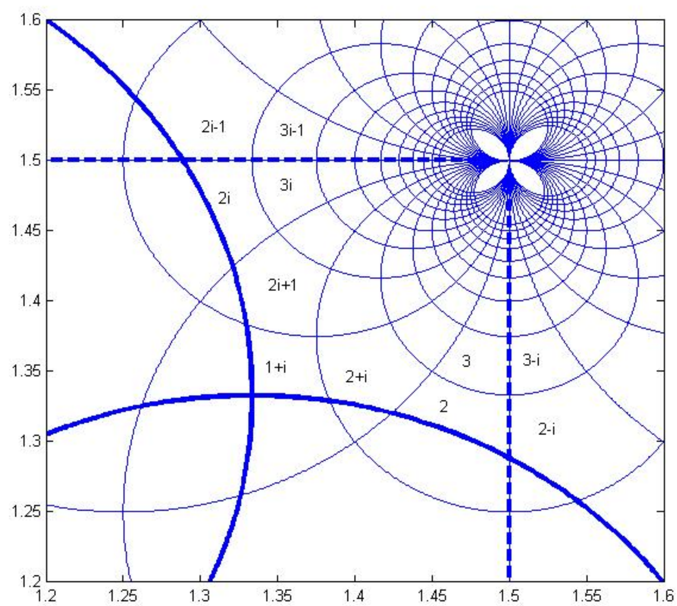


Figure 6: $1+i+1/(1-i+1/z)$ *for* $z$ *the lines in Figure* (1)

**Theorem 3.3.** *For all* $N \in \mathbb{N}$ *there is an* $n > N \in \mathbb{N}$, $x_{n+1}, y_{n+1} \in 1/F$ *and* $b_0, a_0, \ldots, a_n \in \mathbb{Z}[i]$ *such that if* $x = [a_0, \ldots, a_n, x_{n+1}]$ *and* $y = [a_0, \ldots, a_n, y_{n+1}]$ *then HCF*$(x) = [a_0, \ldots, a_n, x_{n+1}]$ *and HCF*$(y) = [b_0, \ldots]$ *with* $b_0 \neq a_0$.

This theorem says that for an insecure partial quotient it can take arbitrarily long before one can say that it is correct as part of a Hurwitz-expansion.This is proved by example:

$$V_n = [0, -2i+1, \underbrace{-2i-2, -2i+2, \ldots, -2i-2, -2i+2}_{n \text{ times } -2i-2,\, -2i+2}]$$

$$W_n = [0, -2i+1, \underbrace{-2i-2, -2i+2, \ldots, -2i-2, -2i+2}_{n \text{ times } -2i-2,\, -2i+2}, -2i-1]$$

$$Z_n = [0, -2i+1, \underbrace{-2i-2, -2i+2, \ldots, -2i-2, -2i+2}_{n \text{ times } -2i-2,\, -2i+2}, -2i-1, -3i-3]$$

Then the imaginary part of $V_n$ is smaller then $\frac{1}{2}$ for every $n \in \mathbb{N}$. While the imaginary part of $Z_n$ is bigger then $\frac{1}{2}$ for every $n \in \mathbb{N}$. The reason is that the imaginary part of $W_n$ is equal to $n$ for every $n \in \mathbb{N}$. We see that if we add $-2i-1$ and $-3i-3$ as partial quotients to $V_n$, that the imaginary part suddenly becomes greater then $\frac{1}{2}$. Thus the first partial quotient of $Z_n$ is wrong, and must be $i$ instead of 0. We see that it if you retrieve the partial quotients one by one, it might take arbitrarily long to find out whether even the first partial quotient is correct! The partial quotient $-2i+1$ is not admissable after 0 (or any Gaussian number) if it is followed by the chain of partial quotients in this example. Because it can be arbitrarily long, describing in general which partial quotients are secure requires some effort. Especially behaviour at the edges of the different regions in for example Figure 4 can be difficult to grasp.

# 4 Continued Fractions and Matrices

The idea of trying to apply Möbius-transforms to continued fractions is not new. Hall [3] devised an algorithm to do so, and so did Raney [9] and Liardet and Stambul [8]. Raney and Liardet and Stambul have one thing in common, they transform continued fractions into series of matrices in $SL_2(\mathbb{R})$. The matrices then represent the convergents.

## 4.1 Matrix-Form

We can write the rules from (1.5) in matrix-form:

$$\begin{aligned} \begin{pmatrix} p_{n-1} & p_{n-2} \\ q_{n-1} & q_{n-2} \end{pmatrix} \begin{pmatrix} 1 & a_n \\ 0 & 1 \end{pmatrix} &= \begin{pmatrix} p_{n-1} & p_{n-1}a_n + p_{n-2} \\ q_{n-1} & q_{n-1}a_n + q_{n-2} \end{pmatrix} \\ &= \begin{pmatrix} p_{n-1} & p_n \\ q_{n-1} & q_n \end{pmatrix} \end{aligned} \tag{4.1}$$

$$\begin{aligned} \begin{pmatrix} p_{n-1} & p_n \\ q_{n-1} & q_n \end{pmatrix} \begin{pmatrix} 1 & 0 \\ a_{n+1} & 1 \end{pmatrix} &= \begin{pmatrix} p_n a_{n+1} + p_{n-1} & p_n \\ q_n a_{n+1} + q_{n-1} & q_n \end{pmatrix} \\ &= \begin{pmatrix} p_{n+1} & p_n \\ q_{n+1} & q_n \end{pmatrix} \end{aligned} \tag{4.2}$$

If we note that

$$\begin{pmatrix} p_{-1} & p_{-2} \\ q_{-1} & q_{-2} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

we see that $p_n$ and $q_n$ appear in these matrices because of multiplication by matrices of two different forms. In fact, they are multiples of two matrices. Define $L$ and $R$, and multiply them with themselves to get the following result. For any $n \in \mathbb{Z}$:

$$L = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, R = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, L^n = \begin{pmatrix} 1 & 0 \\ n & 1 \end{pmatrix}, R^n = \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix}.$$

This means we can represent our continued fraction $a$ in a different way. If we calculate the matrix $R^{a_0}L^{a_1}\dots R^{a_{2n}}$ we can read out $p_n$ and $q_n$ in the right column. If we end with $L^{a_{2n+1}}$ we read them out in the left column. We now say the product of matrices $R^{a_0}L^{a_1}\dots R^{a_{2n}}$ represents the continued fraction $[a_0, a_1, \dots, a_{2n}]$. In other words we could say that the 'word' $R^{a_0}L^{a_1}\dots R^{a_{2n}}$ represents the continued fraction. In this way an infinite word can also respresent an infinite continued fraction.

A geometrically more explanatory matrix representation is found in Finch [2]. he leaves out the matrix $L$, and introduces a new matrix $B = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. Liardet and Stambul use a matrix $C_{a_n} = \begin{pmatrix} a_n & 1 \\ 1 & 0 \end{pmatrix}$. A few computations show that:

$$BR^nB = BC_n = L^n$$

We can now see that $R^{a_0}L^{a_1}R^{a_2} = C_{a_0}C_{a_1}C_{a_2} = R^{a_0}BR^{a_1}BR^{a_2}$, and so we have three different matrix forms representing the same fraction. If you look at $z = p/q$ as a vector $\begin{pmatrix} p \\ q \end{pmatrix}$ then $Bz = \begin{pmatrix} q \\ p \end{pmatrix} = 1/z$, and $R^{-n}z = \begin{pmatrix} p-nq \\ q \end{pmatrix} = (p-nq)/q$. Basically, if $F = [0,1)$ or $F = [-\frac{1}{2}, \frac{1}{2})$ is the fundamental area, some power of $R$ shifts any number back to $F$, and $B$ inverses it out to $1/F$ again.

