Radboud University Nijmegen

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Continued Root Fractions

Alternative Continued Fraction and its properties

Thesis BSc Mathematics

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Table of Contents

Introduction

When Pythagoras discovered his Pythagorean theorem he probably never imagined how famous it would become. This also gave us some famous corollaries, one of them being the discovery of irrational numbers, like the square root of 2. Even although this has greatly expanded our mathematical knowledge, it also provided new problems. One of them is the fact that irrational numbers have an infinite number of decimals. If we therefore are working with an irrational number in decimal form, we are always working with an estimation instead of with the actual value.

One of the methods to try and tackle this is by changing the way how we write down numbers. This gives rise to *continued fractions*, which are expressions of the form

$$a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{...}}} \tag{1}$$

These expressions can both have a finite or infinite length and since the expansion is entirely defined by its coefficients, we will simply denote it as $[a_0; a_1, a_2, ...]^{CF}$. Where the CF is to make clear that we are dealing with a continued fraction. If we have a finite length up to a_n, we say that we have an expansion of length n. One important thing to note is that we require $a_i \in \mathbb{Z}_{\geq 1}$ and $a_0 \in \mathbb{Z}$. This is important since this gives us some nice properties, for example convergence [4].

However, there also other ways that we can rewrite numbers. We could also take the square roots of our coefficients, instead of taking fractions. Which gives us expressions of the form

$$a_0 + \sqrt{a_1 + \sqrt{a_2 + \sqrt{a_3 + ...}}} \tag{2}$$

We call these kind of expressions *continued roots* and since they are also determined by its coefficients, we will denote them as $\sqrt{}(a_0; a_1, a_2, ...)^{CR}$. We now only require that $a_i \in \mathbb{Z}_{\geq 0}$, where the restriction for a_0 stay the same.

We have now seen two ways to rewrite any real number, however, there are even more ways to rewrite numbers. Each with their own property and functionality. One of these ways is to combine the previously mentioned methods into one new expansion. This would give us expressions of the form:

$$a_0 + \cfrac{1}{\sqrt{a_1 + \cfrac{1}{\sqrt{a_2 + \cfrac{1}{\sqrt{...}}}}}} \tag{3}$$

We will refer to this as a *continued root fraction*, which we will denote by $\sqrt{}[a_0; a_1, a_2, ...]$. Where the restrictions on the coefficients are the same as for the continued fraction.

Historically not a lot is known about continued root fractions. It was shortly mentioned by Kevin O'Bryant where he asked the question whether or not 3/4 has a finite continued root expansion. Which was also the inspiration behind this thesis. It is also mentioned in certain other articles, for example by D. J. Jones in "Continued reciprocal roots" [2]. However, no major works can be found on it.

Meanwhile, more can be found on continued fractions, the oldest mention known originates from 300 BCE by Euclid where he creates a continued fraction when he wrote his famous Euclidean algorithm. Another major work on continued fractions is a book by A. M. Rockett and P. Szüsz called "Continued Fractions" [4]. We will also use this book as inspiration for ideas throughout this thesis.

In this book a lot of properties for continued fractions are mentioned and discussed. We shall take some of these and see if something similar holds for our continued root fraction. Or whether we have something entirely different.

One of the main properties for continued fractions is that its expansion is finite if and only if the number we are expanding is a rational number [4]. We are interested if we have something similar for our continued root fraction, this is unfortunately not the case. Instead we have the following theorem for an expansion with a length of 2, which we shall explore more in Chapter 2:

Theorem. *Let* $a, b \in \mathbb{Z}_{\geq 0}$ *where* $\gcd(a, b) = 1$*, then the following are equivalent*

(i) $\dfrac{a}{b}$ *has length 2*

(ii) $a^2 \mid b^2 - 1$

(iii) $\frac{a}{b} = \sqrt{}[0; \frac{b^2-1}{a^2}, a^4]$*, where* $\frac{b^2-1}{a^2}, a^4 \in \mathbb{Z}_{\geq 1}$

As we can see in this theorem, we do have certain similarities between continued fractions and continued root fractions. However there are also differences.

As we already mentioned earlier, our continued fractions always have convergence if we put a restriction on the coefficient values. If we on the other hand look at continued roots we do not always have convergence. This can be seen by looking at the expansion $\sqrt{}(2; 2^3, 2^6, 2^9, ...)^{CR}$ which diverges. So the question remains: what applies for our continued root fraction. We will explore this in the next Chapter.

Chapter 1 Convergence

When we are speaking of convergence for a continued root fraction we want convergence if we look at $\sqrt{[}a_0; a_1, a_2, ..., a_n]$ where we let n go to infinity. For this we first introduce the function $C_n = \sqrt{[}a_0; a_1, a_2, ..., a_n]$. We can see that if we only have a finite number of coefficients it is clear that we have convergence.

1.1 Subsequences of C_n

Instead of looking at the sup C_n we shall say more about convergence by looking at the difference between consecutive terms. For this we will first see what happens when we change the value of only our last coefficient.

Proposition 1.1. *Let $\sqrt{[}a_0; a_1, ..., a_n]$ be a finite continued root fraction and suppose that $\widetilde{a}_n \in \mathbb{R}_{\geq 1}$ where $\widetilde{a}_n \geq a_n$.*

(i) If n is even then $\sqrt{[}a_0; a_1, ..., a_n] \leq \sqrt{[}a_0; a_1, ..., \widetilde{a}_n]$

(ii) If n is odd then $\sqrt{[}a_0; a_1, ..., a_n] \geq \sqrt{[}a_0; a_1, ..., \widetilde{a}_n]$

Proof. (i) We shall prove it by induction on n. Let $n = 0$, and assume that $\widetilde{a}_0 \geq a_0$. Then we have that $\sqrt{[}a_0] = a_0 \leq \widetilde{a}_0 = \sqrt{[}\widetilde{a}_0]$.

