Radboud Universiteit

Bachelor Thesis

Self-Complementary Graphs and Digraphs

In search of a natural bijection

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Abstract

The number of self-complementary digraphs with $2n$ vertices is equal to the number of self-complementary graphs with $4n$ vertices, for arbitrary $n \in \mathbb{N}$. It is likely that there exists a natural bijection between those sets. In this thesis, a class of algorithms is considered. Each step of the algorithms seems to lead to a natural bijection. However, it is proven that these algorithms can not be part of the desired natural bijection.

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1 Introduction

One subject in graph theory is the study of self-complementary directed and undirected graphs. These are graphs which are isomorphic to their complementary graph. For a given $n \in \mathbb{N}$ one might ask how many self-complementary graphs and directed graphs – also called digraphs – exist on n vertices. In 1963, R. C. Read published a paper [5] in which these numbers are enumerated for arbitrary $n \in \mathbb{N}$. The enumeration discloses the curious fact that, for each $n \in \mathbb{N}$, the number of self-complementary graphs with $4n$ vertices is equal to the number of self-complementary digraphs with $2n$ vertices. In 1975, D. Wille discovered a similar result. He proved that the number of self-complementary graphs with $4n + 1$ vertices is equal to the number of self-complementary relations over $2n$ elements [6].

In 1987, Read claims that it is likely that a natural bijection between the self-complementary digraphs with $2n$ vertices and the self-complementary graphs with $4n$ vertices exists [4]. This means that a general procedure exists which maps the self-complementary digraphs with $2n$ vertices onto the self-complementary graphs with $4n$ vertices for all $n \in \mathbb{N}$. Several mathematicans tried to find such a natural bijection, but so far no one succeeded (consider for example [1],[7]). In this thesis we will try to find such a natural bijection.

This thesis is structured as follows. In Chapter 2 we will review some basic definitions. Chapter 3 gives more details about related published results. In Chapter 4 a class of algorithms will be considered. Each step of these algorithms seems to be an obvious step towards a natural bijection. However, we will prove that none of the resulting algorithms leads to the desired natural bijection. Finally, this thesis gives ideas for further research.

At this point I would like to thank my supervisor Wieb Bosma for his enjoyable guidance and for giving helpful new ideas and suggestions regarding the topic.

2 Preliminaries

In this section we briefly review some basic notions of graph theory.

We will start with the fundamental ideas of an undirected graph and a directed graph.

Definition 1. *An undirected graph, also called a* ***graph****, is an ordered pair* $G = (V, E)$ *where* V *is a finite set of vertices and* E *a finite set of undirected pairs* $\{a, b\}$ *with* $a, b \in V$*. These undirected pairs are called edges.*

Definition 2. *A directed graph, also called a* ***digraph****, is an ordered pair* $D = (V, E)$ *where* V *is a finite set of vertices and* E *a finite set of directed pairs* (a, b) *with* $a, b \in V$*. These directed pairs are called arrows.*

Figure 2.1 shows a graph and a digraph along with their visual representations.

G={{1,2,3,4},{{1,2},{1,3},{3,4}}}

D={{1,2,3,4},{(1,2),(3,2),(2,3),(3,1),(3,4),(2,4)}}

Figure 2.1: A graph (left) and a digraph (right).

The considered graphs and digraphs of this thesis have to satisfy certain properties. These are determined in the following definitions.
Note, that every pair of vertices shown in Figure 2.1 is connected by a path. This leads to the following definitions for connected graphs and weakly connected digraphs.

Definition 3. *A graph* $G = (V, E)$ *is called* ***connected*** *if there exist a path between each pair of vertices. Thus, for each pair of vertices* $a, b \in V$ *there exists* $x_1, x_2, \dots, x_m \in V$*, for* $m \in \mathbb{N}$*, such that* $\{a, x_1\}, \{x_1, x_2\}, \dots, \{x_m, b\} \in E$*.*

Definition 4. *A digraph* $D = (V, E)$ *is called* ***weakly connected*** *if for each pair of vertices* $a, b \in V$ *there exist vertices* $x_0, x_1, \dots, x_m \in V$*, for* $m \in \mathbb{N}$*, such that* $x_0 = a$*,* $x_m = b$ *and* $\forall i : 1 \leq i \leq m$ *either* $(x_{i-1}, x_i) \in E$ *or* $(x_i, x_{i-1}) \in E$*.*

Hence, the graph given in Figure 2.1 is connected and the digraph given in Figure 2.1 is weakly connected.
Another property of the graph given in Figure 2.1 is that every edge joins two distinct vertices. The same is true for the digraph given in Figure 2.1, with arrows instead of edges. Each arrow points from a certain vertex to a different vertex. To describe this rigorously, an edge, and analogously an arrow, which joins a vertex to itself is called a loop.

Definition 5. *Let G be a graph, let D be a digraph, let v be a vertex in G and let w be a vertex in D. Then a **loop** in v is an edge $\{v, v\}$, and a loop in w is an arrow (w, w).*

G={{1,2,3,4},{{1,3},{1,3},{2,2},{3,4},{2,4}}} D={{1,2,3,4},{(2,1),(1,3),(1,3),(3,4),(4,3)(2,2)}}

Figure 2.2: A graph with a loop (left) and a digraph with a loop (right).

Figure 2.2 shows a graph with a loop in vertex $v = 2$ and a digraph with a loop in vertex $w = 2$.

In this thesis, we assume graphs to be connected and digraphs to be weakly connected, if not mentioned otherwise. Furthermore, we assume that graphs and digraphs do not contain loops.
The following definitions are required to define a self-complementary graph and digraph.

Definition 6. *Let $G = (V, E)$ be a graph, let E^{tot} be the set of all possible edges and let E^c be $E^{tot} \setminus E$. Then the complementary graph of G is defined as (V, E^c) and is denoted as G^c.*

The following definition shows how to compare graphs.

Definition 7. *Two graphs $G = (V, E)$ and $G' = (V', E')$ are called **isomorphic** if there is a bijection $f : V \to V'$ such that $\{a, b\} \in E$ if and only if $\{f(a), f(b)\} \in E'$. If G and G' are isomorphic, this is denoted as $G \cong G'$.*

Two analogous definitions are needed for digraphs.

Definition 8. *Let $D = (V, E)$ be a digraph, let E^{tot} be the set of all possible arrows and let E^c be $E^{tot} \setminus E$. Then the complementary digraph of D is defined as (V, E^c) and is denoted as D^c.*

Definition 9. *Two digraphs $D = (V, E)$ and $D' = (V', E')$ are called **isomorphic** if there is a bijection $f : V \to V'$ such that $(a, b) \in E$ if and only if $(f(a), f(b)) \in E'$. If D and D' are isomorphic, then this is denoted as $D \cong D'$.*

Combining these definitions we obtain the definition for a self-complementary graph and digraph.

Definition 10. *A **self-complementary graph** is a graph G that is isomorphic to its complementary graph G^c. A **self-complementary digraph** is a digraph D that is isomorphic to its complementary digraph D^c.*

The graph and digraph given in Figure 2.1 are both self-complementary.

To prove certain properties about self-complementary graphs and digraphs, it is useful to define the degree of a vertex as the number of edges, or arrows, which touch this vertex.

Definition 11. *Let G be a graph and let v be a vertex of G. Then the* ***degree*** *of vertex v is the number of $e \in E$ such that $v \in e$.*

Translating this definition to digraphs, we have to differentiate whether the arrow points towards or away from the considered vertex. Therefore, each vertex of a digraph has an in- and an outdegree.

Definition 12. *The* ***indegree*** *of a vertex v of a digraph D is the number of arrows which end in v.*

Definition 13. *The* ***outdegree*** *of a vertex v of a digraph D is the number of arrows which start in v.*

The degrees of all vertices of a graph can be noted efficiently as a sequence.

Definition 14. *Let $G = (V, E)$ be a graph with $\#V = p$. Let, for $1 \leq i \leq p$, the degree of the i^{th} vertex be γ_i. Then is $(\gamma_1, \gamma_2, ..., \gamma_p)$ the* ***degree sequence*** *of G.*

Translated to the digraphs, we have to differentiate between the indegree and the outdegree. This leads to the following definition.

Definition 15. *Let $D = (V, E)$ a digraph with $\#V = p$. Let, for $1 \leq i \leq p$, the indegree of the i^{th} vertex be α_i and the outdegree β_i. Then $(\alpha_1, \alpha_2, ..., \alpha_p)$ is defined as the* ***indegree sequence*** *and $(\beta_1, \beta_2, ..., \beta_p)$ is defined as the* ***outdegree sequence*** *of D.*

Finally, some further definitions are given that are required for subsequent proofs.

Definition 16. *A* ***complete graph*** *is a graph $G = (V, E)$ such that for each pair $a, b \in V$ it holds that $\{a, b\} \in E$.*

Definition 17. *A* ***complete digraph*** *is a digraph $D = (V, E)$ such that for each pair $a, b \in V$ it holds that $(a, b) \in E$ and $(b, a) \in E$.*

Definition 18. *Let $D = (V, E)$ be a digraph. Let E^r be a set such that $(a, b) \in E^r$ if and only if $(b, a) \in E$. Then the* ***reverse graph*** *of D is defined as $D^r = (V, E^r)$.*

Definition 19. *Let $G = (V, E)$ be a graph. Each graph $G^\star = (V^\star, E^\star)$ with $V^\star \subset V$ and $E^\star \subset E$ is called a* ***subgraph*** *of G.*

3 Theoretical approach

A theorem by Read [5] states that, for arbitrary $n \in \mathbb{N}$, the set of self-complementary graphs with $4n$ vertices is equinumerous to the set of self-complementary digraphs with $2n$ vertices. Figure 3.1 shows the number of graphs for some $n \in \mathbb{N}$.

n	# sc-graphs with $4n$ vertices = # sc-digraphs with $2n$ vertices
1	1
2	10
3	720
4	703760
5	9168331776
6	1601371799340544

Figure 3.1: Number of self-complementary (sc) graphs and digraphs for some $n \in \mathbb{N}$.

Read is not the only person who has proven this equinumerosity. In 1984, C. R. J. Clapham published a paper [1] wherein the same fact is proven with another method. While Read uses 'a special case of De Bruijn's generalisation of Polya's theorem that involved the cycle-index of G_n, the group of permutations of pairs of vertices induced by permutations of the vertices', Clapham uses only 'well-known elementary facts about self-complementary graphs and their complementing permutations'.[1]

Given two equinumerous sets there exists, per definition, a bijection between these sets. But does also a *natural* bijection exist? That would be a procedure which maps, for each $n \in \mathbb{N}$, the self-complementary digraphs with $2n$ vertices in one way or another to the self-complementary graphs with $4n$ vertices. We are interested in finding such a procedure.