## 4.2 Complex Matrices

It seems easy to extend Finch' representation to the complex plane. Just shift the number to the fundamental area, like in the real case. The matrix $R$ shifts everything on the real axis, so a new matrix is needed which shifts everything on the complex axis. The matrix $B$ still does its trick. A new matrix is defined:

$$R_i = \begin{pmatrix} 1 & i \\ 0 & 1 \end{pmatrix}$$

Observe that in this case the following also holds:

$$\begin{aligned} R_i^n &= \begin{pmatrix} 1 & ni \\ 0 & 1 \end{pmatrix} \\ RR_i = R_iR &= \begin{pmatrix} 1 & 1+i \\ 0 & 1 \end{pmatrix} \end{aligned}$$

Now a complex continued fraction also follows the rules as in equation (1.5), and thus $a_n$ and $a_{n+1}$ in equations (4.1) and (4.2) might as well be complex. A complex continued fraction can now also be represented using matrices. At the same time, if we define $L_i$ similar to $R_i$, we can also extend Raneys representation. Let $a_n, b_n \in \mathbb{Z}$ for all $n \in \mathbb{N}$.

$$\begin{aligned} p_n/q_n &= [a_0 + b_0 i, a_1 + b_1 i, \ldots, a_n + b_n i] \\ \begin{pmatrix} p_{n-1} & p_n \\ q_{n-1} & q_n \end{pmatrix} &= R^{a_0}R_i^{b_0}BR^{a_1}R_i^{b_1}B \ldots R^{a_n}R_i^{b_n} \end{aligned}$$

Now if using the $LR$ notation, depending on the parity of $n$:

$$\begin{aligned} \begin{pmatrix} p_{n-1} & p_n \\ q_{n-1} & q_n \end{pmatrix} &= R^{a_0}R_i^{b_0}L^{a_1}L_i^{b_1} \ldots R^{a_n}R_i^{b_n} \text{ for } n \text{ even} \\ \begin{pmatrix} p_n & p_{n-1} \\ q_n & q_{n-1} \end{pmatrix} &= R^{a_0}R_i^{b_0}L^{a_1}L_i^{b_1} \ldots R^{a_{n-1}}R_i^{b_{n-1}}L^{a_n}L_i^{b_n} \text{ for } n \text{ odd} \end{aligned}$$

## 4.3 Möbius Transformations

A different way of looking at Möbius-transformations is through matrices:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = \begin{pmatrix} az_1 + bz_2 \\ cz_1 + dz_2 \end{pmatrix}$$

If now $z = z_1/z_2$ then:

$$\frac{az_1 + bz_2}{cz_1 + dz_2} = \frac{a(z_1/z_2) + b}{c(z_1/z_2) + d} = \frac{az + b}{cz + d}$$

We see that applying a Möbius transformation $M$ to a number $z = z_1/z_2$, is essentially the same as multiplying the vector $(z_1, z_2)^\top$ by the corresponding matrix $M$. We also note that the composition of two Möbius transformations yields a new one.

$$\frac{a\dfrac{ex+f}{gx+h} + b}{c\dfrac{ex+f}{gx+h} + d} = \frac{a(ex+f) + b(gx+h)}{c(ex+f) + d(gx+h)} = \frac{(ae+bg)x + (af+bh)}{(ce+dg)x + (cf+dh)}$$

This coincides with the product of matrices:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} e & f \\ g & h \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} ae+bg & af+bh \\ ce+dg & cf+dh \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

We can now apply a Möbius transformation to a continued fraction using matrices. Note that $R, R_i, L, L_i$ and $B$ can also be seen as Möbius transformations. $R$ for example coincides with $x \mapsto x + 1$.

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} p_n & p_{n+1} \\ q_n & q_{n+1} \end{pmatrix} = \begin{pmatrix} ap_n + bq_n & ap_{n+1} + bq_{n+1} \\ cp_n + dq_n & cp_{n+1} + dq_{n+1} \end{pmatrix}$$

Now suppose $z = p_{n+1}/q_{n+1}$, then

$$\frac{ap_{n+1} + bq_{n+1}}{cp_{n+1} + dq_{n+1}} = \frac{ap_{n+1}/q_{n+1} + b}{cp_{n+1}/q_{n+1} + d} = \frac{az + b}{cz + d}$$

If $z$ is irrational, then $z$ is the limit of a series of rational convergents $p_n/q_n$ for $n \to \infty$, and if $M$ is a Möbius transformation it follows that the limit of $M(p_n/q_n)$ goes to $M(z)$ for $n \to \infty$. Again, you could see the matrix with the convergents in it as a Möbius transformation itself. We see that it might be a good idea to take a look at matrices in finding out the continued fraction for $(az+b)/(cz+d)$. This is exactly what Raney has done for real-valued continued fractions and Möbius transformations.

## 4.4 Hall

The following paragraph is just a quick overview of the algorithm by Hall. A better understanding is obtained by reading Aldenhoven [1]. When applying Möbius transformations to continued fractions, a good result is obtained in a specific real case. The following theorem holds:

**Theorem 4.3.** *Let $x, y \in \mathbb{R}\backslash\mathbb{Q}$ and $RCF(x) = [a_0, a_1, a_2, \ldots]$.*
*There is a Möbius transformation $M$ with determinant $1$ such that $x = M(y) \Leftrightarrow$*
*There exist $j$ and $k \in \mathbb{N}$, $b_0 \in \mathbb{Z}$ and $b_1, \ldots, b_k \in \mathbb{N}$ such that*
*$RCF(y) = [b_0, b_1, \ldots, b_k, a_j, a_{j+1}, a_{j+2}, \ldots]$*

Not only is this theorem a nice result in itself, but Hall also uses it in an algorithm to calculate any real Möbius transformation of a real continued fraction. He derives a method of calculating $b_0, \ldots, b_k$ and $j$. Let $M$ be a Möbius transformation with determinant $N$, $x \in \mathbb{R}\backslash\mathbb{Q}$, $y = M(x)$. Now, he finds a transformations $M'$ and $M''$ such that

$$\begin{aligned}
M' = \begin{pmatrix} e & f \\ g & h \end{pmatrix} &\quad M'' = \begin{pmatrix} t & u \\ v & w \end{pmatrix} \\
eh - fg = \pm 1 &\quad tw - uv = \pm 1 \\
x' = M'(x) &\quad y' = M''(y) \\
&\quad y' = Nx'
\end{aligned}$$

So he uses Theorem (4.3) to reduce the problem to multiplying by an integer. He then defines a finite set $\mathcal{C}$ of Möbius transformations for which he can calculate certain rules. For example, if $N = 2$ the following matrices are part of $\mathcal{C}$: (cf. [1])

$$\begin{aligned}
&y' = Ax', \quad y' = A^{-1}x', \quad A = \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}, \qquad A^{-1} = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}, \\
&y' = Bx', \quad y' = B^{-1}x', \quad B = \begin{pmatrix} 1 & 0 \\ -1 & 2 \end{pmatrix}, \quad B^{-1} = \begin{pmatrix} 2 & 0 \\ 1 & 1 \end{pmatrix}, \\
&y' = Cx', \quad y' = C^{-1}x', \quad C = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}, \quad C^{-1} = C, \\
&y' = Dx', \quad y' = D^{-1}x', \quad D = \begin{pmatrix} 2 & 1 \\ 0 & 1 \end{pmatrix}, \qquad D^{-1} = \begin{pmatrix} 1 & -1 \\ 0 & 2 \end{pmatrix}, \\
&y' = Ex', \quad y' = E^{-1}x', \quad E = \begin{pmatrix} 3 & 1 \\ 1 & 1 \end{pmatrix}, \qquad E^{-1} = \begin{pmatrix} -1 & 1 \\ 1 & -3 \end{pmatrix}, \\
&y' = Fx', \quad y' = F^{-1}x', \quad F = \begin{pmatrix} 1 & 0 \\ 2 & -2 \end{pmatrix}, \quad F^{-1} = \begin{pmatrix} 2 & 0 \\ 2 & -1 \end{pmatrix}, \\
&y' = Gx', \quad y' = G^{-1}x', \quad G = \begin{pmatrix} 2 & -1 \\ 1 & 0 \end{pmatrix}, \quad G^{-1} = \begin{pmatrix} 0 & -1 \\ 1 & -2 \end{pmatrix}.
\end{aligned}$$