Now let us assume that it holds for $2n-2$, where $n \in \mathbb{Z}_{>0}$. And suppose that $\widetilde{a}_{2n} \geq a_{2n}$ then we have that

$$\sqrt{\widetilde{a}_{2n}} \geq \sqrt{a_{2n}}$$

So we have

$$a_{2n-1} + \frac{1}{\sqrt{\widetilde{a}_{2n}}} \leq a_{2n-1} + \frac{1}{\sqrt{a_{2n}}}$$

And we can conclude that

$$\frac{1}{\sqrt{a_{2n-1} + \frac{1}{\sqrt{\widetilde{a}_{2n}}}}} \geq \frac{1}{\sqrt{a_{2n-1} + \frac{1}{\sqrt{a_{2n}}}}}$$

Let us now take

$$\widetilde{a}_{2n-2} = a_{2n-2} + \frac{1}{\sqrt{a_{2n-1} + \frac{1}{\sqrt{\widetilde{a}_{2n}}}}}$$

and

$$\overline{a}_{2n-2} = a_{2n-2} + \frac{1}{\sqrt{a_{2n-1} + \frac{1}{\sqrt{a_{2n}}}}}$$

This gives us that

$$\sqrt{[}a_0; ..., \widetilde{a}_{2n}] = \sqrt{[}a_0; ..., \widetilde{a}_{2n-2}] \geq \sqrt{[}a_0, ..., \overline{a}_{2n-2}] = \sqrt{[}a_0; ..., a_{2n}]$$

by our Induction Hypothesis, completing our proof for (i).
(ii) We shall prove it again by induction on n. So let $n = 1$, and assume that $\widetilde{a}_1 \geq a_1$. We have that

$$\sqrt{\widetilde{a}_1} \geq \sqrt{a_1}$$

therefore we have

$$\frac{1}{\sqrt{\widetilde{a}_1}} \le \frac{1}{\sqrt{a_1}}$$

which gives us

$$\sqrt{}[a_0, \widetilde{a}_1] = a_0 + \frac{1}{\sqrt{\widetilde{a}_1}} \le a_0 + \frac{1}{\sqrt{a_1}} = \sqrt{}[a_0, a_1]$$

Where the induction to $2n-1$ goes the same as before, completing our proof. □

Remark that if we take $\widetilde{a_n} > a_n$, we would also have that $\sqrt{}[a_0; a_1, ..., a_n] < \sqrt{}[a_0; a_1, ..., \widetilde{a}_n]$ or $\sqrt{}[a_0; a_1, ..., a_n] > \sqrt{}[a_0; a_1, ..., \widetilde{a}_n]$. This could also be proven by adapting the previous proof and simply changing all the signs.

Proposition 1.1 now gives us some form of alternating system, where we alternate between a higher and a lower value. In fact if we look at the first few terms we can see that we have $C_0 < C_2 < C_3 < ...$ and $... < C_5 < C_3 < C_1$. So in order for us to say something about the convergence of C_n we can first look at the convergence of the subsequences of the even and the odd terms. All we need for this is to prove that they are monotone and bounded.

Corollary 1.2. *Suppose that n is even and positive, then $C_{n-1} > C_n$.*

Proof. Let $C_n = \sqrt{}[a_0, ..., a_n]$, we can take $\widetilde{a}_{n-1} = a_{n-1} + \frac{1}{\sqrt{a_n}} > a_{n-1}$ giving us $C_n = \sqrt{}[a_0, ..., \widetilde{a}_{n-1}]$. Then by Proposition 1.1 we have that $C_{n-1} = \sqrt{}[a_0, ..., a_{n-1}] > \sqrt{}[a_0, ..., \widetilde{a}_{n-1}] = C_n$. □

Corollary 1.3. *Let $n \in \mathbb{Z}_{>0}$, then we have:*

(i) If n is even then $C_{n+2} > C_n$.

(ii) If n is odd then $C_{n+2} < C_n$.

Proof. (i) Since $a_{2n+2} \ge 1$ we have that $0 \le \frac{1}{\sqrt{a_{2n+2}}} \le 1$ therefore we also have

$$\widetilde{a}_{2n+1} = a_{2n+1} + \frac{1}{\sqrt{a_{2n+2}}} \ge a_{2n+1}$$

from which we can conclude that

$$\widetilde{a}_{2n} = a_{2n} + \frac{1}{\sqrt{\widetilde{a}_{2n+1}}} \ge a_{2n}$$

Thus we get that $C_{2n+2} = \sqrt{}[a_0; a_1, ..., \widetilde{a}_{2n}] \ge C_{2n}$ by Proposition 1.1.
(ii) This proof is done in a similar way as (i) □

Proposition 1.4. *The sequences $\{C_{2n}\}_{n\ge0}$ and $\{C_{2n+1}\}_{n\ge0}$ both converge.*

Proof. By Corollary 1.3 we see that the sequence $\{C_{2n}\}_{n\ge0}$ is increasing and similarly that the sequence $\{C_{2n+1}\}_{n\ge0}$ decreasing. If the sequences are also bounded, they will both converge. Let us take C_1, we will show that this is a bound for $\{C_{2n}\}_{n\ge0}$. We now have for any even positive n that $C_1 > C_{n-1} > C_n$ by Corollary 1.2 & 1.3 as wanted. Let m now be odd, then we also have that $C_0 < C_{m-1} < C_m$. Therefore C_0 is a bound for $\{C_{2n+1}\}_{n\ge0}$. So we have two monotone functions that are both bounded, and so they are both converging. □

1.2 Convergence of C_n

Lemma 1.5. *Suppose that $x, y \in \mathbb{R}_{\geq 1}$ and $p \in \mathbb{R}$ where $0 \leq p \leq 1$. Then we have that:*

$$x \geq y \Rightarrow x - y \geq x^p - y^p$$

Proof. Assume that $x \geq y \geq 1$, then there is an $a \in \mathbb{R}$ such that $x = ay$ where $a \geq 1$. This gives us that $a^p \leq a$. Since we have that $x \geq 1$, we also have that $x - x^p \geq 0$. So have that

$$x - x^p = ay - (ay)^p = a(y - \frac{a^p}{a}y^p) \geq y - \frac{a^p}{a}y^p \geq y - y^p$$

so we can conclude $x - y \geq x^p - y^p$. □

Corollary 1.6. *Let $x, y, c, p \in \mathbb{R}$ where $0 \leq p \leq 1, c \geq 0$ and $x, y \geq 1$, then:*

(i) If $x \geq y$ then $(x + c)^p - (y + c)^p \leq x - y$.