One attempt to find a natural bijection has been made by B. Zelinka [7]. Unfortunately, the paper was only found in a badly scanned version and it was impossible to reconstruct some distinguishing symbols such as indices. A review [2] of this paper, written by Clapham, claims that the algorithm given by B. Zelinka is not well-defined, since the complementing permutations, upon which the algorithm is based, are not unique. Hence, the algorithm is not a natural bijection.

Another approach of finding a natural bijection is to look whether one of the enumerations induces a natural bijection.
Read enumerates the graphs and digraphs in [5] as follows. First, the total number of graphs with n vertices is determined. Call this number T. Then, an equivalence relation is defined such that a graph and its complement are equivalent. Call the number of different equivalence classes C. Hence, every self-complementary graph is counted once in T and once in C, and every graph which is not self-complementary is counted twice in T and once in C. So the number of non-isomorphic self-complementary graphs is given by $2 \cdot C - T$. The number of self-complementary digraphs is enumerated analogically. It seems unlikely that this leads to a natural bijection.

Another method to prove the equinumerosity is given in [1]. Clapham mentions himself that it does not seem to induce a natural bijection. By looking at the enumeration, I suspect it as well.

We conclude that no natural bijection is found yet and that the enumerations do not give a clue about a natural bijection.

4 A class of algorithms

As mentioned, the literature does not give a viable suggestion on how to produce a natural bijection between the self-complementary digraphs with $2n$ vertices and the self-complementary graphs with $4n$ vertices. In this chapter we derive some reasonable steps for algorithms which may lead to the desired natural bijection. Before deriving such steps, we construct the graphs and digraphs for $n = 1$ and $n = 2$. Thereafter we apply the obtained algorithms on the two cases $n = 1$ and $n = 2$. Finally, it will be proven that none of these algorithms leads to the desired natural bijection.

4.1 Self-complementary digraphs with $2n$ vertices and graphs with $4n$ vertices

To construct the digraphs and graphs efficiently, we first determine the number of arrows of a digraph and the number of edges of a graph as a function of the number of vertices. This narrows down the number of self-complementary digraphs and graphs that have to be checked.

Lemma 1. *Each self-complementary digraph D on $2n$ vertices has $n(2n-1)$ edges.*

Proof. The complete digraph with $2n$ vertices has $2n(2n-1)$ arrows. A self-complementary graph has exactly half of all possible arrows, i.e. $n(2n-1)$. □

Lemma 2. *Each self-complementary graph G on $4n$ vertices has $n(4n-1)$ edges.*

Proof. A complete graph with $4n$ vertices has $\frac{4n(4n-1)}{2}$ edges. A self-complementary graph has exactly half of all possible edges, i.e. $n(4n-1)$. □

4.1.1 Self-complementary digraphs and graphs for $n = 1$

The table in Figure 3.1 shows that there is only one self-complementary digraph with two vertices. From Lemma 1 it follows that the digraph has one arrow. Hence, the only self-complementary digraph with two vertices is the one which is shown in Figure **??**.

$$1 \longrightarrow 2$$

Figure 4.1: Self-complementary digraph for $n = 1$.

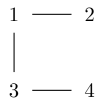


Figure 4.2: Self-complementary graph for $n = 1$.

There is only one self-complementary graph with four vertices. It follows from Lemma 2 that this graph has three edges. Figure 4.1 shows this graph.

A natural bijection should map the digraph in Figure **??** onto the graph in Figure 4.1. In the next section, a construction will be made which is based on these two graphs. However, we will first construct the self-complementary digraphs for $n = 2$. They have several new properties which have to be considered additionally.

4.1.2 Self-complementary digraphs and graphs for $n = 2$

In order to simplify the search for the self-complementary digraphs and graphs for $n = 2$ we first consider the following lemmas. The first one shows that only a few in- and outdegree sequences are possible for a self-complementary digraph with four vertices. The second lemma states that for a self-complementary digraph its reverse digraph is also self-complementary. So the number of digraphs which have to be checked for being self-complementary can be reduced.

Lemma 3. *The only possible in-/outdegree sequences for a self-complementary digraph with four vertices are* $(0, 1, 2, 3), (1, 1, 2, 2)$ *and* $(3, 3, 0, 0)$.

Proof. Let $(\gamma_1, \gamma_2, \gamma_3, \gamma_4)$ be an in- or outdegree sequence. As mentioned prior, the self-complementary digraphs with four vertices have six arrows. By assumption, they do not contain loops. Therefore, the maximum in-/outdegree of a vertex is three. Hence, the number six has to be subdivided into four parts $\gamma_1, \gamma_2, \gamma_3, \gamma_4$ with $0 \leq \gamma_1, \gamma_2, \gamma_3, \gamma_4 \leq 3$. The only options are:

$$\begin{aligned} 6 &= 3 + 3 + 0 + 0 \\ &= 3 + 2 + 1 + 0 \\ &= 3 + 1 + 1 + 1 \\ &= 2 + 2 + 2 + 0 \\ &= 2 + 2 + 1 + 1 \end{aligned} \tag{4.1}$$

So $(3, 3, 0, 0), (3, 2, 1, 0), (3, 1, 1, 1), (2, 2, 2, 0)$ and $(2, 2, 1, 1)$ are potential in-/outdegree sequences. We will prove now that $(3, 1, 1, 1)$ and $(2, 2, 2, 0)$ cannot be in-/outdegree sequences of a self-complementary digraph. Assuming that $(3, 1, 1, 1)$ is the indegree (or outdegree) sequence of a digraph D with four vertices, then the complementary digraph D^c has the indegree (or outdegree) sequence $(0, 2, 2, 2)$. Because these sequences do not coincide, D is not self-complementary. The same argument is true for the indegree (or outdegree) sequence $(2, 2, 2, 0)$. □

In order to illustrate the proof of the prior named statement, that the reverse digraph of a self-complementary digraph is self-complementary as well, we will first consider an example. The example describes the procedure which is also applicable to prove the statement.

Example 1. *Figure 4.1.2 will be used to illustrate example-based that the reverse digraph of a self-complementary digraph is as well self-complementary.*

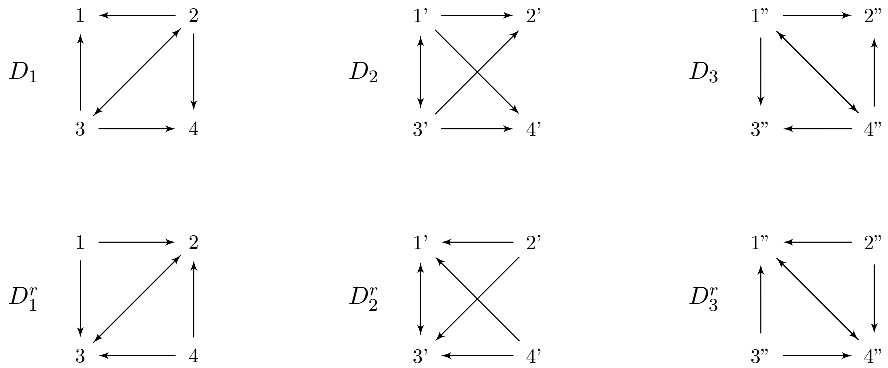


Figure 4.3: Example showing the algorithm used to prove Lemma 4.

D_1 is a self-complementary graph. Its complementary graph is called D_3. The bijection $f : \{1, 2, 3, 4\} \to \{1'', 2'', 3'', 4''\}$ with

$$\begin{aligned} f(1) &= 2'', \\ f(2) &= 4'', \\ f(3) &= 1'', \\ f(4) &= 3'' \end{aligned}$$

proves that D_1 and D_3 are isomorphic. Every permutation can be written as a product of transpositions. Applying this on the bijection f, we get two transpositions f_1 and f_2 with $f_1 : \{1, 2, 3, 4\} \to \{1', 2', 3', 4'\}$ defined by

$$\begin{aligned} f_1(1) &= 2', \\ f_1(2) &= 1', \\ f_1(3) &= 3', \\ f_1(4) &= 4' \end{aligned}$$

and $f_2 : \{1', 2', 3', 4'\} \to \{1'', 2'', 3'', 4''\}$ defined by

$$\begin{aligned} f_2(1') &= 1'', \\ f_2(2') &= 2'', \\ f_2(3) &= 4'', \\ f_2(4) &= 3''. \end{aligned}$$

Then $f_2 \circ f_1 = f$. By applying f_1 on the vertices of D_1 we obtain D_2. And by applying f_2 on the vertices of this digraph, we obtain D_3. Now, let us take a look at D_1^r. As the name suggests it is the reverse graph of D_1. The digraph D_2^r can be obtained from D_1^r by applying f_1 on the vertices of D_1^r. Another way to obtain D_2^r is to construct the reverse digraph of D_2. Analogously, there are two options to obtain D_3^r. The first option is to apply f_2 on the vertices of D_2^r and the second option is to construct the reverse digraph of D_3. So $f_2 \circ f_1 = f$ is a bijection of the vertices from D_1^r to the vertices of D_3^r. Because D_1 and D_3 are complementary digraphs of each other, also the reverse digraphs D_1^r and D_3^r are complementary digraphs of one another. So we have found a bijection between the reverse digraph and its complementary digraph. Hence, the reverse digraph is also self-complementary.

The following lemma proves that for each self-complementary digraph also the reverse digraph is self-complementary.