Mind that an inverse here doesn't mean the inverse as matrix, but the inverse as Möbius transformation. For example $\begin{pmatrix} 4 & 0 \\ 0 & 4 \end{pmatrix}$ is as a Möbius transformation equal to $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$. The inverse of $\begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}$ as Möbius transformation is

$\begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$. He now defines the following rules:

$$\begin{array}{rlcl}
A \mapsto A^{-1}, & x' = [a, x_1'], x_1' > 2 & \to & y' = [2a, y_1'] \\
A \mapsto C, & x' = [a, 1, x_2'] & \to & y' = [2a+1, y_1'] \\
A^{-1} \mapsto A, & x' = [2a, x_1'] & \to & y' = [a, y_1'] \\
A^{-1} \mapsto C, & x' = [2a+1, x_1'] & \to & y' = [a, 1, y_2'] \\
C \mapsto D, & x' = [1, x_1'] & \to & y' = [y'] \\
C \mapsto C, & x' = [2, x_1'] & \to & y' = [2, y_1'] \\
C \mapsto D^{-1}, & x' = [x'], x' > 2 & \to & y' = [1, y_1'] \\
D \mapsto G^{-1}, & x' = [a, 1, x_2'] & \to & y' = [2a+2, y_1'] \\
D \mapsto A^{-1}, & x' = [a, x_1'], x_1' > 2 & \to & y' = [2a+1, y_1'] \\
D^{-1} \mapsto A, & x' = [2a+1, x_1'] & \to & y' = [a, y_1'] \\
D^{-1} \mapsto G, & x' = [2a+2, x_1'] & \to & y' = [a, 1, y_2'] \\
G \mapsto C, & x' = [a, 1, x_2'] & \to & y' = [2a, y_1'] \\
G \mapsto A^{-1}, & x' = [a, x_1'], x_1' > 2 & \to & y' = [2a-1, y_1'] \\
G^{-1} \mapsto A, & x' = [2a-1, x_1'] & \to & y' = [a, y_1'] \\
G^{-1} \mapsto C, & x' = [2a, x_1'] & \to & y' = [a, 1, y_2']
\end{array}$$

The way these rules have to be applied is almost the same as how the rules in (2.1) and (2.2) have to be applied. Suppose the $x = [3, 6, 7, 1, \ldots]$, then:

$$\begin{array}{rlrl}
2x &= y_0' & y_0' &= A[3, 6, 7, 1, \ldots] \\
&= [6, y_1'] & y_1' &= A^{-1}[6, 7, 1, \ldots] \\
&= [6, 3, y_2'] & y_2' &= A[7, 1, \ldots] \\
&= [6, 3, 15, y_3'] & y_3' &= C[\ldots]
\end{array}$$

So $2x = [6, 3, 15, \ldots]$.

## 4.5 Raney

A more stylized way is developed by Raney. For his algorithm, Raney uses only doubly-balanced matrices as Möbius transformations, and later proves that the other cases can be converted into one with a doubly-balanced matrix. He calls a matrix $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ row-balanced when $a, b, c$ and $d$ are nonnegative integers, the determinant $n$ is positive, and $a > c$ and $d > b$. A matrix is column-balanced when $a > b$ and $d > c$. It is doubly-balanced when it is both column-balanced and row-balanced. Let $\mathfrak{RB}_n$, $\mathfrak{CB}_n$ and $\mathfrak{DB}_n$ th denote the row-balanced, column-balanced and doubly-balanced matrices with determinant $n$, and $\mathfrak{D}_n$ all matrices with determinant $n$. Now the following theorem is very important in Raney's algorithm.

**Theorem 4.4.** *Let $Q \in \mathfrak{CB}_n$. There is a unique $P \in \mathfrak{D}_n$ such that $PQ \in \mathfrak{RB}_n$*

The proof involves subtracting matrix-rows from each other. Define $M = PQ$. When $M \notin \mathfrak{RB}_n$, it means that both entries of one row are bigger then the corresponding entries of the other. Then subtract the row as many times as needed until either the other row is bigger, or $M$ is row-balanced. When the other row is now bigger, continue on the same foot until the matrix is row-balanced. It is easy to prove that the column-balancedness isn't lost in this proces. Now $P$ is the product of the $L$ and $R$ matrices corresponding with the row-subtractions, and thus is of determinant 1.

Now suppose $M_1 \in \mathfrak{DB}_n$. Then there is an initial part $P_1$ of the matrix-representation of $x$ such that $M_1P_1 \in L \cdot \mathfrak{CB}_n$ or $M_1P_1 \in R \cdot \mathfrak{CB}_n$. Together with the previous theorem, we can now make an $M_2 \in \mathfrak{DB}_\mathfrak{n}$ and a finite product of $LR$-matrices $v_1$ such that $M_1P_1 = v_1M_2$. For any $k \in \mathbb{N}$ there exist $v_1 \dots v_k$, $P_1 \dots P_k$ and $M_k \in \mathfrak{DB}_n$, such that each $v_i$ and $P_i$ is a finite product of $LR$-matrices and

$$M(x) = MP_1P_2P_3 \dots = v_1v_2 \dots v_kM_kP_{\Pi(k)}P_{\Pi(k)+1} \cdots$$

for some permutation $\Pi$. He then makes a transducer which works for any Möbius transformation with determinant $n$, in which the transitions describe a small step in the form of $M_1v_1 = v_2M_2$. Using this transducer you can transform one sequence of matrices representing $x$ into another representing $y = M(x)$. Essentially Raneys algorithm only has a number $n$ as input, and then creates an algorithm which works for any transformation with determinant $n$. The upside of this is that for given $n$, it becomes very clear how the Möbius transformation affects continued fractions. The downside is, for large $n$, the transducer becomes big, and you might end up calculating loads of transitions you don't need.

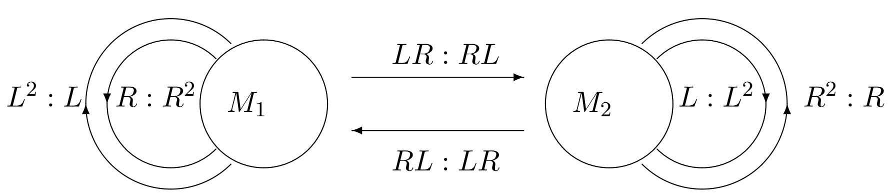


Figure 7: *Transducer for the real row-balanced Möbius transformations with determinant* 2

Let $N$ be 2. Define matrices $M_1 = \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}$ and $M_2 = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$. We now have the only 2 row-balanced matrices with determinant 2. The transducer belonging to those two matrices is shown in Figure (7). When we take $x = [3, 6, 7, 1, \dots]$, and we want to know what $2x$ is, we have to use the matrix-form of our continued fraction. Now using the transducer in Figure (7) we can calculate the following:

$$\begin{aligned}
x &= [3, 6, 7, 1, 5, \ldots] \\
M_1(x) &= M_1 \cdot R^3 L^6 R^7 L R^5 \ldots \\
&= R^6 \cdot M_1 \cdot L^6 R^7 L R^5 \ldots \\
&= R^6 L^3 \cdot M_1 \cdot R^7 L R^5 \ldots \\
&= R^6 L^3 R^{14} \cdot M_1 \cdot L R^5 \ldots \\
&= R^6 L^3 R^{15} L \cdot M_2 \cdot R^4 \ldots \\
&= R^6 L^3 R^{15} L R^2 \cdot M_2 \cdot \ldots \\
M(x) &= [6, 3, 15, 2, 2, \ldots]
\end{aligned}$$

# 5 A Complex Algorithm

The algorithm by Raney covers only the regular continued fraction, and thus it does not work for the Hurwitz expansion. That is, not in the same form. In this chapter I will describe two algorithms that do. They do not build transducers, and only calculate what they need at a given time. They produce continued fractions that converge to the right complex number, but the result is not a Hurwitz continued fraction. In the next chapter a different algorithm does produce Hurwitz continued fractions.