(ii) If $x \leq y$ then $(x + c)^p - (y + c)^p \leq y - x$.

Proof. (i) This follows directly from Lemma 1.5. (ii) This can be seen from the fact that $(x + c)^p - (y + c)^p \leq 0 \leq y - x$. □

We can now start to work on our main part, namely to show that both our increasing sequences and decreasing sequences convergence to the same point. Which will then result in convergence for our continued root fraction. However, for this we first need to introduce a new variable for which we only want to look at a subsection of the expansion.

Definition 1.7. For $n \geq i \geq 0$ we define

$$\phi_i^n(a_i, a_{i+1}..., a_n) := a_i + \cfrac{1}{\sqrt{a_{i+1} + \cfrac{1}{\sqrt{... + \cfrac{1}{\sqrt{a_n}}}}}}$$

where we define the entire tail as

$$\phi_i^\infty(a_i, a_{i+1}...) := a_i + \cfrac{1}{\sqrt{a_{i+1} + \cfrac{1}{\sqrt{...}}}}$$

We first note that if $i \geq 1$ then we have that $\phi_i^n \geq 1$, since we have that $a_i \geq 1$ for all $i \geq 1$. We will also simply write ϕ_i^n and ϕ_i^∞ since the function parameters already determine which coefficients we take and they can therefore be omitted.

Finally we define $\Phi_i^n := \phi_{i+1}^n \cdot \phi_{i+1}^{n+1}$ as the product of two consecutive terms.

In order for us to say more about the convergence of C_n, we will first analyze Φ_i^n. It has some nice properties that we can use, one of them being that $\Phi_i^n \geq 1$, since $\phi_{i+1}^{n+1} \geq 1$ and $\phi_{i+1}^n \geq 1$. We shall now see in the next proposition that if we take the product of two terms, we can conclude that they are large enough for some kind of convergence property.

Proposition 1.8. *Let $n, i \in \mathbb{Z}_{>0}, i < n$, then $\Phi_i^n \cdot \Phi_{i+1}^n > (\frac{3}{2})^2$.*

Proof. From Definition 1.7 we know that we have

$$\Phi_i^n = \phi_{i+1}^n \cdot \phi_{i+1}^{n+1}$$

and

$$\Phi_{i+1}^n = \phi_{i+2}^n \cdot \phi_{i+2}^{n+1}$$

where all the terms are greater than or equal to 1.

Now let $\epsilon = \dfrac{1}{2}$, if we have that $\phi_{i+1}^n > 1+\epsilon$ we are done. So assume now that $\phi_{i+1}^n \leq 1+\epsilon$. Since we have

$$\phi_{i+1}^n = a_{i+1} + \frac{1}{\sqrt{\phi_{i+2}^n}} \leq 1 + \epsilon$$

we can see that we must have that $a_{i+1} = 1$ and

$$\frac{1}{\sqrt{\phi_{i+2}^n}} \leq \epsilon$$

which results in

$$\phi_{i+2}^n \geq 1/\epsilon^2 = 4 > 1 + \epsilon$$

So we have that either $\phi_{i+1}^n > 1 + \epsilon$ or $\phi_{i+2}^n > 1 + \epsilon$. Note that we can do the same for ϕ_{i+1}^{n+1} and ϕ_{i+2}^{n+1}. So we can now conclude that $\Phi_i^n \cdot \Phi_{i+1}^n > (1+\epsilon)^2 = (\frac{3}{2})^2$. □

Proposition 1.9. *Let $n \in \mathbb{Z}_{\geq 1}$, then*

$$|C_n - C_{n-1}| \leq |\frac{1}{\sqrt{\prod_{i=0}^{n-2} \Phi_i^{n-1}}}|.$$

Proof. Assume that we have $k \in \mathbb{Z}_{\geq 0}$ where $k < n-2$, we will first show that the following inequality holds

$$|\phi_k^n - \phi_k^{n-1}| \leq \frac{1}{\sqrt{\Phi_k^{n-1}}} |\phi_{k+1}^n - \phi_{k+1}^{n-1}| \tag{4}$$

To see this, first notice that

$$\begin{aligned}
|\phi_k^n - \phi_k^{n-1}| &= |(a_k + \frac{1}{\sqrt{\phi_{k+1}^n}}) - (a_k + \frac{1}{\sqrt{\phi_{k+1}^{n-1}}})| \\
&= |\frac{1}{\sqrt{\phi_{k+1}^n}} - \frac{1}{\sqrt{\phi_{k+1}^{n-1}}}| \\
&= |\frac{\sqrt{\phi_{k+1}^{n-1}} - \sqrt{\phi_{k+1}^n}}{\sqrt{\phi_{k+1}^n \phi_{k+1}^{n-1}}}| \\
&= \frac{1}{\sqrt{\Phi_k^{n-1}}} |\sqrt{\phi_{k+1}^{n-1}} - \sqrt{\phi_{k+1}^n}|
\end{aligned}$$

When k is even, we have that $n-k-2$ is even and so by Proposition 1.1 we have $\phi_{k+1}^{n-1} \leq \phi_{k+1}^n$. Now Corollary 1.6 gives us that

$$|\sqrt{\phi_{k+1}^{n-1}} - \sqrt{\phi_{k+1}^n}| \;\; \leq \;\; |\phi_{k+1}^n - \phi_{k+1}^{n-1}|$$

When k is odd, we instead have $\phi_{k+1}^{n-1} \geq \phi_{k+1}^{n}$. And we can now use that fact that $|a-b| = |b-a| \ \forall a, b \in \mathbb{R}$ to get the same result as in the even case by simply switching the signs.