Lemma 4. *The reverse digraph of a self-complementary digraph is self-complementary.*

Proof. Let $D = (V, E)$ be a self-complementary digraph with $D^c = (V, E^c)$ the complementary digraph. Then the reverse graph of D^c, say D^{cr}, is the complementary graph of D^r, which is the reverse graph of D. Hence, to prove that D^r is self-complementary, we have to prove that $D^r \cong D^{cr}$.
Because D and D^c are isomorphic, there exists a bijection $f : V \to V$ such that $(a, b) \in E$ if and only if $(f(a), f(b)) \in E^c$. Every bijection between vertices can be written as a product of transpositions. Let $f = f_m \circ \ldots \circ f_1$ with $f_1, \ldots, f_m$ transpositions. It is suffices to prove that first constructing the reverse graph of D and then applying f_1 results in the same graph as first applying f_1 and then constructing the reverse graph.
For certain $a, b \in V$, the transposition $f_1 : V \to V$ can be written as:

$$f_1(v) = \begin{cases} b & \text{if } v = a \\ a & \text{if } v = b \\ v & \text{otherwise} \end{cases}$$

For an arrow (v_1, v_2), with $v_1, v_2 \in V$, the arrow that is obtained by applying f_1 is noted as $(f_1(v_1), f_1(v_2)) = f_1((v_1, v_2))$.
To construct the reverse graph, each arrow (w_1, w_2), for $w_1, w_2 \in V$, is replaced by (w_2, w_1). This is noted as $r(w_1, w_2) = (w_2, w_1)$.
It can be checked that

$$\begin{aligned} r \circ f_1((a, b)) &= f_1 \circ r((a, b)), \\ r \circ f_1((a, v)) &= f_1 \circ r((a, v)), \\ r \circ f_1((v, a)) &= f_1 \circ r((v, a)), \\ r \circ f_1((v, b)) &= f_1 \circ r((v, b)), \\ r \circ f_1((b, v)) &= f_1 \circ r((b, v)), \\ r \circ f_1((v, v')) &= f_1 \circ r((v, v')), \end{aligned}$$

for arbitrary $v, v' \in V$. Hence, constructing the reverse graph and applying a transposition is commutative. Thus, the reverse graph of a self-complementary graph is again self-complementary.

□

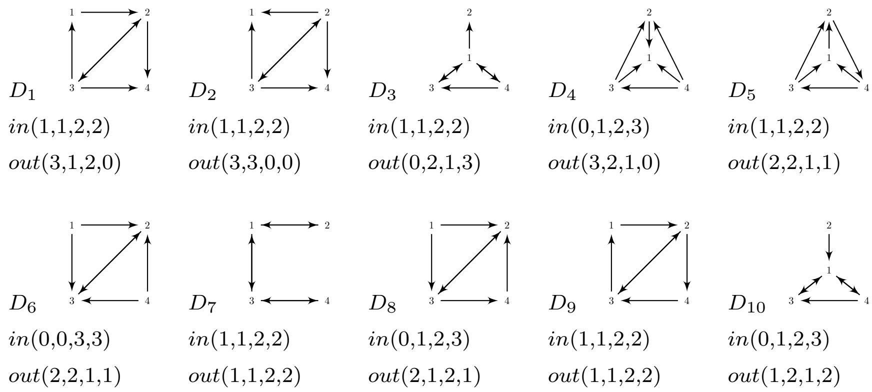


Figure 4.4: Self-complementary digraphs for $n = 2$.

Now we start with determining the self-complementary digraphs with four vertices. Figure 3.1 shows that there are ten self-complementary digraphs with four vertices. By Lemma 1 these digraphs have six arrows. And Lemma 3 gives the corresponding possible in- and outdegree sequences. After finding a self-complementary digraph, it is checked whether the digraph is non-isomorphic to the reverse digraph. If so, this one is according to Lemma 4 also self-complementary. With this, the ten self-complementary digraphs with four vertices are computed manually. Figure 4.2 shows them in visual representation. Furthermore, the in- and outdegree sequences are given accordingly.

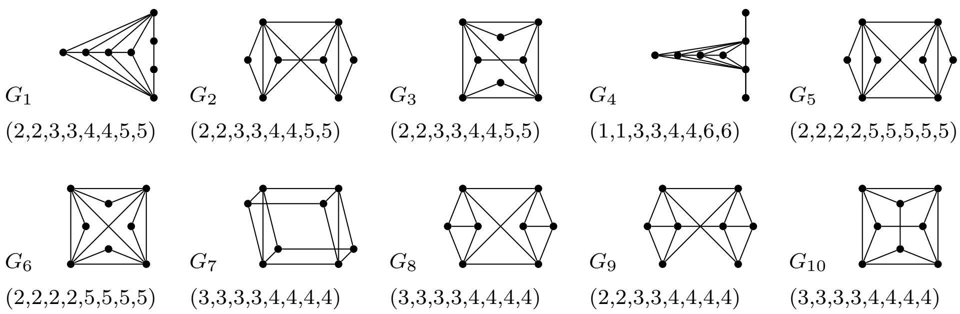


Figure 4.5: Self-complementary graphs for $n = 2$.

We will now determine all self-complementary graphs with eight vertices. There are ten of these graphs, which are given in Figure 4.3. Moreover, Figure 4.3 also lists the corresponding degree sequences. With the software MATLAB I verified that these graphs are indeed self-complementary and pairwise non-isomorphic.

4.2 Approaches to find a natural bijection

The desired natural bijection should transform the digraph in Figure 4.4 into the graph given in Figure 4.4. This is the digraph given in Figure **??** and the graph given in Figure 4.1. Furthermore, the procedure should be applicable for the self-complementary digraphs and graphs for $n = 2$. Thus, the digraphs given at the left hand side of Figure 4.5 should in one way or another be translated into the graphs at the right hand side. These digraphs and graphs are the same as in Figure 4.2 and 4.3.

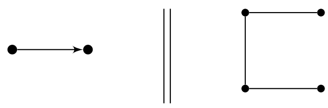


Figure 4.6: Self-complementary digraph (left) and graph (right) for $n = 1$.

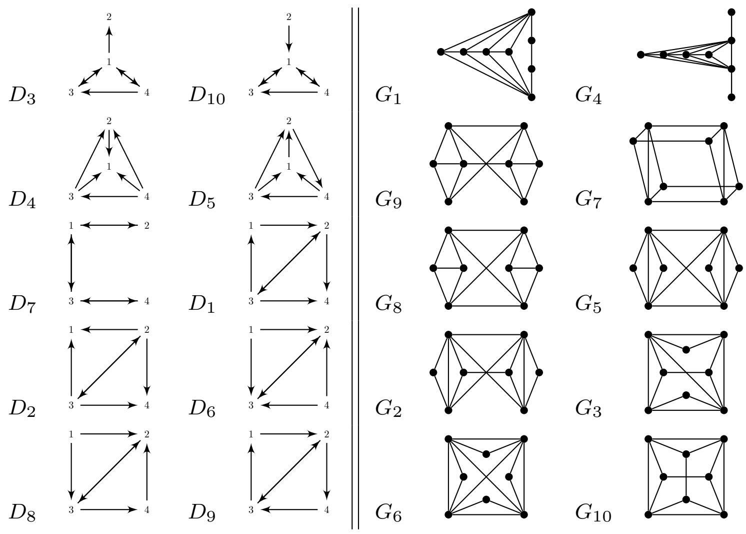


Figure 4.7: Self-complementary digraphs and graphs for n=2.

Since the digraphs with $2n$ vertices will be mapped onto graphs with $4n$ vertices an obvious first step is to duplicate each vertex. Thus, one vertex of a digraph will be associated with two vertices for a graph. Furthermore, the case for $n = 1$ suggests that we should transform each arrow of a digraph into three edges for a graph. The following lemma states that, for $n > 1$, transforming each arrow into three edges results in a graph that has too many edges to be self-complementary.

Lemma 5. *Let D be a self-complementary digraph with $2n$ vertices. Associate each arrow of the digraph with three edges for a graph. Then the obtained graph has $2n(2n-1)$ more edges than a self-complementary graph with $4n$ vertices.*

Proof. It follows from Lemma 1 that a self-complementary digraph with $2n$ vertices has $n(2n-1)$ arrows. A self-complementary graph with $4n$ vertices has $n(4n-1)$ edges (see Lemma 2). If each arrow of the digraph is associated with three edges, then the obtained graph has $3 \cdot n(2n-1) = n(4n-1) + 2n(n-1)$ edges. Hence, the resulting graph has $2n(n-1)$ more edges than a self-complementary graph with $4n$ vertices. □

Therefore, if we replace each arrow by three edges we need to remove $2n(n-1)$ edges to obtain possibly a self-complementary graph. Let us now look what happens when we transform each arrow of the digraph into two edges for the graph.

Lemma 6. *Let D be a self-complementary digraph with $2n$ vertices. Associate with each arrow of the digraph two edges for a graph. Then the obtained graph has n edges less than a self-complementary graph with $4n$ vertices.*

Proof. Again, the number of arrows of a self-complementary digraph with $2n$ vertices is $n(2n-1)$ and the number of edges of a self-complementary graph with $4n$ vertices is $n(4n-1)$. Associating each arrow with two edges gives $2 \cdot n(2n-1) = n(4n-1) - n$. Hence, the graph has n fewer edges than a self-complementary graph. □

We conclude that there are multiple options in order to get the right number of edges. One option is to associate each arrow with three edges and then remove $2n(n-1)$ edges. Another possibility is to associate each arrow with two edges and then add n edges. In this thesis we will consider the approach of adding n edges, i.e. the approach where we transform each arrow of the digraph into two edges for the graph.

The next step is to determine *how* to associate one arrow with two edges. One possibility is shown in Figure 4.6.

Figure 4.8: Possible association of one arrow of a digraph with two edges of a graph.

Looking at Figure 4.5 we notice that there are digraphs $D = (V, E)$ with, for some $a, b \in V$, $(a, b) \in E$ and $(b, a) \in E$. See for example digraph D_1 with $a = 2$ and $b = 3$. First associating each vertex of the digraph with two vertices of the graph, and then applying the construction as in Figure 4.6 for all arrows of the digraph, we would obtain pairs of vertices which are connected by more than one edge. According to the definition of a graph, only one of these edges can be considered. Therefore, in order to effectively associate each arrow with two edges – such that each other arrow is associated with two other edges – we let the associated structure depend on the direction of an arrow. However, in a digraph there is not a clear orientation of the arrows. By changing two vertices, the orientation of the arrow will change as well, but it is still the same digraph. In order to augment the digraphs with an orientation we order the vertices. Then each arrow points to the right or to the left and the associations as in Figure 4.7 can be made without resulting in multiple edges between one pair of vertices.

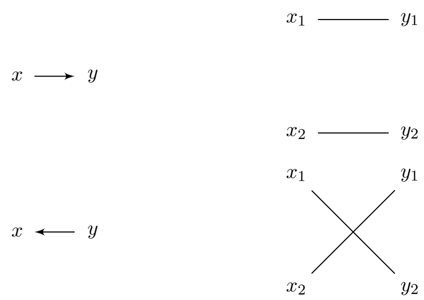


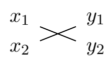
Figure 4.9: Association of orientated arrows of a digraph with two edges of a graph.

4.2.1 Class of algorithms

The procedure can be summarised by the following steps.