## 5.1 First Algorithm

In the first algorithm I circumvent the need for generalising row-balanced matrices to the complex numbers. Because I don't use them, I can't prove a theorem like theorem (4.4). Instead I will prove a similar theorem. For all $n \in \mathbb{Z}[i]$ we now define $\mathfrak{D}_n$ as the set of matrices in $\mathrm{GL}_2(\mathbb{Z}[i])$ with determinant $n$.

**Theorem 5.1.** *Let* $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ *be a matrix in* $\mathfrak{D}_n$*. There exist a matrix* $P$ *in* $\mathfrak{D}_1$*, and a matrix* $M'$ *in* $\mathfrak{D}_n$ *such that* $P_1 M' = M$ *and the first column in* $M'$ *equals either* $\begin{pmatrix} a' \\ 0 \end{pmatrix}$ *or* $\begin{pmatrix} 0 \\ a' \end{pmatrix}$ *for some* $a' \in \mathbb{Z}[i]$.

*Proof.* We start to apply the Gcd-algorithm to $a$ and $c$. Let's say we now want to subtract $c\, y_0$ times from $a$. We then have to subtract the second row of $M$ $y_0$ times from the first. Which equals to multiplying $M$ on the left by $R^{-y_{01}} R_i^{-y_{02}}$.

$$M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 1 & y_0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} a - y_0 c & b - y_0 d \\ c & d \end{pmatrix}$$

In the next step we subtract $a - y_0 c$ from $c$, which translates to multiplying with $L$ and $L_i$. After a finite number, of steps, let's say $m+1$ we end up in the following situation:

$$M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 1 & y_0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ y_1 & 1 \end{pmatrix} \cdots \begin{pmatrix} 1 & y_m \\ 0 & 1 \end{pmatrix} \begin{pmatrix} gcd(a,c) & b' \\ gcd(a,c) & d' \end{pmatrix}$$

for some $b'$ and $d'$ in $\mathbb{Z}[i]$. $r_m/s_m$ and $r_{m-1}/s_{m-1}$ are the convergents of $a/c$. The last matrix could also have $y_m$ in the lower left corner instead of the upper right, depending on the parity of $n$. In this case, it is a multiple of $L$ and $L_i$, instead of $R$ and $R_i$. Also, $r_n$ and $s_n$ are then in the left column. Notice that $R, R_i, L$ and $L_i$ all have determinant 1. If we replace $y_m$ by $y_m + 1$ to create an entry with 0 instead of $gcd(a,c)$, we're done. □

Note that we can do exactly the same with the second column, and extend theorem (5.1) accordingly.

Now let $x = [a_0 + b_0 i, a_1 + b_1 i, a_2 + b_2 i, \ldots]$, and $M$ a Möbius transformation. Let's take the first $n+1$ coefficients of $x$, with $n$ odd. We can now apply the previous theorem to $MR^{a_0}R_i^{b_0}\ldots L^{a_n}L_i^{b_n}$. We then get

$$R^{y_0}R_i^{u_0}\ldots L^{y_m}L_i^{u_m}\begin{pmatrix} gcd(a,c) & c' \\ 0 & d'' \end{pmatrix} = MR^{a_0}R_i^{b_0}\ldots L^{a_n}L_i^{b_n}$$

$$\begin{pmatrix} r_m & r_{m-1} \\ s_m & s_{m-1} \end{pmatrix}\begin{pmatrix} gcd(a,c) & c' \\ 0 & d'' \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} p_n & p_{n-1} \\ q_n & q_{n-1} \end{pmatrix}$$

$$\begin{pmatrix} gcd(a,c)r_m & r_m c' + r_{m-1}d' \\ gcd(a,c)s_m & s_m c' + s_{m-1}d' \end{pmatrix} = \begin{pmatrix} ap_n + bq_n & ap_{n-1} + bq_{n-1} \\ cp_n + dq_n & cp_{n-1} + dq_{n-1} \end{pmatrix}$$

Again the parity of $m$ decides whether 0 is in the first or the secon row. We now see that $\frac{r_m}{s_m} = \frac{ap_n+bq_n}{cp_n+dq_n}$. Since $\frac{r_m}{s_m}$ is a convergent calculated with $y_i$ and $u_i$, the continued fraction $[y_0 + u_0 i, y_1 + u_1 i, \ldots, y_m + u_m i]$ is a continued fraction representing $M(\frac{p_n}{q_n})$. So if we let $n$ go to infinity, we would get the continued fraction of $M(x)$. Since we can't really calculate that, we have to take steps. We can do this all again for the next $n+1$ coefficients of $x$. the result is the same:

$$\begin{aligned} R^{u_0}R_i^{v_0}\ldots L^{u_m}L_i^{v_m}M' &= MR^{a_0}R_i^{b_0}\ldots L^{a_n}L_i^{b_n} \\ M' = R^{-u_m}R_i^{-v_m}\ldots L^{-u_0}L_i^{-v_0}&MR^{a_0}R_i^{b_0}\ldots L^{a_n}L_i^{b_n} \\ R^{u_{m+1}}R_i^{v_{m+1}}\ldots L^{u_{m+l}}L_i^{v_{m+l}}M'' &= M'R^{a_{n+1}}R_i^{b_{n+1}}\ldots L^{a_{2n+1}}L_i^{b_{2n+1}} \\ R^{u_0}R_i^{v_0}\ldots L^{u_{m+l}}L_i^{v_{m+l}}M'' &= MR^{a_0}R_i^{b_0}\ldots L^{a_{2n+1}}L_i^{b_{2n+1}} \end{aligned}$$

$$\begin{pmatrix} r_{m+l} & r_{m+l-1} \\ s_{m+l} & s_{m+l-1} \end{pmatrix}\begin{pmatrix} gcd(a',c') & c'' \\ 0 & d'' \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} p_{2n+1} & p_{2n} \\ q_{2n+1} & q_{2n} \end{pmatrix}$$

And now we see that $\frac{r_{m+l}}{s_{m+l}} = \frac{ap_{2n+1}+bq_{2n+1}}{cp_{2n+1}+dq_{2n+1}}$. And thus we have found a continued fraction for $M(p_{2n+1}/q_{2n+1})$. If we repeat this, it will converge to a continued fraction for $M(x)$. However, the resulting continued fraction doesn't need to be the Hurwitz expansion of $M(x)$.

### 5.2 Greatest Common Divisor

I mentioned the greatest common divisor algorithm in the last paragraph, but the word 'the' might be a bit out of place. There are some choices to be made. If we for example want to know what the gcd of 14 and 3 is we normally do the following:

$$\begin{aligned} 14 &= 4 * 3 + 2 \\ 3 &= 1 * 2 + 1 \\ 2 &= 2 * 1 + 0 \end{aligned}$$

And we see that the gcd is 1. But now choose to go in a different direction:

$$\begin{aligned} 14 &= 5 * 3 - 1 \\ 3 &= -3 * -1 + 0 \end{aligned}$$

This illustrates two different ways of calculating the continued fraction of $14/3$. It can be both $[4,1,2]$ and $[5,-3]$. The regular continued fraction and the nearest integer continued fraction. The ambiguity lies within the choice of the fundamental area. In all cases, if you start subtracting a number $a$ from $b$, you want to find a $k$ such that $b-ka$ is 'smaller' then $a$. If you include negative or complex numbers, you will have to translate 'smaller' to some sort of norm, because you want $-5$ to be bigger then 3. The Euclidian norm seems fit for the job. If we now subtract 3 from 14, we see that both 2 and $-1$ are smaller then 3. Which one to choose? We look at where the quotient of the two lies. In the regular continued fraction, the quotient must lie in the interval $[0,1)$. In the nearest integer case, the quotient must lie in the interval $[-1/2,1/2)$. In the complex case, the quotient must lie in the region $F$ of Figure 2.