This means that we can conclude that

$$\frac{1}{\sqrt{\Phi_k^{n-1}}}|\sqrt{\phi_{k+1}^{n-1}} - \sqrt{\phi_{k+1}^{n}}| \ \leq \ \frac{1}{\sqrt{\Phi_k^{n-1}}}|\phi_{k+1}^{n-1} - \phi_{k+1}^{n}|$$

as wanted.

We now have that

$$\begin{aligned}
|C_n - C_{n-1}| &= |\phi_0^n - \phi_0^{n-1}| \\
&= |\phi_0^n - \phi_0^{n-1}| \\
&\leq \frac{1}{\sqrt{\Phi_0^{n-1}}}|\phi_1^n - \phi_1^{n-1}| \\
&\leq \dots \\
&\leq \frac{1}{\sqrt{\prod_{i=0}^{n-2} \Phi_i^{n-1}}}|\phi_{n-1}^n - \phi_{n-1}^{n-1}| \\
&= \frac{1}{\sqrt{\prod_{i=0}^{n-2} \Phi_i^{n-1}}}|\frac{1}{\sqrt{a_n}}| \\
&\leq \frac{1}{\sqrt{\prod_{i=0}^{n-2} \Phi_i^{n-1}}}
\end{aligned}$$

□

Now we can finally say something about convergence, for this we shall first make a remark on how we can use the previous two propositions. If we namely look at $|C_n - C_{n-1}|$ we can see that we get

$$|C_n - C_{n-1}| \leq \frac{1}{\sqrt{(1+\epsilon)^{n-1}}}$$

where we can take $\epsilon = \frac{1}{2}$. And we see that this inequality goes to 0 as n goes to ∞.

Theorem 1.10. *Every continued root fraction converges.*

Proof. From the previous remark we see that $|C_n - C_{n-1}| \xrightarrow{\to\infty} 0$. We have also already seen that the sequences $\{C_{2n}\}_{n\geq 0}$ and $\{C_{2n+1}\}_{n\geq 0}$ both converge. So we can conclude now that they must converge to the same limit. Which is exactly the limit of our continued root fraction which must also converge. □

1.3 Alternative idea for a proof of convergence

We have proved now the convergence by looking at the expansion as a whole and by looking how we can estimate it. However, instead of this we could also start looking at the expansion in detail. For this we could ask ourselves the question what it means for a coefficient to be the value of a number $k \in \mathbb{Z}_{\geq 1}$.

So let us take $x = \sqrt{}[a_0; a_1, a_2, a_3, ...]$, we can take a_0 here to be 0, since the value of a_0 does not have any influence on our convergence. This gives us that $x \in [0, 1]$, and let us now assume that $a_1 = k \in \mathbb{Z}_{\geq 1}$. Then we have

$$x = \frac{1}{\sqrt{k + \phi_1^\infty}} \leq \frac{1}{\sqrt{k}}$$

Could we now also have that $x \leq \frac{1}{\sqrt{k+1}}$? No we can't, since if this were the case we would have chosen $a_1 = k + 1$. So we can conclude that we must have that $x \in [\frac{1}{\sqrt{k+1}}, \frac{1}{\sqrt{k}}]$. This process can be seen in the following figure:

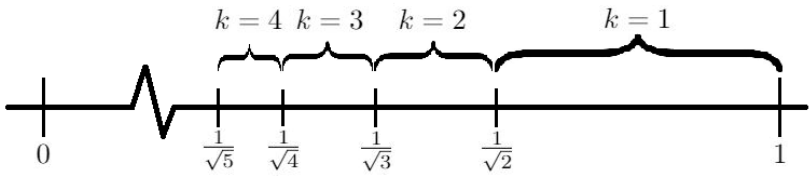

Figure 1: Alternative proof, first step

Since we are interested in the convergence of our expansion, we want to see what happens to the size of the interval where x lies inside. If this interval keeps shrinking in size by a constant factor, we retrieve convergence for the expansion. Let us remark that the first interval is the largest, so instead of taking a random k value, we simply take $k = 1$. Because this gives us that $x \in [\frac{1}{\sqrt{2}}, 1]$, we can say that we now have $|C_n - C_{n-1}| \leq 1 - \frac{1}{\sqrt{2}}$ for $n > 1$.

Now we could ask ourselves what it means for $a_2 = q = 1$. So that we have

$$x = \frac{1}{\sqrt{1 + \frac{1}{\sqrt{q + \phi_2^\infty}}}} \geq \frac{1}{\sqrt{1 + \frac{1}{\sqrt{q}}}} = \frac{1}{\sqrt{1 + \frac{1}{\sqrt{1}}}}$$

So could we again have that

$$x \geq \frac{1}{\sqrt{1 + \frac{1}{\sqrt{1+1}}}} \quad ?$$

No, because if this was the case then we would have taken $q = 2$. So we now have that

$$x \in [\frac{1}{\sqrt{1 + \frac{1}{\sqrt{1}}}}, \frac{1}{\sqrt{1 + \frac{1}{\sqrt{2}}}}]$$

giving us that

$$|C_n - C_{n-1}| \leq \frac{1}{\sqrt{1 + \frac{1}{\sqrt{2}}}} - \frac{1}{\sqrt{1 + \frac{1}{\sqrt{1}}}} \quad \text{for} \quad n > 2$$

Now we can start to see a pattern here, where we wish to show that $|C_n - C_{n-1}| \leq (1-c)^n$ for some constant c which has yet to be determined. A possible solution would be by using induction to n. We already have the induction step, since we know that

$|C_n - C_{n-1}| \leq 1 - \frac{1}{\sqrt{2}}$.