1. Fix an ordering for the vertices of a self-complementary digraph with $2n$ vertices and arrange them in a sequence $(v_1, v_2, \ldots, v_{2n})$.
2. Associate with each vertex x of the digraph two vertices x_1 and x_2.
3. Make the following association:
   Assume that $1 \leq i < j \leq 2n$ and let $v_i = x$ and $v_j = y$ be vertices of the digraph.

   Then replace $x \longrightarrow y$ by $\begin{array}{l} x_1 \;—\; y_1 \\ x_2 \;—\; y_2 \end{array}$

   and $x \longleftarrow y$ by 
4. Add n edges.

Note that these steps represent a class of algorithms instead of a single algorithm. Although Steps 2 and 3 are unambiguous, Steps 1 and 4 are still not explicit. Not only can we choose different orderings of the vertices, we can also choose which edges to add in Step 4. One might wonder whether the choice in Step 1 is important for the rest of the algorithm. The following lemma states that applying Steps 2 and 3 to digraphs with different orderings can result in non-isomorphic graphs. Furthermore, the following lemma also states that for some orderings we could have been fixed, applying Steps 2 to 4 cannot result in a self-complementary graph.

Lemma 7. *Applying the above method on a digraph can produce non-isomorphic graphs by choosing different orderings at Step* 1*. Furthermore, there are digraphs for which a certain ordering results in non-self-complementary graphs.*

Proof. Figure 4.8 shows digraph D_5 where two different orderings of the vertices are chosen for Step 1. Step 2 and Step 3 are applied on both orderings. Rearranging the vertices shows that only one of the obtained graphs is connected. Therefore, they can not be isomorphic. Looking at the self-complementary graphs for $n = 2$ (see Figure 4.5) we see that there is no self-complementary graph with a complete subgraph on four vertices. Hence, we cannot obtain a self-complementary graph by adding two edges to the upper graph.

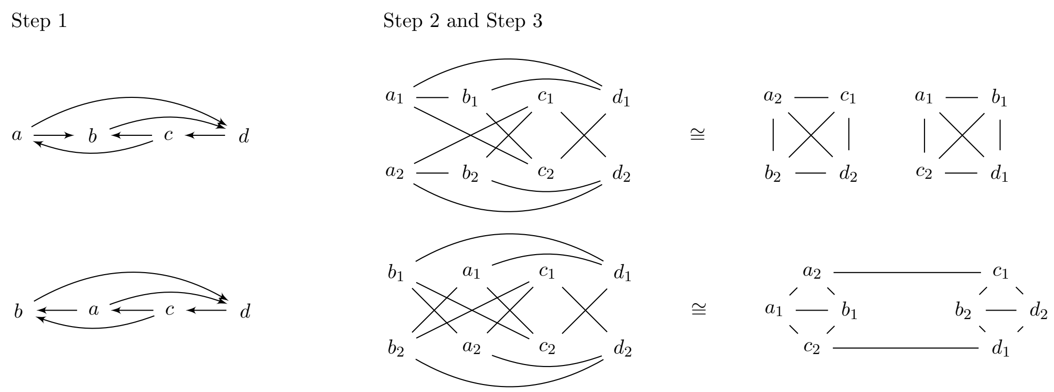


Figure 4.10: Application of Step 1 to 3 on digraph D_5.

$\square$

Note that the lower graph in Figure 4.8 can be extended to a self-complementary graph by adding for example the two edges $\{b_1, b_2\}$ and $\{a_1, d_2\}$. So possibly there exists a possibility to extend Step 1 such that only self-complementary graphs can be obtained.

4.3 Application of the class of algorithms

In Section 4.2.1 a class of algorithms is defined. In the following section we will check whether one of these algorithms will lead to the desired natural bijection. Hence, we will investigate the cases $n = 1$, $n = 2$ and $n = 3$. It is likely that one of the algorithms maps the self-complementary digraphs onto the self-complementary graphs for $n = 1$ and $n = 2$. For $n = 3$ we will see that this is not true. Thus none of the algorithms defined in 4.2.1 can be extended to the natural bijection we are interested in.

4.3.1 The case for $n = 1$

For $n = 1$ the digraph on the left hand side of Figure 4.4 should be mapped onto the graph on the right hand side. Figure 4.9 shows the construction, described in 4.2.1, applied on the digraph for $n = 1$. Since the digraph has two vertices, there are two possible orderings at Step 1. At Step 4, there are for both orderings four possibilities to add one edge. All the obtained graphs are pairwise isomorphic and self-complementary. Therefore, independent on how Step 1 and Step 4 are performed, the algorithm will map the digraph given in Figure 4.4 onto the graph given in Figure 4.4.

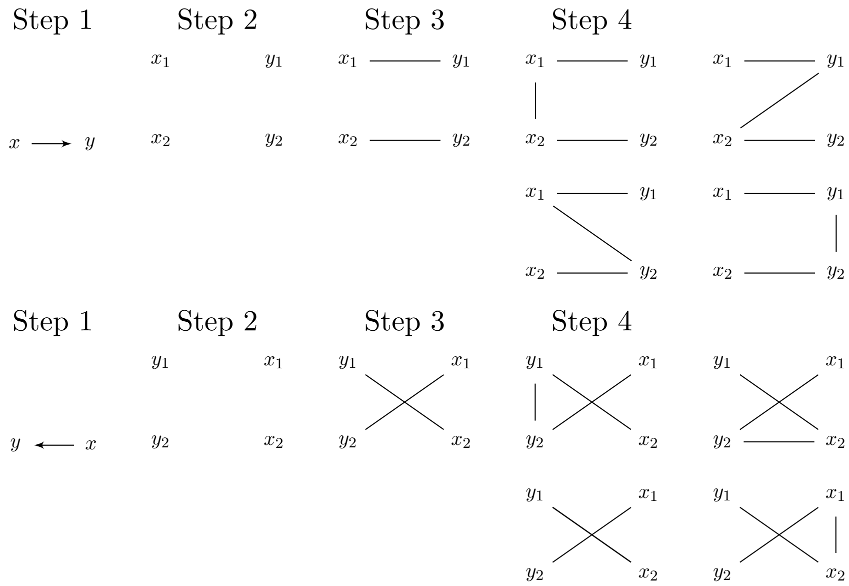


Figure 4.11: Construction applied on the self-complementary digraph for $n = 1$.

4.3.2 The case for $n = 2$

Figure 4.10 shows a table where, for each digraph, those graphs are listed which can be obtained by applying the construction of Section 4.2.1. Looking at this table, the first question which arises is: Is it possible to extend the construction such that it becomes bijective? Figure 4.11 shows that it is possible. By mapping the digraphs to the gray marked graphs, each graph will be reached exactly once. However, it is questionable whether this bijection is natural.
In the remainder of this section it will be demonstrate that the content of Figure 4.10 is correct. Thus, that exactly those graphs are listed which can be obtained by applying the steps of the construction of Section 4.2.1 to a certain digraph.

To prove that the graphs which are listed in Figure 4.10 are those ones, which can be obtained, five new graphs will be introduced. By applying Step 1 to 3 of the construction

Digraph	Graphs obtained from the steps									
D_3	G_1			G_4						
D_{10}	G_1			G_4						
D_4							G_7	G_8	G_9	
D_5							G_7	G_8	G_9	
D_7					G_5			G_8		
D_1		G_2	G_3		G_5	G_6	G_7			G_{10}
D_2		G_2	G_3		G_5	G_6	G_7			G_{10}
D_6		G_2	G_3		G_5	G_6	G_7			G_{10}
D_8		G_2	G_3		G_5	G_6	G_7			G_{10}
D_9		G_2	G_3		G_5	G_6	G_7			G_{10}

Figure 4.12: Graphs per digraph which can be obtained by use of the construction in Section 4.2.1.

Digraph	Graphs obtained from the steps									
D_3	**G_1**			G_4						
D_{10}	G_1			**G_4**						
D_4							G_7	**G_8**	G_9	
D_5							G_7	G_8	**G_9**	
D_7					**G_5**			G_8		
D_1		**G_2**	G_3		G_5	G_6	G_7			G_{10}
D_2		G_2	**G_3**		G_5	G_6	G_7			G_{10}
D_6		G_2	G_3		G_5	**G_6**	G_7			G_{10}
D_8		G_2	G_3		G_5	G_6	**G_7**			G_{10}
D_9		G_2	G_3		G_5	G_6	G_7			**G_{10}**

Figure 4.13: A bijection between the digraphs and graphs for $n = 2$ which admits the construction in Section 4.2.1.

of Section 4.2.1 on a self-complementary digraph, always at least one of these five graphs can be obtained. And all of these five graphs can be extended to self-complementary graphs by adding two edges. These two statements will be proven in the following lemmas. The five mentioned graphs are shown in Figure 4.12.

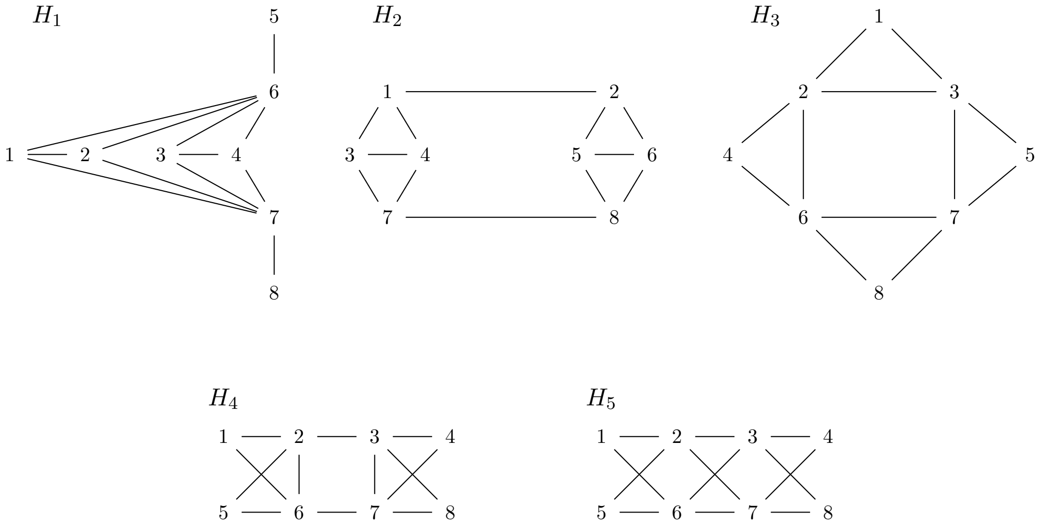


Figure 4.14: Auxiliary graphs H_1, H_2, H_3, H_4 and H_5.