### 5.3 Example

Let's calculate an example. Let $x=\sqrt{3}+\sqrt{2}i$. Then $x=[i+2,-2i-1,-3i-1,20i-40,4i-1,-i+2,-3i-2,-i+2,\dots]$. Define:

$$
\begin{aligned}
P_1 &= R_iR^2L_i^{-2}L^{-1}R_i^{-3}R^{-1}L_i^{20}L^{-40} \\
P_2 &= R_i^4R^{-1}L_i^{-1}L^2R_i^{-3}R^{-2}L_i^{-1}L^2
\end{aligned}
$$

Then $x=P_1P_2\dots$. Now let's take multiplication by 3 as our Möbius-transformation. Now the following happens:

$$
\begin{aligned}
M(x) &= \begin{pmatrix} 3 & 0 \\ 0 & 1 \end{pmatrix} P_1P_2\dots \\
&= \begin{pmatrix} -1215i+1503 & 9i-42 \\ -282i+59 & 5i-4 \end{pmatrix} S_2\dots \\
&= R_i^4R^5L_i^{-2}L^2R_i^2L_i^{-6}L^{13}R_i^2R^2L_i^1L^{-1}\begin{pmatrix} i & -2i+2 \\ 0 & -3i \end{pmatrix} P_2\dots \\
&= R_i^4R^5\dots L_i^1L^{-1}\begin{pmatrix} 42i+23 & 24i+3 \\ 42i-9 & 18i-12 \end{pmatrix}\dots \\
&= R_i^4R^5\dots L_i^1L^{-1}R_i^{-1}R^1L_i^{-3}L^{-2}R_i^{-i}R^2L_i^2L^{-6}\begin{pmatrix} i & 0 \\ 0 & -3i \end{pmatrix}\dots
\end{aligned}
$$

Which gives us: $M(x)=[4i+5,-2i+2,2i,-6i+13,2i+2,i-1,-i+1,-3i-2,-i+2,2i-6,\dots]$. If $r_n$ and $s_n$ are the resulting convergents, and we compare $r_9/s_9$ to what $3x$ is in decimal form we see that they are indeed very close.

$$
\begin{aligned}
r_9/s_9 &\approx 5.196152417+4.2426406896i \\
3x &\approx 5.196152422+4.2426406871i
\end{aligned}
$$

However, if we compare our continued fraction with the actual Hurwitz expansion of $3x$, we notice some differences:

$$
\begin{aligned}
M(x) &= [4i+5,-2i+2,2i,-6i+13,2i+2,i-1,-i+1,-3i-2,\dots] \\
\mathrm{HCF}(3x) &= [4i+5,-2i+2,2i,-6i+13,i+2,-2i-1,2i+2,i-2,\dots]
\end{aligned}
$$

And it might even get worse! If we define $P_1$ to come from the first 2 partial quotients of $x$, instead of the first 4, and define $P_2, P_3$ and $P_4$ accordingly, we get the following result:

$$M(x) = [4i+5, -i+2, i, 3i, -6i+13, 2i+2, i-1, i+1, i-1, \ldots]$$
$$\mathrm{HCF}(3x) = [4i+5, -2i+2, 2i, -6i+13, i+2, -2i-1, 2i+2, i-2, \ldots]$$

This is exactly as expected. The more information we put into our Matrix before working on the rows, the more we can be sure that the first partial quotients will be correct! The problem is that there has to be a bound on how many partial quotients to take into account at one given time. And if you do pose bounds, how to know how many of the given partial quotients will be correct?

## 5.4 Second Algorithm

The algorithm described in the last paragraph sort of resembles the algorithm by Raney, but it is not a direct generalisation. A better generalisation is described here. First, I need a different definition for $\mathfrak{RB}_n$. Let $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, with $a, b, c, d \in \mathbb{Z}[i]$ and $ad - bc = n$. Then $M$ is row-balanced, or $M \in \mathfrak{RB}_n$ if ($|a| > |c|$ and $|d| > |b|$) or ($|c| > |a|$ and $|b| > |d|$). $\mathfrak{DB}_n, \mathfrak{DC}_n$ and $\mathfrak{D}_n$ are extended in the same way. Unfortunately the proof of theorem (4.4) doesn't work in the complex case. That is, when making something row-balanced, the column-balancedness isn't necessarily preserved. Therefore only row-balancedness is obtained. Theorem (5.1) is now adapted into the following theorem. Let $M \in \mathfrak{D}_n$. Then

**Theorem 5.2.** *Llet $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. There is a matrix $P \in \mathfrak{D}_1$ and a matrix $M' \in \mathfrak{RB}_n$ such that $M = PM'$*

*Proof.* Without loss of generality let $|a| \geq |c|$ and $|b| \geq |d|$. Now $M$ is not row-balanced. There are $w_1$ and/or $w_2 \in \mathbb{Z}[i]$ such that $|a - w_1 c| < |c|$ and/or $|b - w_2 d| < |d|$. Let $a' \underset{\mathrm{D}}{=} a - w_1 c$. Without loss of generality assume that $|a'| < |c|$. Now $b' \underset{\mathrm{D}}{=} a - w_1 d$. Remark that if $w_1 = w_{11} + w_{12} i$ then

$$\begin{aligned}
\begin{pmatrix} a - w_1 c & b - w_1 d \\ c & d \end{pmatrix} &= \begin{pmatrix} 1 & -w_1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} \\
&= \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}^{-w_{11}} \begin{pmatrix} 1 & i \\ 0 & 1 \end{pmatrix}^{-w_{12}} \begin{pmatrix} a & b \\ c & d \end{pmatrix} \\
&= R^{w_{11}} R_i^{w_{12}} M
\end{aligned}$$

If now $|b'| > |d|$, we are done. Now suppose $|b'| \leq |d|$. Then$|a'| < |c|, |b'| \leq |d|$, thus $|a'| + |b'| < |c| + |d|$. We can now start over, but now we subtract the first row from the second. In each step the sum of absolute vales of the entries of one row is reduced. Because the entries in the matrix are Gauss-integers, they can only be reduced finitely many times. This concludes that eventually the matrix must be row-balanced. And there is a product $P$ of $L$ and $R$ matrices, and a row-balanced matrix $M'$ such that $M = PM'$. $\square$

This proof is based on reducing the sum of the entries in one row, but there are more ways to achieve a row-balanced matrix. For example, we could reduce the maximum entry of one row, which can also be done only finitely many times. We can also create a zero, like we did in the previous paragarph, and then subtract the row with the zero in it from the other one untill row-balance is achieved. In every case, the algorithm is the same; Multiply the Möbius transformation with a number of $LR$-matrices on the right, apply row-balanced algorithm, repeat. Although computational results suggested that the resulting continued fractions converge to $M(x)$, I couldn't find a proof. The algorithm which reduces the sum of the rows does achieve a good result on the example in the previous paragraph. When multiplying $M$ with one partial quotient before balancing it, the algorithm gets the first 40 or so coefficients correct!

# 6 An Exact Algorithm

As the algorithm in section (5) showed, it is possible to create an algorithm that, given Möbius transformation $M$, and continued fraction $x$, returns a continued fraction which converges to $M(x)$. The problem is that we want to be sure that the resulting continued fraction is of the Hurwitz-form, so that the unicity and other nice properties remain. There is a method that works.

## 6.1 Möbius transformations on squares

For the final algorithm, we need to know a bit more about Möbius transformations.