Now for the n-th case we would need to introduce some new kind of function or variable which takes the known expansion with only 1's and one 2 at the end. For now we will simply define

$$\varsigma_n := \sqrt{}[0; 1, ..., 1, 2]$$

where $a_n = 2$. The important step to note is that instead of adding elements to the back, we instead add elements to front:

$$\varsigma_{n+1} = \frac{1}{\sqrt{1 + \varsigma_n}}$$

where we now have multiple ways to rewrite ς_n, to get it into a desired form to estimate it. We shall give a few ways to rewrite it, but since this is an open problem there may be other more desirable forms.

$$\begin{aligned}
|C_{n+1} - C_n| &\leq |\varsigma_{n+1} - \varsigma_n| \\
&= |\varsigma_n - \frac{1}{\sqrt{1 + \varsigma_n}}| \\
&= \frac{\varsigma_n \sqrt{1 + \varsigma_n} - 1}{\sqrt{1 + \varsigma_n}} \\
&= \frac{\sqrt[4]{\varsigma_{n-1}}}{\sqrt{\sqrt{\varsigma_{n-1}} - 1}}
\end{aligned}$$

Where the last task now is to give a proper estimation of this equation.

1.4 Unique representation

Now that we know that a continued root fraction always exits, we can also say something about the uniqueness of its coefficients. We do not have total uniqueness, since we can always extend its length by one by a simple trick. So let us assume that we have the expansion given by $\sqrt{}[a_0; a_1, ..., a_n]$, then we also have that $\sqrt{}[a_0; a_1, ..., a_n] = \sqrt{}[a_0; a_1, ..., a_n - 1, 1]$. However we are still able to say something about the uniqueness if we exclude this trick.

Let us first look at the case where two expansions are the same except for one coefficient. If we can show that they then always expand towards a different number we can also conclude that the expansions must be unique.

Proposition 1.11. *Let $a_n \neq b_n$, then*

$$\sqrt{}[a_0; a_1, ..., a_{n-1}, a_n, a_{n+1}, ...] \neq \sqrt{}[a_0; a_1, ..., a_{n-1}, b_n, a_{n+1}, ...].$$

Proof. Assume, without loss of generality, that $a_n > b_n$, then we can find $m \in \mathbb{Z}_{\geq 1}$ such that $a_n = b_n + m$. Now let us look at the two tails as defined by Defintion 1.7, which gives us

$$\begin{aligned}
\phi_n^\infty(a_n, a_{n+1}, ...) &= a_n + \frac{1}{\sqrt{\phi_{n+1}^\infty(a_{n+1}, ...)}} \\
&= b_n + m + \frac{1}{\sqrt{\phi_{n+1}^\infty(a_{n+1}, ...)}} \\
&> b_n + \frac{1}{\sqrt{\phi_{n+1}^\infty(a_{n+1}, ...)}} \\
&= \phi_n^\infty(b_n, a_{n+1}, ...)
\end{aligned}$$

And since we have that

$$\sqrt{}[a_0; a_1, ..., a_{n-1}, a_n, a_{n+1}, ...] = \sqrt{}[a_0; a_1, ..., a_{n-1}, \phi_n^\infty(a_n, a_{n+1}, ...)]$$

and

$$\sqrt{}[a_0; a_1, ..., a_{n-1}, b_n, a_{n+1}, ...] = \sqrt{}[a_0; a_1, ..., a_{n-1}, \phi_n^\infty(b_n, a_{n+1}, ...)]$$

we now either have that $\sqrt{}[a_0; a_1, ..., a_{n-1}, a_n, a_{n+1}, ...] < \sqrt{}[a_0; a_1, ..., a_{n-1}, b_n, a_{n+1}, ...]$ or $\sqrt{}[a_0; a_1, ..., a_{n-1}, a_n, a_{n+1}, ...] > \sqrt{}[a_0; a_1, ..., a_{n-1}, b_n, a_{n+1}, ...]$ by the remark from after Proposition 1.1. □

This does not mean that we also have uniqueness, since we could have that later on there might be also be another different coefficient which causes us to convergence to the same number once more. However we will see now that this is not possible since we demanded that all our coefficients are elements from $\mathbb{Z}_{\geq 1}$.

Theorem 1.12. *Let* $\sqrt{}[a_0; a_1, ..., a_n, ...]$ *and* $\sqrt{}[b_0; b_1, ..., b_n, ...]$ *be two expansions where* $a_n \neq b_n$. *Then* $\sqrt{}[a_0; a_1, ..., a_n, ...] \neq \sqrt{}[b_0; b_1, ..., b_n, ...]$.

Proof. Let m be the smallest number such that $a_m \neq b_m$, this does exist since we know that $a_n \neq b_n$ and we have a finite number of possibilities of other unequal coefficients.

Let us assume again without loss of generality that $a_m > b_m$, then there is a number $c \in \mathbb{Z}_{\geq 1}$ such that $a_m = b_m + c$. Since we have $a_{m+1}, b_{m+1} \in \mathbb{Z}_{\geq 1}$ we have that $\phi_{m+1}^\infty(a_{m+1}, ...) > a_{m+1} \geq 1$ and $\phi_{m+1}^\infty(b_{m+1}, ...) > b_{m+1} \geq 1$. So therefore we have that

$$\frac{1}{\sqrt{\phi_{m+1}^\infty(a_{m+1}, ...)}} < 1$$

and

$$\frac{1}{\sqrt{\phi_{m+1}^\infty(b_{m+1}, ...)}} < 1$$

This gives us

$$\begin{aligned}
\phi_m^\infty(a_m, a_{m+1}, ...) &= a_m + \frac{1}{\sqrt{\phi_{m+1}^\infty(a_{m+1}, ...)}} \\
&= b_m + c + \frac{1}{\sqrt{\phi_{m+1}^\infty(a_{m+1}, ...)}} \\
&> b_m + c \\
&> b_m + c - 1 + \frac{1}{\sqrt{\phi_{m+1}^\infty(b_{m+1}, ...)}} \\
&\geq \phi_m^\infty(b_m, b_{m+1}, ...)
\end{aligned}$$