The following lemma shows that from each digraph one of the five graphs H_1, H_2, H_3, H_4 and H_5 from Figure 4.12 can be obtained by applying the steps described in Section 4.2.1.

Lemma 8. *For each self-complementary digraph with* 4 *vertices, there is an ordering of the vertices such that, by applying Step* 2 *and Step* 3 *of the construction, at least one of the graphs* H_1, H_2, H_3, H_4 *and* H_5 *can be obtained. In particular:* D_3 *and* D_{10} *can result in* H_1*,* D_4 *and* D_5 *can result in* H_2*,* D_7 *can result in* H_5*, and the digraphs* D_1, D_2, D_6, D_8 *and* D_9 *can result in both graphs* H_3 *and* H_4*.*

Proof. This lemma is proven graphically. For a given digraph an ordering of the vertices is fixed. After applying Step 2 and Step 3 of the construction the vertices are rearranged to show that they are isomorphic to one of H_1, H_2, H_3, H_4 and H_5. The graphical representations can be found in the appendix. □

The next lemma shows that each self-complementary graph with eight vertices can be obtained by adding two edges to one of the five graphs H_1, H_2, H_3, H_4 and H_5.

Lemma 9. *Each self-complementary graph with eight vertices can be obtained by adding two edges to one of the graphs in Figure 4.12.*

Proof. G_1 can be obtained by adding edges $(2,3)$ and $(5,8)$ in H_1. G_2 can be obtained by adding edges $(4,5)$ and $(2,7)$ in H_4. G_3 can be obtained by adding edges $(2,7)$ and $(4,5)$ in H_3. G_4 can be obtained by adding edges $(2,3)$ and $(6,7)$ in H_1. G_5 can be obtained by adding edges $(2,7)$ and $(3,6)$ in H_4 or by adding edges $(2,6)$ and $(3,7)$ in H_5. G_6 can be obtained by adding edges $(2,7)$ and $(3,6)$ in H_3. G_7 can be obtained by adding edges $(4,5)$ and $(1,8)$ in H_4 or by adding edges $(4,5)$ and $(3,6)$ in H_2. G_8 can be obtained by adding edges $(1,8)$ and $(2,7)$ in H_2 or by adding edges $(1,5)$ and $(4,8)$ in H_5. G_9 can be obtained by adding edges $(1,8)$ and $(4,5)$ in H_2. G_{10} can be obtained by adding edges $(1,8)$ and $(4,5)$ in H_3. □

From these two lemmas it follows that each self-complementary graph, listed in Figure 4.10 actually can be obtained from a self-complementary digraph by applying the construction.

It remains to prove that those graphs which are not listed really can not be obtained. Therefore, a new definition will be introduced. After that, a lemma will be given which claims that after Step 3 of the algorithms in Section 4.2.1 each graph has this property.

Definition 20. *A graph $G = (V, E)$, possibly not connected, with $\#V = 2m$ is called* ***mirror-symmetric*** *if it is possible to split the vertices in two sets X and Y with the following properties:*

- *$\#X = \#Y = m$ and $X \cap Y = \emptyset$*
- *There are sequences $(x_i)_{i=1}^m$ and $(y_j)_{j=1}^m$ with $x_i \in X$ and $y_j \in Y$, for $1 \le i, j \le m$, such that*
  - *for $i \neq j$:*

    $(x_i, y_j) \in E_H$ *if and only if* $(y_i, x_j) \in E_H$,

    and $(x_i, x_j) \in E_H$ *if and only f* $(y_i, y_j) \in E_H$.
  - *for $i = j$:*

    $(x_i, y_j) \notin E_H$.

Two vertices $a, b \in V$ are called a ***pair*** $[a, b]$ *when they have the same sequence index.*

Informally, the definition stated that the vertices can be drawn in such a way that the graph is mirror-symmetric with respect to a line between two rows of vertices and the additional condition that vertically aligned vertices are not connected by an edge. Two vertically aligned vertices are called a pair.

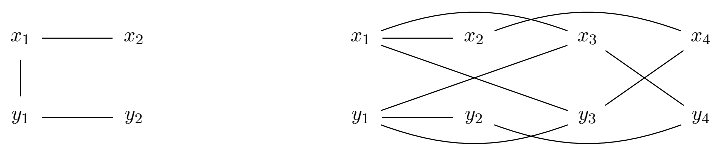


Figure 4.15: A graph which is not mirror-symmetric graphs (left) and a graph which is mirror-symmetric (right).

Figure 4.13 shows an example and a counterexample of a mirror-symmetric graph. The graph at the left hand side is not mirror-symmetric. Independent of how the vertices are arranged in two rows, there is always an edge which does not satisfy the requirements of the definition.

Next, it will be proven that each graph that is obtained from the algorithms of Section 4.2.1 is mirror-symmetric after Step 3.

Lemma 10. *Applying Step* 1 *to Step* 3 *on a self-complementary digraph results in a mirror-symmetric graph.*

Proof. In Step 2, each vertex of the digraph is associated with two vertically aligned vertices. From this, two sets – the two rows – of vertices are obtained. In Step 3 the edges are constructed. Obviously, the resulting graph is mirror-symmetric with respect to a line between the two rows of vertices. □

Notice that a graph does not need to be mirror-symmetric after Step 4, when n edges are added. The graph at the left hand side of Figure 4.13 is isomorphic to the self-complementary graph with four vertices, but it is not mirror-symmetric.

We conclude this section with the following proposition which proves that the table in Figure 4.10 is indeed correct.

Proposition 1. *Figure 4.10 lists for each digraph exactly those graphs which can be obtained by applying the construction of Section 4.2.1.*

Proof. For all digraphs it follows from Lemma 8 and 9 that the listed graphs can be obtained by applying the construction. So it remains to prove that the graphs which are not listed by a digraph can not be obtained by applying the construction.
Figure 4.2 shows the application of Step 1 to 3 on an arbitrary ordering of the vertices of digraph D_7. Because of the symmetry of D_7, we obtain, independent of the ordering in Step 1, the graph which is shown in Figure 4.2 after Step 3. It follows that only G_5 and G_8 can be obtained from D_7.

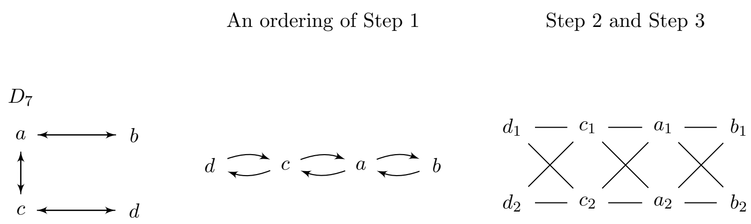
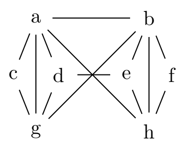


Figure 4.16: Application of Step 1 to 3 on digraph D_7.

To prove for the other digraphs that certain graphs can not be obtained by applying the construction, it is helpful to take a look at the degree sequences. For each arrow (x, y) which is replaced by two edge, as in Figure 4.6, the degree of the associated vertices of x and y increases by one. Hence, the degree sequence of the graph which is obtained after Step 3 is unique. And because of the mirror-symmetry occurs each degree an even number of times. To predict the degree sequence of a possible self-complementary graph, which can be obtained from a certain digraph, we have to add two edges. So let us take a closer look at what it exactly means for the degree sequence when two edges are added. One possibility is to increase one degree by two, and two other degrees by one. However, this does not result in a sequence of a self-complementary graph. According to [2], in every potential degree sequence of a self-complementary graph each degree occurs an even number of times. Another possibility is to increase the degree of two distinct vertices by two. Actually, this is only possible when two edges are added between one pair of vertices, but this is not allowed. The final possibility is to increase the degree of four vertices by one. This will result in possible degree sequences of self-complementary graphs.
Figure 4.2 shows for each digraph firstly the degree sequences of the graph which is obtained after Step 3, secondly the degree sequences of self-complementary graphs which can be obtained by adding two edges, and thirdly the graphs which belong to these degree sequences.

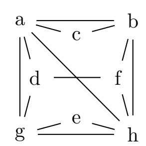
It follows from Figure 4.2 that it is left over to prove that the graphs G_2 and G_3 can not be obtained from one of the digraphs D_3 and D_{10}, that G_{10} can not be obtained from one of D_4 and D_5, and that G_1, G_8 and G_9 can not be obtained from one of D_1, D_2, D_6, D_8 and D_9. It will be shown step by step under consideration that a graph can be obtained by applying the construction steps. This leads to a contradiction.

G_2



G_2 has the degree sequence $(5, 5, 2, 3, 3, 2, 4, 4)$. Assume that G_2 is obtained from D_3 or D_{10}. Then $(5, 5, 2, 3, 3, 2, 4, 4)$ must become $(1, 1, 5, 5, 3, 3, 3, 3)$ when two edges are removed in G_2. The only possibility to obtain this degree sequence is to remove edges $\{c, g\}$ and $\{f, h\}$. Then, vertices a and h are the only vertices of degree five. Hence, they must form a pair. However, $\{a, h\}$ is an edge. That contradicts that $[a, h]$ can be a pair. Hence, there is no possibility to obtain G_2 from D_3 or D_{10}.

G_3

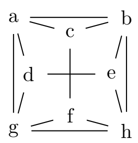


G_3 has the degree sequence $(5, 4, 2, 3, 3, 2, 4, 4)$. Assume this graph is obtained from D_3 or D_{10}. Then $(5, 4, 2, 3, 3, 2, 4, 4)$ must become $(1, 1, 5, 5, 3, 3, 3, 3)$ when two edges are removed. Hence, $\{c, b\}$ and $\{g, e\}$ are the only edges which can be removed. Then, a and h are the only vertices of degree five but can not be a pair because $\{a, h\}$ is an edge in G_3. Hence, also G_3 can not be obtained from D_3 or D_{10}.