**Definition 6.1.** *Given a Gauss-integer $g$, we define the square $S_g$ as the square bounded by the vertices $g+\frac{1+i}{2}, g+\frac{-1+i}{2}, g+\frac{1-i}{2}, g+\frac{-1-i}{2}$, including the line segments $l_{g1} = (g+\frac{1-i}{2}, g+\frac{1+i}{2}]$ and $l_{g2} = (g+\frac{-1+i}{2}, g+\frac{1+i}{2}]$, and excluding the rest of the boundary.*

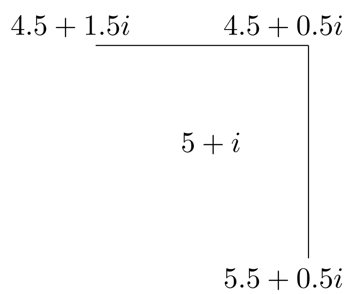


Figure 8: *The square $S_{5+i}$ in Figure* (1)

Note that $g+\frac{1+i}{2}$ belongs to $S_g$, and that $g+\frac{1-i}{2}$ and $g+\frac{-1+i}{2}$ do not. Given a transformation $M = \frac{ax+b}{cx+d}$, we now want to know whether $M$ applied to the square $S_g$ lies within the square $S_k$ for some Gaussian integer $k$.

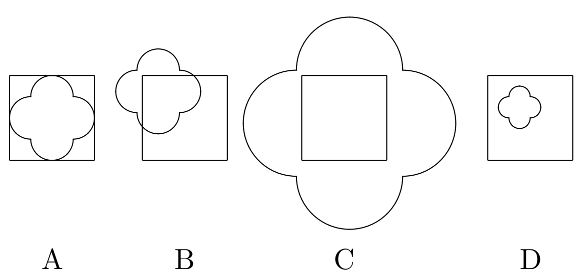


Figure 9: *Different possibilities for $M(S_g)$ related to $S_k$. Although a Möbius tranformation of a square has a clover-like appearance, most of the time it will not be as regular as it is here.*

If $g = 0$ then the Image of $M$ applied to the area $S_g$ will be unbounded, so it will most definitely not lie in $S_k$ for any $k$. Thus we can assume $g \neq 0$.

We now calculate whether the borders of $M(S_g)$ and $S_k$ intersect(B). If they dont intersect, and at least one point from the border of $M(S_g)$ lies within $S_k$, then $S_g$ lies within(D) $S_k$. If they do intersect, but the intersections are just tangents(A), and those tangents are between the lines $M(l_{g1}), M(l_{g2})$ and $l_{k1}, l_{k2}$, then the same result as the previous sentence holds. If they intersect and the intersections are not tangents, it means that part of $M(S_g)$ lies within $S_k$ and part of $M(S_g)$ lies outside $S_k$. When there are no intersections and at least one point of the border of $M(S_g)$ lies outside $S_k$ we know that $M(S_g)$ does not lie inside $S_k$.

The different possibilities for $M(S_g)$ and $S_k$ are shown in Figure (9).

Now the details. For any complex number $c$ we define $c = c_1 + c_2 i$, with $c_1, c_2 \in \mathbb{R}$.

$$\begin{aligned}
&\frac{ax+b}{cx+d} = \\
&\frac{(a_1+a_2i)(x_1+x_2i)+b_1+b_2i}{(c_1+c_2i)(x_1+x_2i)+d_1+d_2i} = \\
&\frac{(x_1a_2+b_2+x_2a_1)i+x_1a_1+b_1-x_2a_2}{(x_1c_2+d_2+x_2c_1)i+x_1c_1+d_1-x_2c_2} = \\
&\frac{((x_1a_2+b_2+x_2a_1)i+(x_1a_1+b_1-x_2a_2))}{((x_1c_2+d_2+x_2c_1)i+(x_1c_1+d_1-x_2c_2))} \cdot \\
&\frac{((x_1c_2+d_2+x_2c_1)i-(x_1c_1+d_1-x_2c_2))}{((x_1c_2+d_2+x_2c_1)i-(x_1c_1+d_1-x_2c_2))} = \\
&\left(\begin{array}{c} \frac{(x_1a_1+b_1-x_2a_2)(x_1c_1+d_1-x_2c_2)}{(x_1c_2+d_2+x_2c_1)^2+(x_1c_1+d_1-x_2c_2)^2} \\ +\frac{(x_1a_2+b_2+x_2a_1)(x_1c_2+d_2+x_2c_1)}{(x_1c_2+d_2+x_2c_1)^2+(x_1c_1+d_1-x_2c_2)^2} \end{array}\right) + \\
&\left(\begin{array}{c} \frac{(x_1a_2+b_2+x_2a_1)(x_1c_1+d_1-x_2c_2)}{(x_1c_2+d_2+x_2c_1)^2+(x_1c_1+d_1-x_2c_2)^2} \\ -\frac{(x_1a_1+b_1-y_2a_2)(x_1c_2+d_2+x_2c_1)}{(x_1c_2+d_2+x_2c_1)^2+(x_1c_1+d_1-x_2c_2)^2} \end{array}\right) \cdot i
\end{aligned} \tag{6.2}$$

We have now split $M$ in a real and an imaginary part. Let's call them $ReM(x_1, x_2)$ and $ImM(x_1, x_2)$. Both are functions of two real variables, $x_1$ and $x_2$. When applying $M$ to the border of $S_g$, the search for intersections comes down to 16 quadratic polynomials in one variable. Because we are looking at the border of a square, we can fix either the imaginary or the real part of $x$ , leaving one variable. We then equate it to one of the four edges of the square around $k$. If we take the top edge, we equate $ImM$ to $k_2 + \frac{1}{2}$, if we take the left edge, we equate $ImM$ to $k_1 - \frac{1}{2}$. If we for example want to know whether $l_{g1}$ intersects $l_{k2}$, we can take $ImM(g_1 + \frac{1}{2}, x_2) = k_2 + \frac{1}{2}$. If now for example $ReM(x_1, g_2 + \frac{1}{2}) = k_1 - \frac{1}{2}$, then we only have to check whether $g_1 - \frac{1}{2} \le x_1 < g_1 + \frac{1}{2}$, and $k_2 - \frac{1}{2} \le ImM(x_1, g_2 + \frac{1}{2}) < k_2 + \frac{1}{2}$. We now found an intersection point. If the mulitplicity of the resulting root is 1, then it is not a tangent. If we try to apply $M$ to the square $S_g$, and define $k$ as $M(g)$ we get the following 16 equations to solve:

$$\begin{array}{ll} ReM(g_1+\frac{1}{2},x_2)=k_1+\frac{1}{2} & ImM(g_1+\frac{1}{2},x_2)=k_2+\frac{1}{2} \\ ReM(g_1+\frac{1}{2},x_2)=k_1-\frac{1}{2} & ImM(g_1+\frac{1}{2},x_2)=k_2-\frac{1}{2} \\ ReM(x_1,g_2+\frac{1}{2})=k_1+\frac{1}{2} & ImM(x_1,g_2+\frac{1}{2})=k_2+\frac{1}{2} \\ ReM(x_1,g_2+\frac{1}{2})=k_1-\frac{1}{2} & ImM(x_1,g_2+\frac{1}{2})=k_2-\frac{1}{2} \\ ReM(g_1-\frac{1}{2},x_2)=k_1+\frac{1}{2} & ImM(g_1-\frac{1}{2},x_2)=k_2+\frac{1}{2} \\ ReM(g_1-\frac{1}{2},x_2)=k_1-\frac{1}{2} & ImM(g_1-\frac{1}{2},x_2)=k_2-\frac{1}{2} \\ ReM(x_1,g_2-\frac{1}{2})=k_1+\frac{1}{2} & ImM(x_1,g_2-\frac{1}{2})=k_2+\frac{1}{2} \\ ReM(x_1,g_2-\frac{1}{2})=k_1-\frac{1}{2} & ImM(x_1,g_2-\frac{1}{2})=k_2-\frac{1}{2} \end{array} \tag{6.3}$$

for $g_1-\frac{1}{2}\leq x_1<g_1+\frac{1}{2}$ and $g_2-\frac{1}{2}\leq x_2<g_2+\frac{1}{2}$.