So we can conclude that $\phi_m^\infty(a_m, a_{m+1}, ...) > \phi_m^\infty(b_m, b_{m+1}, ...)$ and since we still have that

$$\sqrt{}[a_0; a_1, ..., a_m, ...] = \sqrt{}[a_0; a_1, ..., a_m, \phi_m^\infty(a_m, a_{m+1}, ...)]$$

and

$$\sqrt{}[b_0; b_1, ..., b_m, ...] = \sqrt{}[b_0; b_1, ..., b_{m-1}, \phi_m^\infty(b_m, b_{m+1}, ...)]$$

We can once again say that we have either $\sqrt{}[a_0; a_1, ..., a_m, ...] < \sqrt{}[b_0; b_1, ..., b_m, ...]$ or $\sqrt{}[a_0; a_1, ..., a_m, ...] > \sqrt{}[b_0; b_1, ..., b_m, ...]$ by the remark from after Proposition 1.1. □

Now we can conclude that expansions are unique, because if we have two expansions that are different at one or multiple positions they will always converge towards different numbers.

Chapter 2 Finite Expansions

In the introduction we shortly mentioned that a continued fraction is finite if and only if the number we are expanding is a rational number. We also know that every continued fraction expansion always converges [4]. Which now also holds for continued root fractions as we have seen in the previous chapter. So we could ask the question whether the finite expansion property also holds for our continued root fraction, which we shall explore in this Chapter.

2.1 Expansion of length 1

We shall start by looking at the simple case where we have an expansion of length 1. Let us first remark that there certainly are rational numbers that have a finite expansion. For example we have that $\frac{1}{2} = \surd[0; 4]$ and $\frac{1}{3} = \surd[0; 9]$. Something that stands out is the last coefficient, which seems to be the denominator squared. We can easily see why this happens, since we can rewrite the fractions by:

$$\frac{1}{n} = 0 + \frac{1}{\sqrt{n^2}} = \surd[0; n^2]$$

However not every finite expansion is also rational, because $\surd[0; 2] = \frac{1}{\sqrt{2}}$, which is not a rational number. This means that the property mentioned in the introduction for continued fractions does not hold for continued root fractions.

We shall analyze the previously mentioned property in the following theorem:

Theorem 2.1. *Let $n \in \mathbb{Z}_{\geq 1}$, then*

$$\surd[0; n] \in \mathbb{Q} \iff n \text{ is a square.}$$

Proof. We have

$$\begin{aligned} \surd[0; n] = 0 + \frac{1}{\sqrt{n}} &\in \mathbb{Q} \\ \iff \sqrt{n} &\in \mathbb{Z} \\ \iff n &\text{ is a square.} \end{aligned}$$

□

2.2 Expansion of length 2

Now let us look at rational numbers with an expansion of length 2. By first looking at some examples of fractions with a finite expansion we can already see some noteworthy things:

$\frac{2}{3} = \surd[0; 2, 16]$

$\frac{2}{5} = \surd[0; 6, 16]$

$\frac{2}{7} = \surd[0; 12, 16]$

Here we can clearly see some kind of correlation, since all the expansions look similar. The only difference is the second coefficient. We can further see here that when the denominator is increasing in size, so does the second coefficient. We could ask ourselves if these kind of properties hold for more rational numbers.

For the next section we shall use the convention that if we are referring to a rational number of the form $x = \frac{a}{b}$. We assume that $0 \leq x \leq 1$, since if this is not the case we can always return back to this case by adjusting a_0, our first coefficient. Because we already know that not every expansion of length 2 is a rational number, we shall first say something about the value that the coefficients such that the expansion is a fraction.

Proposition 2.2. *Let $\frac{a}{b} = \sqrt{}[0; a_1, a_2]$ be a finite continued root fraction of length 2. Then $a_1 = \frac{b^2-1}{a^2}$ and $a_2 = a^4$, where $a_1, a_2 \in \mathbb{Z}_{\geq 1}$.*

Proof. We shall give two different ways to prove it, where it depends on how we rewrite our fraction.

For the first proof we first note that we already know that the continued root fraction has length 2. So we can take

$$\frac{a}{b} = \frac{1}{\sqrt{a_1 + \frac{1}{\sqrt{a_2}}}} = \frac{1}{\sqrt{\frac{a_1\sqrt{a_2}+1}{\sqrt{a_2}}}} = \frac{\sqrt[4]{a_2}}{\sqrt{a_1\sqrt{a_2}+1}}$$

We can now take $a = \sqrt[4]{a_2}$ and $b = \sqrt{a_1\sqrt{a_2}+1}$ as a possible solution, which results in $a_2 = a^4$ and $a_1 = \frac{b^2-1}{a^2}$. However since we now have a solution and we know that our solutions are unique by Chapter 1, we also have that this is the correct solution.

An alternate way to prove this is by rewriting our fraction as

$$\frac{a}{b} = \frac{1}{\sqrt{\frac{b^2}{a^2}}} = \frac{1}{\sqrt{\frac{b^2-1}{a^2} + \frac{1}{\sqrt{a^4}}}}$$

Here we see that this also results in $a_1 = \frac{b^2-1}{a^2}$ and $a_2 = a^4$ as wanted. □

Let us now look again at the previous examples. We had

$\frac{2}{3} = \sqrt{}[0; 2, 16]$

$\frac{2}{5} = \sqrt{}[0; 6, 16]$

If we now use the formula from Proposition 2.2, we find that we have

$$\begin{cases} \frac{3^2-1}{2^2} = \frac{8}{4} = 2 \\ 2^4 = 16 \end{cases}$$

and

$$\begin{cases} \frac{5^2-1}{2^2} = \frac{24}{4} = 6 \\ 2^4 = 16 \end{cases}$$

as we expected.