Digraph	Degree sequence after Step 3	Possible degree sequences of sc graphs after adding two edges	Sc graphs with this degree sequence
D_3	$(1,1,3,3,3,3,5,5)$	$(1,1,3,3,4,4,6,6)$, $(2,2,3,3,4,4,5,5)$	G_4, G_1,G_2,G_3
D_{10}	$(1,1,3,3,3,3,5,5)$	$(1,1,3,3,4,4,5,5)$, $(2,2,3,3,4,4,5,5)$	G_4, G_1,G_2,G_3
D_4	$(3,3,3,3,3,3,3,3)$	$(3,3,3,3,4,4,4,4)$	G_7,G_8,G_9,G_{10}
D_5	$(3,3,3,3,3,3,3,3)$	$(3,3,3,3,4,4,4,4)$	G_7,G_8,G_9,G_{10}
D_7	$(2,2,2,2,4,4,4,4)$	$(2,2,2,2,5,5,5,5)$, $(3,3,3,3,4,4,4,4)$, $(2,2,3,3,4,4,5,5)$	G_5,G_6, G_7,G_8,G_9,G_{10}, G_1,G_2,G_3
D_1	$(2,2,2,2,4,4,4,4)$	$(2,2,2,2,5,5,5,5)$, $(3,3,3,3,4,4,4,4)$, $(2,2,3,3,4,4,5,5)$	G_5,G_6, G_7,G_8,G_9,G_{10}, G_1,G_2,G_3
D_2	$(2,2,2,2,4,4,4,4)$	$(2,2,2,2,5,5,5,5)$, $(3,3,3,3,4,4,4,4)$, $(2,2,3,3,4,4,5,5)$	G_5,G_6, G_7,G_8,G_9,G_{10}, G_1,G_2,G_3
D_6	$(2,2,2,2,4,4,4,4)$	$(2,2,2,2,5,5,5,5)$, $(3,3,3,3,4,4,4,4)$, $(2,2,3,3,4,4,5,5)$	G_5,G_6, G_7,G_8,G_9,G_{10}, G_1,G_2,G_3
D_8	$(2,2,2,2,4,4,4,4)$	$(2,2,2,2,5,5,5,5)$, $(3,3,3,3,4,4,4,4)$, $(2,2,3,3,4,4,5,5)$	G_5,G_6, G_7,G_8,G_9,G_{10}, G_1,G_2,G_3
D_9	$(2,2,2,2,4,4,4,4)$	$(2,2,2,2,5,5,5,5)$, $(3,3,3,3,4,4,4,4)$, $(2,2,3,3,4,4,5,5)$	G_5,G_6, G_7,G_8,G_9,G_{10}, G_1,G_2,G_3

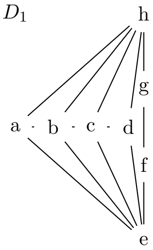
Figure 4.17: Self-complementary (sc) graphs which possibly can be obtained from a certain self-complementary digraph.

G_{10}



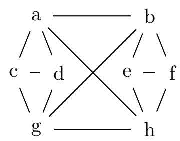
Assume that G_{10} is obtained from D_4 or D_5. Then it must be possible to split the vertices into pairs. The degree sequence of G_{10} is $(4, 4, 3, 3, 3, 3, 4, 4)$. It follows from Figure 4.2 that the vertices of degree four have to become vertices of degree three, when two edges are deleted. The only possibilities for this are – such that each degree occurs an even number of times – to delete $\{a, b\}$ and $\{g, h\}$, or $\{a, g\}$ and $\{b, h\}$. In both cases it is impossible to arrange the vertices in pairs such that the properties of mirror-symmetric graphs are satisfied. Hence, G_{10} can not be obtained from D_4 or D_5.

D_1



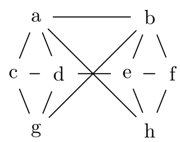
G_1 has the degree sequence $(5, 2, 2, 5, 3, 4, 4, 3)$. Assume that the graph is obtained from one of D_1, D_2, D_6, D_8 and D_9. Then $(5, 2, 2, 5, 3, 4, 4, 3)$ must become $(2, 2, 2, 2, 4, 4, 4, 4)$ when two edges are removed. Hence, the vertices of degree five have to become vertices of degree four and those of degree three have to become vertices of degree two. This can only be obtained by removing edges $\{a, e\}$ and $\{d, h\}$, or edges $\{a, h\}$ and $\{d, e\}$. In both cases it is impossible to arrange the vertices in pairs such that the properties of mirror-symmetric graphs are satisfied. Hence, G_1 can not be obtained from one of D_1, D_2, D_6, D_8 and D_9.

G_8



G_8 has the degree sequence $(4, 4, 3, 3, 3, 3, 4, 4)$. Assume that the graph is obtained from one of D_1, D_2, D_6, D_8 and D_9. Then $(4, 4, 3, 3, 3, 3, 4, 4)$ must become $(2, 2, 2, 2, 4, 4, 4, 4)$ when two edges are removed. This can only be obtained by removing edges $\{c, d\}$ and $\{e, f\}$. The vertices a, b, g and h have the same degree and $\{a, h\}$ and $\{b, g\}$ are edges. Hence, $[a, g]$ and $[b, h]$ must be pairs. Then also $[c, d]$ and $[e, f]$ form pairs. This graph can only be obtained by applying the construction on digraph D_7. Hence, D_1, D_2, D_6, D_8 and D_9 can not result in G_8.

G_9



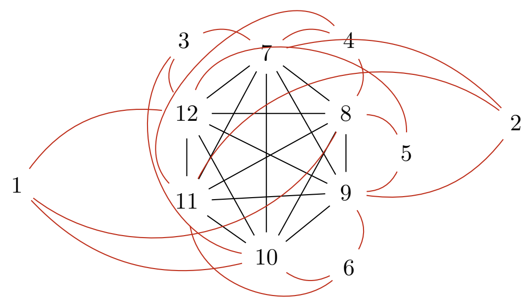
G_9 has the degree sequence $(4, 4, 3, 4, 4, 3, 3, 3)$. Again, the degree sequence must be $(2, 2, 2, 2, 4, 4, 4, 4)$ when two edges are removed, assuming the graph is obtained from one of D_1, D_2, D_6, D_8 and D_9. The only possibility to obtain this degree sequence is to remove edges $\{c, g\}$ and $\{f, h\}$. But then, it is impossible to arrange the vertices in pairs such that the properties of mirror-symmetric graphs are satisfied. Hence, G_9 is not obtained from one of D_1, D_2, D_6, D_8 and D_9.

$\square$

4.4 Proof that the algorithms are not part of a natural bijection

To check whether any of the algorithms of Section 4.2.1 works for $n = 3$, a computer program was written which determines those self-complementary graphs which can be obtained from self-complementary digraphs with six vertices by applying the construction in Section 4.2.1. It turns out that not all of the 720 self-complementary graphs with twelve vertices can be computed by the algorithms. Therefore, we know that none of the algorithms of Section 4.2.1 can be extended to the desired natural bijection.

Although the program already shows that we cannot find the natural bijection by applying the steps of Section 4.2.1, we will devote this subsection to a specific self-complementary graph that cannot be obtained by the steps. This graph will be called $H = (V_H, E_H)$ and is shown in Figure 4.14. In the following part of this section it will be proven rigorously that this graph can not be obtained by the algorithms of Section 4.2.1.



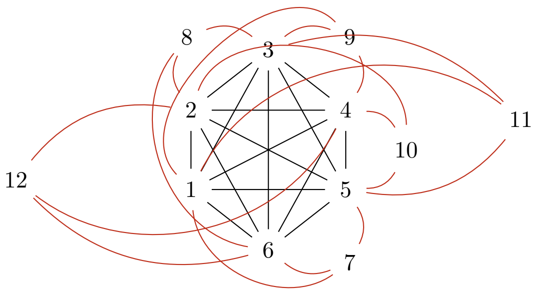
$$\mathit{Graph} < 12|\{\{8,10,12\},\{7,9,11\},\{7,10,12\},\{7,8,11\},\{8,9,12\},\{9,10,11\},$$
$$\{2,3,4,8,9,10,11,12\},\{1,4,5,7,9,10,11,12\},\{2,5,6,7,8,10,11,12\},$$
$$\{1,3,6,7,8,9,11,12\},\{2,4,6,7,8,9,10,12\},\{1,3,5,7,8,9,10,11\}\} >$$

Figure 4.18: Graph H that can not be obtained via the construction.

First, it will be shown that $H = (V, E)$ is self-complementary. Figure 4.15 shows the complementary graph $H^c = (V, E^c)$. To show that H is self-complementary, a bijection between the vertices of H and the vertices of H^c must be given. One possible bijection is $f : V \to V$ given by

$$\begin{aligned} f(1) &= 12 & f(7) &= 3 \\ f(2) &= 11 & f(8) &= 4 \\ f(3) &= 8 & f(9) &= 5 \\ f(4) &= 9 & f(10) &= 6 \\ f(5) &= 10 & f(11) &= 1 \\ f(6) &= 7 & f(12) &= 2 \end{aligned}$$

We conclude that H is self-complementary. Next, we will prove that H can not be obtained by one of the algorithms of Section 4.2.1.



$$\begin{aligned}Graph < 12|\{&\{2,3,4,5,6,7,9,11\},\{1,3,4,5,6,8,10,12\},\{1,2,4,5,6,8,9,11\},\\ &\{1,2,3,5,6,9,10,12\},\{1,2,3,4,6,7,10,11\},\{1,2,3,4,5,7,8,12\},\\ &\{1,5,6\},\{2,3,6\},\{1,3,4\},\{2,4,5\},\{1,3,5\},\{2,4,6\}\} >\end{aligned}$$

Figure 4.19: H^c; the complementary graph of H.

Proposition 2. *The graph $H = (V_H, E_H)$, as shown in Figure 4.14, can not be obtained from a digraph by applying the construction of Section 4.2.1.*

Proof. We assume that the graph can be constructed by one of the algorithms. This will result in a contradiction.
From Lemma 10 we know that, when three edges are left out of account, the vertices of H can be split in two sequences $(x_i)_{i=1}^6$ and $(y_j)_{j=1}^6$ such that

- for $i \neq j$:

  $(x_i, y_j) \in E_H$ if and only if $(y_i, x_j) \in E_H$,

  and $(x_i, x_j) \in E_H$ if and only f $(y_i, y_j) \in E_H$.

- for $i = j$:

  $(x_i, y_j) \notin E_H$.

Furthermore, it follows that two vertices with the same index have the same degree. To determine the three edges which have to be left out of account, the vertices of a subgraph of H will be subdivided into pairs. Let us look at subgraph $H_{sub} = (V_{sub}, E_{sub})$ of H, with $E_{sub} = \{\{a, b\} \in E_H : a, b \in \{7, 8, 9, 10, 11, 12\}\}$ and $V_{sub} = \{7, 8, 9, 10, 11, 12\}$. Hence, this subgraph is complete. The edges of H_{sub} are drawn in black in Figure 4.14. All six vertices $7, 8, 9, 10, 11$ and 12 have degree 8 in H. The other six vertices of H have degree 3. Hence, the two sequences have each three vertices of degree 8 and three vertices of degree 3.