## 6.2 The Algorithm

Let $[a_0,a_1,a_2,\ldots]$ be the continued fraction of $x$, $M(x)=y=[b_0,b_1,b_2,\ldots]$. Let's guess that the coefficients up to $a_n$ will be enough to determine what $b_0$ is. The best value for $b_0$ we can think of with this information would be $k=\lfloor\frac{a\frac{p_n}{q_n}+b}{c\frac{p_n}{q_n}+d}\rceil$.

Now according to (stelling in hfdstk1) $p_n=a_np_{n-1}+p_{n-2}$, and $q_n=a_nq_{n-1}+q_{n-2}$. Let $X_n$ be a variable in $S_{a_n}$. Because $p_n/q_n$ is a convergent of $x$, there must be a $z$ in $S_{a_n}$ such that $x=\frac{zp_{n-1}+p_{n-2}}{zq_{n-1}+q_{n-2}}$. Now there are $a',b',c',d'\in\mathbb{Z}[i]$, $z\in S_{a_n}$ such that

$$\begin{aligned} M(x) &= \frac{a\frac{zp_{n-1}+p_{n-2}}{zq_{n-1}+q_{n-2}}+b}{c\frac{zp_{n-1}+p_{n-2}}{zq_{n-1}+q_{n-2}}+d} \\ &= \frac{a(zp_{n-1}+p_{n-2})+b(zq_{n-1}+q_{n-2})}{c(zp_{n-1}+p_{n-2})+d(zq_{n-1}+q_{n-2})} \\ &= \frac{(ap_{n-1}+bq_{n-1})z+(ap_{n-2}+bq_{n-2})}{(cp_{n-1}+dq_{n-1})z+(cp_{n-2}+dq_{n-2})} \\ &= \frac{a'z+b'}{c'z+d'} \end{aligned}$$

because $p_n/q_n$ is close to $x$, $k$ will probably be close to $y_0$, but it may not be the same. However, we do know that $z\in S_{x_n}$. Thus if $\frac{a'X_n+b'}{c'X_n+d'}\in S_k$ for every $X_n\in S_{a_n}$, then $b_0=k$. The previous paragraph describes a method to do just that. This gives rise to an algorithm to find $b_0$. If a certain $a_n$ doesn't work, try $a_{n+1}$. by iteration we can also calculate the rest of the continued fraction of $y$, by changing the transformation $M$ as follows. Let $y = [b_0, b_1, b_2, \ldots]$, and let $b_0$ be known, and $b_i$ be unknown for all $i \geq 1$. Now

$$\begin{aligned}
y = M(x) = \frac{ax+b}{cx+d} &= [b_0, b_1, b_2, b_3, \ldots] \\
\frac{ax+b}{cx+d} - b_0 &= [0, b_1, b_2, b_3, \ldots] \\
\frac{(a-b_0c)x + (b-b_0d)}{cx+d} &= [0, b_1, b_2, b_3, \ldots] \\
\frac{cx+d}{(a-b_0c)x + (b-b_0d)} &= [b_1, b_2, b_3, \ldots]
\end{aligned}$$

And now we have a new Möbius transformation to calculate $b_1$ with. To relate this with earlier work, we can also write it down in matrix form:

$$\begin{aligned}
M(x) &= MR^{a_{01}}R_i^{a_{02}}L^{a_{11}}L_i^{a_{12}}R^{a_{21}}R_i^{a_{22}}... \\
&= R^{b_{01}}R_i^{b_{02}}R^{-b_{01}}R_i^{-b_{02}}MR^{a_{01}}R_i^{a_{02}}L^{a_{11}}L_i^{a_{12}}R^{a_{21}}R_i^{a_{22}}... \\
&= R^{b_{01}}R_i^{b_{02}}BBR^{-b_{01}}R_i^{-b_{02}}MR^{a_{01}}R_i^{a_{02}}L^{a_{11}}L_i^{a_{12}}R^{a_{21}}R_i^{a_{22}}... \\
&= R^{b_{01}}R_i^{b_{02}}BM'R^{a_{01}}R_i^{a_{02}}L^{a_{11}}L_i^{a_{12}}R^{a_{21}}R_i^{a_{22}}...
\end{aligned}$$

When applying this algorithm, $n$ and the elements of $M$ can become quite large. there is a way to counter that. In the algorithm, $M$ is altered by an $b_i$ you involve, but you can do the same thing with an extra $a_i$. Now:

$$\frac{ax+b}{cx+d} = \frac{a(a_0 + \frac{1}{x_1}) + b}{c(a_0 + \frac{1}{x_1}) + d} = \frac{\frac{a}{x_1} + aa_0 + b}{\frac{c}{x_1} + ca_0 + d} = \frac{(aa_0+b)x_1 + a}{(ca_0+d)x_1 + c}$$

Which is quite easy to see in matrix-form:

$$\begin{aligned}
M(x) &= MR^{a_{01}}R_i^{a_{02}}L^{a_{11}}L_i^{a_{12}}R^{a_{21}}R_i^{a_{22}}L^{a_{31}}L_i^{a_{32}} \\
&= MR^{a_{01}}R_i^{a_{02}}BR^{a_{11}}R_i^{a_{12}}BR^{a_{21}}R_i^{a_{22}}BR^{a_{31}}R_i^{a_{32}}B \\
&= (MR^{a_{01}}R_i^{a_{02}}B)R^{a_{11}}R_i^{a_{12}}BR^{a_{21}}R_i^{a_{22}}BR^{a_{31}}R_i^{a_{32}}B \\
&= (M')R^{a_{11}}R_i^{a_{12}}L^{a_{21}}L_i^{a_{22}}R^{a_{31}}R_i^{a_{32}}B
\end{aligned}$$

## 6.3 Example

To illustrate the mechanism of this algorithm I will apply it to the same example as used in the previous chapter. Let $x = \sqrt{3} + \sqrt{2}i = [i+2, -2i-1, -3i-1, 20i-40, 4i-1, -i+2, -3i-2, -i+2, \ldots]$, and let $M$ be multiplication by 3. Now according to equation (6.2), $ReM = 3z_1$. and $ImM = 3z_2$. $g = i+2$, and $k$ becomes $3 \cdot i + 2 = 3i + 6$. We have to check whether the equations in (6.3) have solutions:

$$\begin{aligned}
2.5 &= 6.5 & z_2 &= 3.5 \\
2.5 &= 5.5 & z_2 &= 2.5 \\
z_1 &= 6.5 & 1.5 &= 3.5 \\
z_1 &= 5.5 & 1.5 &= 2.5 \\
1.5 &= 6.5 & z_2 &= 3.5 \\
1.5 &= 5.5 & z_2 &= 2.5 \\
z_1 &= 6.5 & 0.5 &= 3.5 \\
z_1 &= 6.5 & 0.5 &= 2.5
\end{aligned}$$

for $1.5 \le z_1 < 2.5$ and $0.5 \le z_2 < 1.5$. We see that there are no intersections possible in the right area. Yes there is a solution for $z_1 = 6.5$, but if $z_1 = 6.5$ it is not in the right interval($[1.5, 2.5]$). We now have to check whether the border of the image of $M(S_g)$ is completely in $S_k$ or completely outside $S_k$. We take one vertex from the border, namely $1.5 + 0.5i$ and apply $M = 3z$. The result is $4.5 + 1.5i \notin S_k$. Thus we can conclude nothing yet about the first partial quotient of $3x$.