One important requirement in these cases is the fact that $a_1 = \frac{b^2-1}{a^2}$ is a whole number, so we must have that a^2 divides $b^2 - 1$. We can assume that $a \neq 1$, because then our expansion has a length of 1 and we always prefer the shortest expansion. Therefore we can say that a^2 divides either $b-1$ or $b+1$, since $b^2-1 = (b-1)(b+1)$. So we see that we must have either $b \equiv 1 \bmod a^2$ or $b \equiv -1 \bmod a^2$. Using this we can say something about the existence of finite rational fraction expansions.

Corollary 2.3. *Let $a \in \mathbb{Z}$, then we can find $b \in \mathbb{Z}$, with $\gcd(a, b) = 1$, such that $\frac{a}{b}$ has a finite expansion.*

Proof. Let us take $b = 1 + a^2$, and assume that p is a prime where $p \mid a$. Then $p \mid a^2$, and since $p \neq 1$ we have that $p \nmid a^2 + 1$, so $p \nmid b$. Because this holds for every prime p we have that $\gcd(a, b) = 1$ as required. We now have that $b \equiv 1 \bmod a^2$, therefore we have, by the previous remark, that a^2 divides $b^2 - 1$. So we have $a_1, a_2 \in \mathbb{Z}_{\geq 1}$ for $a_1 = \frac{b^2-1}{a^2}$ and $a_2 = a^4$ where $\frac{a}{b} = \sqrt{[0; a_1, a_2]}$. □

We could also reverse this question, that is if we have $b \in \mathbb{Z}$, can we find an $a \in \mathbb{Z}$, with $\gcd(a, b) = 1$, such that $\frac{a}{b}$ has a finite expansion? Yes we can, because we can simply take $a = 1$ which gives us the trivial expansion discussed in the first section.

If we analyze the requirement that $a \mid b^2 - 1$ further, we can say something even stronger which we shall explore in the next theorem.

Theorem 2.4. *Let $a, b \in \mathbb{Z}_{\geq 0}$ where $\gcd(a, b) = 1$, then the following are equivalent*

(i) $\dfrac{a}{b}$ *expansion has length 2*

(ii) $a^2 \mid b^2 - 1$

(iii) $\frac{a}{b} = \sqrt{[0; \frac{b^2-1}{a^2}, a^4]}$, *where* $\frac{b^2-1}{a^2}, a^4 \in \mathbb{Z}_{\geq 1}$

Proof. *(i)* $\Rightarrow$ *(ii)* This follows directly from Proposition 2.2 if we combine it with the statement before Corollary 2.3.

(ii) $\Leftarrow$ *(i)* Assume that $a^2 \mid b^2 - 1$, then there is a $m \in \mathbb{Z}$ such that $ma^2 = b^2 - 1$. Or in other words, we have that $b = \pm\sqrt{ma^2 + 1}$, or simply $b = \sqrt{ma^2 + 1}$ since we assumed that $a, b > 0$. We can now rewrite our fraction $\frac{a}{b}$ in a similar way as in Proposition 2.2 resulting in

$$\begin{aligned}
\frac{a}{b} &= \frac{1}{\sqrt{\frac{b^2}{a^2}}} \\
&= \frac{1}{\sqrt{\frac{b^2-1}{a^2} + \frac{1}{\sqrt{a^4}}}} \\
&= \frac{1}{\sqrt{\frac{\sqrt{ma^2+1}^2 - 1}{a^2} + \frac{1}{\sqrt{a^4}}}} \\
&= \frac{1}{\sqrt{\frac{ma^2}{a^2} + \frac{1}{\sqrt{a^4}}}} \\
&= \frac{1}{\sqrt{m + \frac{1}{\sqrt{a^4}}}}
\end{aligned}$$

Since we had that $m \in \mathbb{Z}$ we can see that $\frac{a}{b} = \sqrt{[0; m; a^4]}$, therefore $\frac{a}{b}$ has a finite expansion of length 2.

$(i) \Rightarrow (iii)$ This has already been proven in Proposition 2.2.

$(iii) \Leftarrow (i)$ This holds per definition. $\square$

Given either an a or b we can always find a fraction of the form $\frac{a}{b}$ such that it has a finite expansion. However, there is also a correlation between the distance of fractions as seen in the example at the beginning of this section.

Corollary 2.5. *Let $\frac{a}{b}$ be a fraction with a finite expansion of length 2, with $\gcd(a, b) = 1$. Then $\frac{a}{b+a^2}$ also has a finite expansion of length 2.*

Proof. Since $\frac{a}{b}$ has a finite expansion of length 2, we know that $a^2|b^2 - 1$. So we also have that $a^2|b^2 - 1 + a^2(2a + a^2)$. So we have that $\frac{a}{b+a^2} = \sqrt{}[0; (b + a^2)^2 - 1; a^4]$ by Theorem 2.4. $\square$

2.3 Expansion of length 3

Now that we have a better understanding of an expansion of length 2, we can also look at what happens when we have an expansion of length 3. From the fact that we have:

$\frac{3}{7} = \sqrt{}[0; 5, 5, 256]$

$\frac{3}{11} = \sqrt{}[0; 13, 5, 256]$

and

$\frac{11}{119} = \sqrt{}[0; 117, 915, 256]$

$\frac{11}{123} = \sqrt{}[0; 125, 915, 256]$

we can see that fractions with an expansion of length 3 do exist. However we can also see that the last coefficient now appears to be a constant factor. So now the question remains what the new requirements are on our coefficients. For this we can first rewrite a fraction $\frac{a}{b}$ as before, giving us:

$$\frac{a}{b} = \frac{1}{\sqrt{\frac{b^2}{a^2}}} = \frac{1}{\sqrt{\frac{b^2-c}{a^2} + \frac{c}{\sqrt{a^4}}}} = \frac{1}{\sqrt{\frac{b^2-c}{a^2} + \frac{1}{\sqrt{\frac{a^4}{c^2}}}}} = \frac{1}{\sqrt{\frac{b^2-c}{a^2} + \frac{1}{\sqrt{\frac{a^4-1}{c^2} + \frac{1}{\sqrt{c^4}}}}}}$$

All of our coefficients must be elements from $\mathbb{Z}_{\geq 1}$, so we have that $c^2 \mid a^4 - 1$ and $a^2 \mid b^2 - c$. These requirements now determine our value of c, where this value can be different for different fractions.