By looking at Figure 4.16 we comprehend that it does not matter whether the vertices $x_1, x_2, x_3, y_1, y_2, y_3$ are permuted or not. There are always three edges, called α, β and γ, which have to be left out of account to make the graph mirror-symmetric. Thus, these are the $n = 3$ edges, which are added in Step 4 of the algorithms.

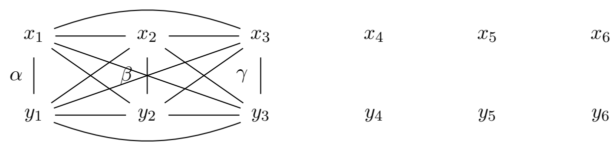


Figure 4.20: Sub-graph H_{sub} of H.

It remains to subdivide the vertices $1, 2, 3, 4, 5$ and 6 into pairs. Each of these vertices has degree 3. And none of them are pairwise connected. Let us have a look at Figure 4.16. Suppose that $(x_s, x_t) \in E_H$ and $(x_s, y_t) \in E_H$, for $4 \leq s \leq 6$ and $1 \leq t \leq 3$. Then also $(y_s, y_t) \in E_H$ and $(y_s, x_t) \in E_H$. Otherwise, the graph would not be mirror-symmetric. So it can be concluded that two vertices, which form a pair have an even number of vertices by which they are both connected, or none.

	$\{8, 10, 12\}$	$\{7, 9, 11\}$	$\{7, 10, 12\}$	$\{7, 8, 11\}$	$\{8, 9, 12\}$	$\{9, 10, 11\}$
vertex 1: $\{8, 10, 12\}$	0	1	1	0	1	0
vertex 2: $\{7, 9, 11\}$	1	0	0	1	0	1
vertex 3: $\{7, 10, 12\}$	1	0	0	0	0	0
vertex 4: $\{7, 8, 11\}$	0	1	0	0	0	0
vertex 5: $\{8, 9, 12\}$	1	0	0	0	0	0
vertex 6: $\{9, 10, 11\}$	0	1	0	0	0	0

Figure 4.21: Possible pairs of vertices.

Figure 4.17 shows a table where two vertices are marked with a 1 when they can form a pair and with a 0 otherwise. It follows that vertex 3 and vertex 5 can only form a pair with vertex 1. So it is not possible to subdivide all the vertices in pairs such that the graph remains mirror-symmetric, with α, β and γ not considered. This proves that the graph in Figure 4.14 can not be obtained from one of the algorithms in Section 4.2.1. □

Proposition **??** shows that there exists a self-complementary graph which can not be obtained from one of the algorithms described in 4.2.1. It follows that the entire class of algorithms do not lead to a natural bijection which maps the self-complementary digraphs with $2n$ vertices onto the self-complementary graphs with $4n$ vertices.

5 Ideas for further research

This thesis studies a specific class of algorithms, defined by steps which could have led to the desired natural bijection. However, at the end of Chapter 4 it was proven that this class of algorithms can not be extended to the desired natural bijection.
This final chapter suggests a further idea that possibly leads to the desired natural bijection.

Figure 4.3 shows the self-complementary graphs with eight vertices and their corresponding degree sequences. Noticeably some degree sequences occur multiple times. Arranging these graphs in sets such that the degree sequence coincides, we obtain four sets with respectively one, two, three and four elements. Figure 4.2 shows the self-complementary digraphs with four vertices and the corresponding in- and outdegree sequences. Again, some combinations of in- and outdegree sequences occur multiple times. If we consider not only those digraphs with the same combination in one set, but also those with an interchanged in- and outdegree, we obtain again four sets. Furthermore, the four sets have again respectively one, two, three and four elements. The graphical representation of these sets of graphs and digraphs are shown in figure 5.1. Now, a natural bijection can possibly be obtained by mapping the equinumerous sets onto each other.
However, it is not yet known whether the sets determined by the degree sequences also match for $n > 2$. A table in [3] shows the number of self-complementary graphs with twelve vertices for each degree sequence. This table is based on the 'Parthasatathy-Sridharan' formula. For further research it might be of interest to calculate the number of self-complementary graphs and digraphs for a given degree sequence.

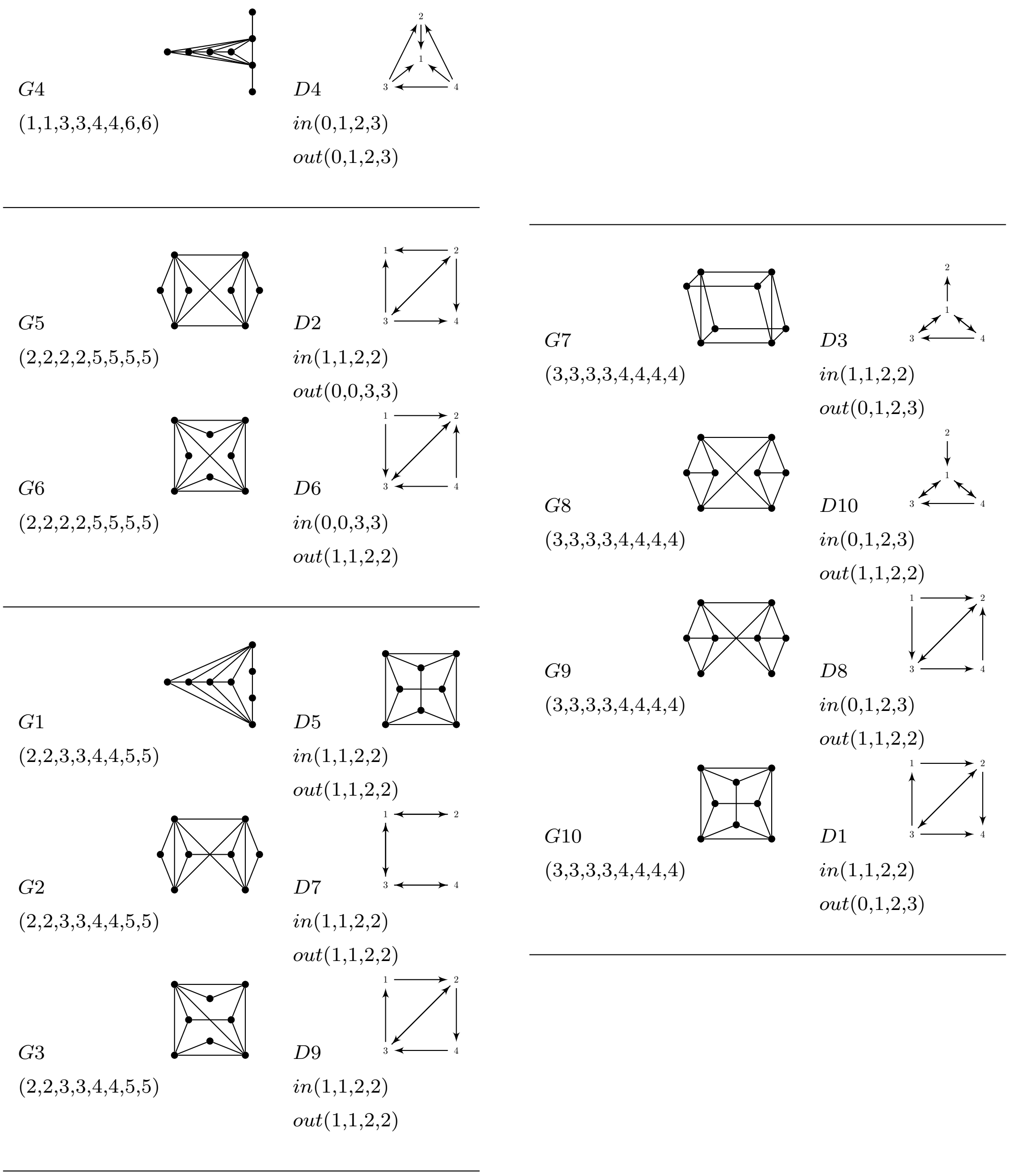
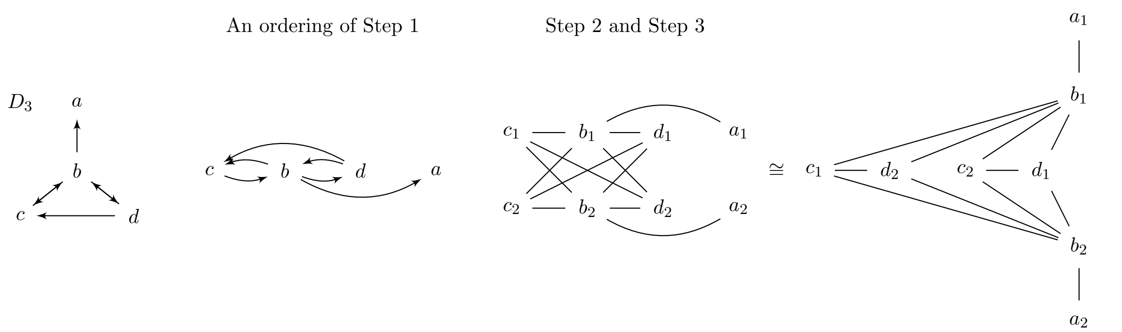
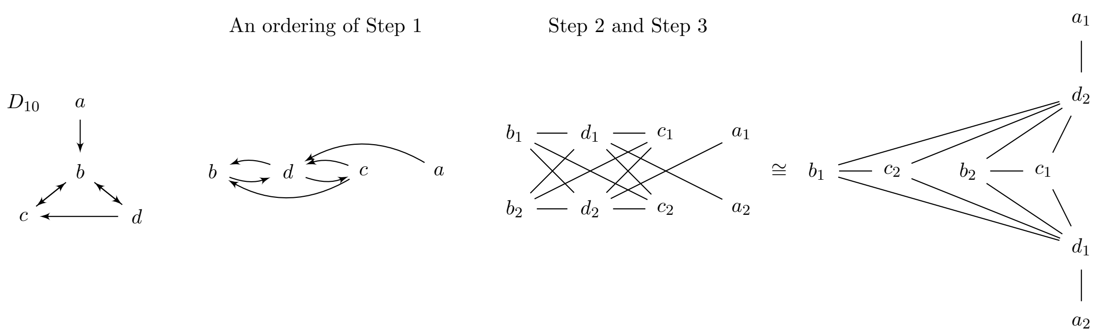