Incorporating the next partial quotient will yield $5 + 4i$ as first partial quotient of $3x$, but it will prove not be enough to be completely certain. I skip checking whether 2 partial quotients of $x$ is enough, and go on to incorporating the third partial quotient of $x$. This will prove sufficient. Now $p_0 = p_{2-2} = i + 2$, $p_1 = p_{2-1} = -5i + 1$, $q_0 = q_{2-2} = 1$, $q_1 = q_{2-1} = -2i - 1$, thus now for some $z = z_1 + z_2 i \in S_{-3i-1}$

$$\begin{aligned}
&M(\sqrt{3} + \sqrt{2}i) = 3\frac{zp_1 + p_0}{zq_1 + q_0} \\
=&\frac{(-15i + 3)z + 3i + 6}{-(2i + 1)z + 1} \\
=&\frac{(3z_1 + 15z_2 + 6)(-z_1 + 2z_2) + (-15z_1 + 3z_2 + 3)(-2z_1 - z_2)}{(-2z_1 - z_2)^2 + (-z_1 + 2z_2 + 1)^2} \\
+&\frac{(-z_1 + 2z_2 + 1)(-15z_1 + 3z_2 + 3) - (3z_1 + 15z_2 + 6)(-2z_1 - z_2)}{(-2z_1 - z_2)^2 + (-z_1 + 2z_2 + 1)^2} i \\
=&\frac{27z_1^2 - 9z_1 + 27z_2^2 + 24z_2 + 6}{5z_1^2 - 2z_1 + 5z_2^2 + 4z_2 + 1} \\
+&\frac{21z_1^2 - 6z_1 + 21z_2^2 + 15z_2 + 3}{5z_1^2 - 2z_1 + 5z_2^2 + 4z_2 + 1} i \\
=&ReM(z_1, z_2) + ImM(z_1, z_2)i
\end{aligned}$$

With $g = -3i - 1$ and $k = 4i + 5$ we solve the 16 equations in 6.3;

$$\begin{aligned}
\frac{108z_2^2 + 96z_2 + 69}{20z_2^2 + 16z_2 + 13} &= \frac{11}{2} & \frac{84z_2^2 + 60z_2 + 45}{20_2^2 + 16z_2 + 13} &= \frac{9}{2} \\
\frac{108z_2^2 + 96z_2 + 69}{20z_2^2 + 16z_2 + 13} &= \frac{9}{2} & \frac{84z_2^2 + 60z_2 + 45}{20_2^2 + 16z_2 + 13} &= \frac{7}{2} \\
\frac{108z_1^2 - 36z_1 + 459}{20z_1^2 - 8z_1 + 89} &= \frac{11}{2} & \frac{84z_1^2 - 24z_1 + 387}{20z_12 - 8z_1 + 89} &= \frac{9}{2} \\
\frac{108z_1^2 - 36z_1 + 459}{20z_1^2 - 8z_1 + 89} &= \frac{9}{2} & \frac{84z_1^2 - 24z_1 + 387}{20z_12 - 8z_1 + 89} &= \frac{7}{2} \\
\frac{108z_2^2 + 96z_2 + 321}{20z_2^2 + 16z_2 + 61} &= \frac{11}{2} & \frac{84z_2^2 + 60z_2 + 237}{20_2^2 + 16z_2 + 61} &= \frac{9}{2} \\
\frac{108z_2^2 + 96z_2 + 321}{20z_2^2 + 16z_2 + 61} &= \frac{9}{2} & \frac{84z_2^2 + 60z_2 + 237}{20_2^2 + 16z_2 + 61} &= \frac{7}{2} \\
\frac{108z_1^2 - 36z_1 + 1011}{20z_1^2 - 8z_1 + 193} &= \frac{11}{2} & \frac{84z_1^2 - 24z_1 + 831}{20z_12 - 8z_1 + 193} &= \frac{9}{2} \\
\frac{108z_1^2 - 36z_1 + 1011}{20z_1^2 - 8z_1 + 193} &= \frac{9}{2} & \frac{84z_1^2 - 24z_1 + 831}{20z_12 - 8z_1 + 193} &= \frac{7}{2}
\end{aligned} \tag{6.4}$$

The roots of these equations are, in the same order as (6.4), with a - indicating there are no real roots:

| | | | | | |
|---|---|---|---|---|---|
| $z_2 =$ | 0.341... | $\vee$ 3.658... | $z_2 =$ | -... | $\vee$ -... |
| $z_2 =$ | - | $\vee$ - | $z_2 =$ | -0.379 | $\vee$ -0.094 |
| $z_1 =$ | - | $\vee$ - | $z_1 =$ | - | $\vee$ - |
| $z_1 =$ | - | $\vee$ - | $z_1 =$ | - | $\vee$ - |
| $z_2 =$ | - | $\vee$ - | $z_2 =$ | - | $\vee$ - |
| $z_2 =$ | - | $\vee$ - | $z_2 =$ | - | $\vee$ - |
| $z_1 =$ | - | $\vee$ - | $z_1 =$ | - | $\vee$ - |
| $z_1 =$ | - | $\vee$ - | $z_1 =$ | - | $\vee$ - |

Now there are no roots with $-1.5 \leq z_1 < -0.5$ or $-3.5 \leq z_2 < -2.5$. Also $ReM(-1/2, -7/2) + ImM(-1/2, -7/2) \cdot i = 5.227\ldots + 4.277\ldots i$, and thus inside $S_{5+4i}$. Thus we can now conclude that the first partial quotient of $3x$ is $5 + 4i$. For the second partial quotient of $3x$ we now use the matrix $R^{-5}R_i^{-4}M$ and apply it to the continued fraction of $x$. However, we can also take the matrix $R^{-5}R_i^{-4}MR^2R_i$ and apply it to the continued fraction of $\frac{1}{x-(i+2)}$. The result is the following:

$$\begin{aligned}
R^{-5}R_i^{-4}M &= \begin{pmatrix} 3 & -4i - 5 \\ 0 & 1 \end{pmatrix} \\
R^{-5}R_i^{-4}MR^2R_i &= \begin{pmatrix} 3 & -i + 1 \\ 0 & 1 \end{pmatrix}
\end{aligned}$$

We see that the entries in second matrix are smaller then the entries in the first. When calculating more partial quotients of $3x$ the difference can get even bigger:

$$
\begin{aligned}
3x &= [4i+5, -2i+2, 2i, \ldots] \\
x &= [2+i, -1-2i, \ldots] \\
R_i^2 L^2 L_i^2 R^{-5} R_i^{-4} M &= \begin{pmatrix} -24i-57 & 122i+71 \\ -24i+18 & 20i-65 \end{pmatrix} \\
R_i^2 L^2 L_i^2 R^{-5} R_i^{-4} M R^2 R_i L^{-1} L_i^{-2} &= \begin{pmatrix} -3i-4 & 17i-19 \\ -4i+3 & -10i-5 \end{pmatrix}
\end{aligned}
$$

We see that it is a good idea to incorporate a part of the continued fraction of $x$ in the new matrix $M$ everytime. How big that part should be to get the smallest possible numbers is not clear. It is also not clear how many partial quotients of $x$ are needed to find out one partial quotient of $y$. Maybe it is possible to create a function $\mathfrak{C}(M)$ that returns the amount of partial quotients needed. In that case the algorithm would become much easier. The current algorithm only provides a check to see whether a partial quotient is correct, If you would know that $N = \mathfrak{C}(M)$ partial quotients are enough you already know that $\lfloor M(p_N/q_N) \rceil$ is the correct first partial quotient of $M(x)$. The following example suggests that it is not possible in every case.

When taking $x = \frac{13+2i\sqrt{(5)}}{6}$, and $M = 3x$, the continued fraction of $M(x)$ starts with $-2i-7$. But $\lfloor \frac{3p_{200}}{q_{200}} \rceil = -2i-6$. In this case the algorithm wouldn't have found even the first partial quotient of $M(x)$ while it has an input of 201 partial quotients of $x$. It is not very likely to find the right partial quotient at all! I suspect this kind of behaviour only happens when $M(x)$ is on the edge of a square $S_g$ for some $g \in \mathbb{Z}[i]$. Nevertheless, if this example would work for any number, and not only 200, this would have serious consequences. There would be **no** algorithm that could find the right partial quotient for every Möbius transformation of every complex number $x$, given only finitely many partial quotients of $x$!

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