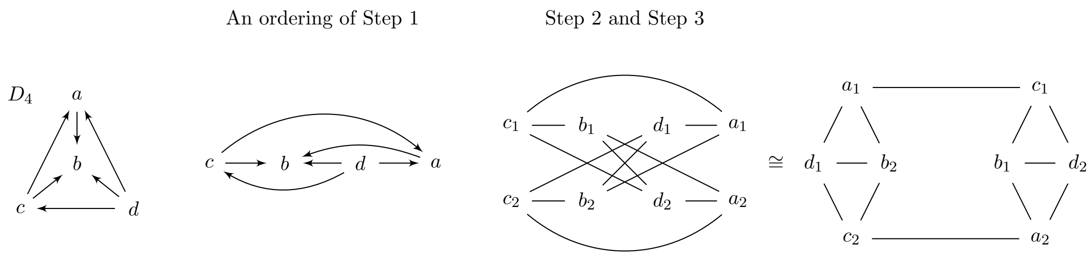
Figure 5.1: Arrangement of the graphs and digraphs according to the degree sequences

6 Appendix

6.1 Graphical representations for the proof of Lemma 8

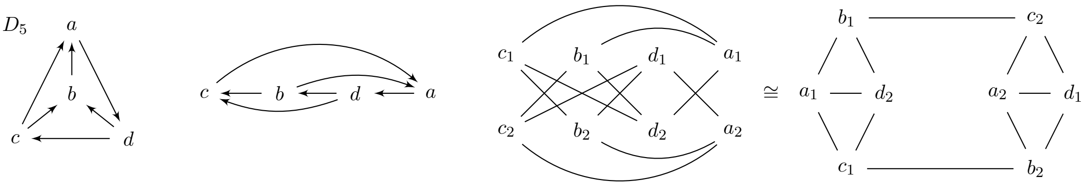






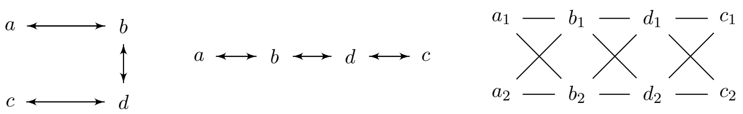
An ordering of Step 1 | Step 2 and Step 3

D_5



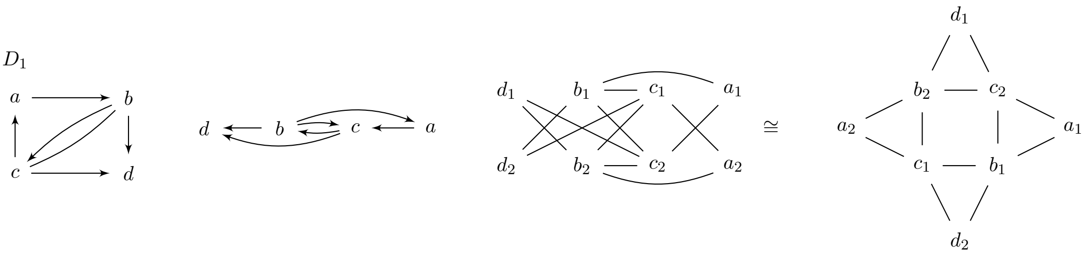
An ordering of Step 1 | Step 2 and Step 3

D_7



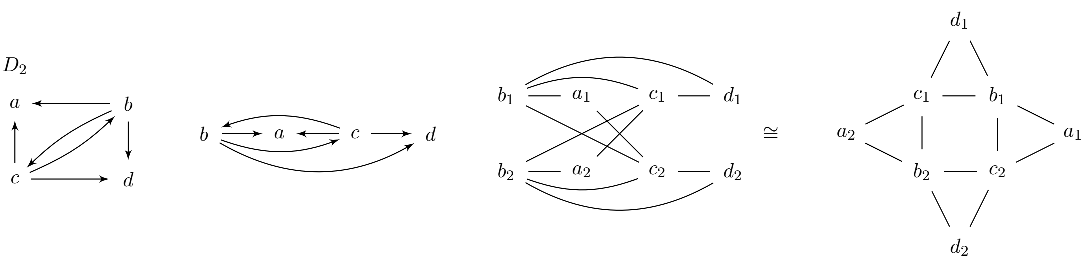
An ordering of Step 1 | Step 2 and Step 3

D_1



An ordering of Step 1 | Step 2 and Step 3

D_2



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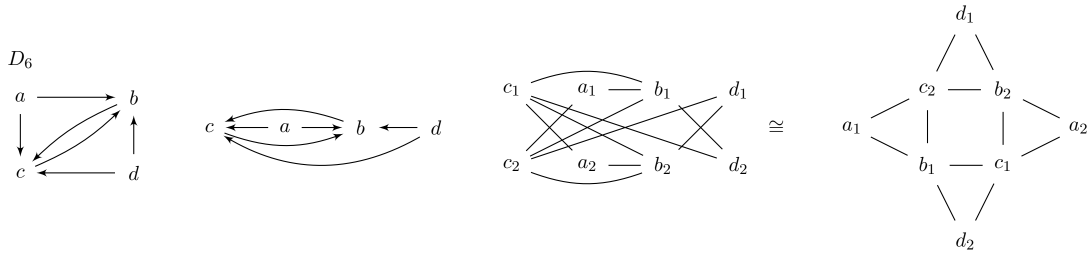

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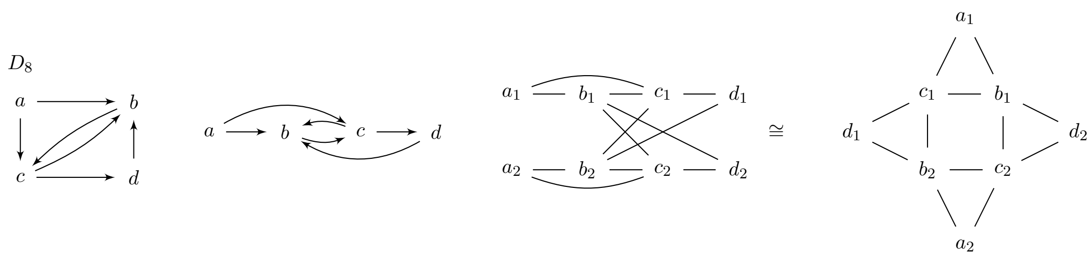

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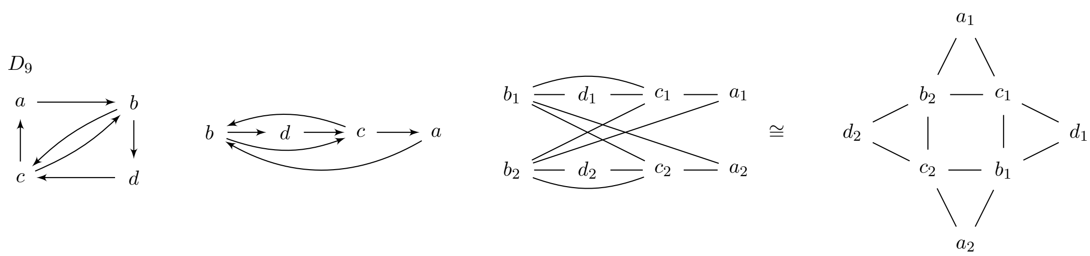
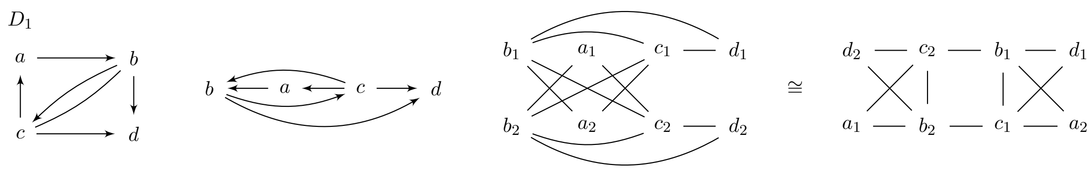
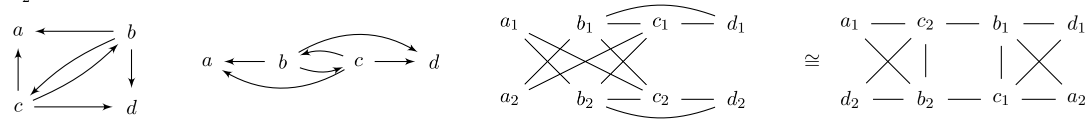


An ordering of Step 1 Step 2 and Step 3



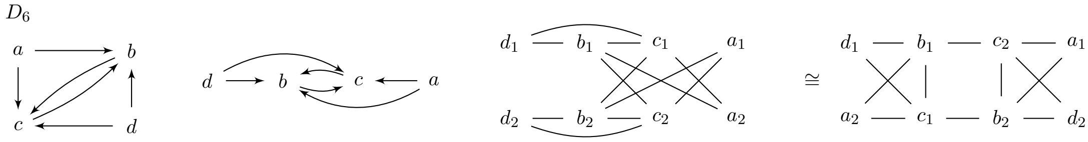
An ordering of Step 1 Step 2 and Step 3

D_2



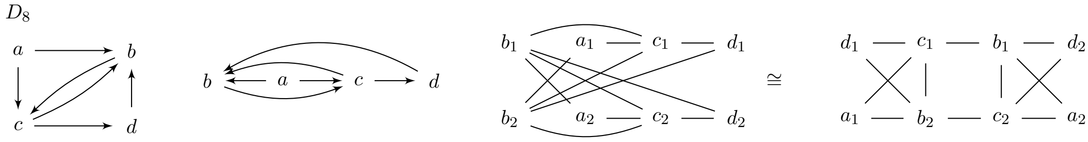
An ordering of Step 1 Step 2 and Step 3

D_6



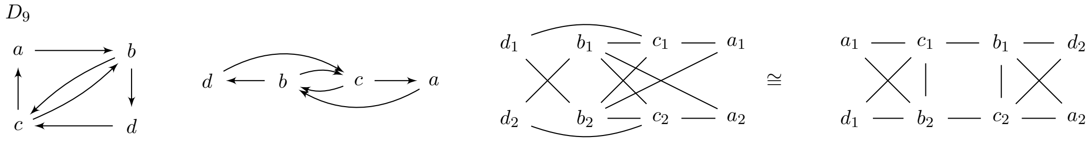
An ordering of Step 1 Step 2 and Step 3

D_8



An ordering of Step 1 Step 2 and Step 3

D_9



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