For the fraction $\frac{3}{7}$, we have that $c^4 = 256$. So $c = \sqrt[4]{256} = 4$, which then gives that

$$a_1 = \frac{b^2 - c}{a^2} = \frac{49 - 4}{9} = 5$$

and

$$a_2 = \frac{a^4 - 1}{c^2} = \frac{81 - 1}{16} = 5$$

as required.

2.4 Expansion of length n

One thing to note from the case where we have an expansion of length 3, is that when we take $c = 1$ we are back with a length of 2. So we can also move back to a lower length, however we can also use this process to see what happens when we have an expansion of length n. So let $n \geq 3$, then we can rewrite $\frac{a}{b}$ in the following way:

$$\frac{a}{b} = \cfrac{1}{\sqrt{\cfrac{b^2 - p_0}{a^2} + \cfrac{1}{\sqrt{\cfrac{a^4}{p_0^2}}}}}$$

$$= \cfrac{1}{\sqrt{\cfrac{b^2 - p_0}{a^2} + \cfrac{1}{\sqrt{\cfrac{a^4 - p_1}{p_0^2} + \cfrac{1}{\sqrt{\cfrac{p_0^4}{p_1^2}}}}}}}$$

$$= \cfrac{1}{\sqrt{\cfrac{b^2 - p_0}{a^2} + \cfrac{1}{\sqrt{\cfrac{a^4 - p_1}{p_0^2} + \cfrac{1}{\sqrt{\cfrac{p_0^4 - p_2}{p_1^2} + \cfrac{1}{\sqrt{\cfrac{p_1^4 - p_3}{p_2^2}}}}}}}}}$$

$$= \cfrac{1}{\sqrt{\cfrac{b^2 - p_0}{a^2} + \cfrac{1}{\sqrt{\cfrac{a^4 - p_1}{p_0^2} + \cfrac{1}{\sqrt{\cfrac{p_0^4 - p_2}{p_1^2} + \cfrac{1}{\sqrt{\cfrac{p_1^4 - p_3}{p_2} + \cfrac{1}{\sqrt{... + \cfrac{1}{\sqrt{\cfrac{p_{n-3}^4}{p_{n-2}^2}}}}}}}}}}}}}$$

Or if we use our coefficient form we get

$$\frac{a}{b} = \sqrt{}[0; \frac{b^2 - p_0}{a^2}, \frac{a^4 - p_1}{p_0^2}, \frac{p_0^4 - p_2}{p_1^2}, ..., \frac{p_{n-4}^4 - p_{n-2}}{p_{n-3}^2}, \frac{p_{n-3}^4}{p_{n-2}^2}]$$

Here the p_i values have to be chosen case by case for different lengths and fractions. One thing that we can already note is the fact that we can take $p_{n-2} = 1$ or it must divide p_{n-3}^4, since we need $\frac{p_{n-3}^4}{p_{n-2}^2}$ to be an element from $\mathbb{Z}_{\geq 1}$. The same argument can be made for all other coefficients. So we have the requirement that $a^2 \mid b^2 - p_0$, $p_0^2 \mid a^4 - p_1$, $p_1^2 \mid p_0^4 - p_2$, ..., $p_{n-3} \mid p_{n-4}^4 - p_{n-2}$ and $p_{n-2}^2 \mid p_{n-3}^4$.

With this now in mind, we can revisit Corollary 2.5 and generalize it to the n-th length.

Theorem 2.6. *Let $\frac{a}{b}$ be a fraction with a finite expansion of length n, with* $\gcd(a, b) = 1$*. Then $\frac{a}{b+a^2}$ also has a finite expansion of length n.*

Proof. The case when $n = 1$ follows directly from Theorem 2.1 and the case when $n = 2$ has already been proven in Corollary 2.5. So let us now assume that $n \geq 3$. Then we know that $\frac{a}{b}$ must be of the form

$$\frac{a}{b} = \sqrt{}[0; \frac{b^2 - p_0}{a^2}, \frac{a^4 - p_1}{p_0^2}, \frac{p_0^4 - p_2}{p_1^2}, ..., \frac{p_{n-4}^4 - p_{n-2}}{p_{n-3}^2}, \frac{p_{n-3}^4}{p_{n-2}^2}]$$

for p_i still to be chosen. And as mentioned before, we must have that $a^2 \mid b^2 - p_0$, $p_0^2 \mid a^4 - p_1$, $p_1^2 \mid p_0^4 - p_2$, ..., $p_{n-3} \mid p_{n-4}^4 - p_{n-2}$ and $p_{n-2}^2 \mid p_{n-3}^4$. Since we assumed that $\frac{a}{b}$ already has an expansion of length n. We can also assume that all of these conditions hold. Therefore if we look at

$$\frac{a}{b+a^2} = \sqrt{}[0; \frac{(b+a^2)^2 - p_0}{a^2}, \frac{a^4 - p_1}{p_0^2}, \frac{p_0^4 - p_2}{p_1^2}, ..., \frac{p_{n-4}^4 - p_{n-2}}{p_{n-3}^2}, \frac{p_{n-3}^4}{p_{n-2}^2}]$$

All we need to prove is that $a^2 \mid (b+a^2)^2 - p_0$, where $(b+a^2)^2 - p_0 = b^2 - p_0 + 2a^2b + a^4$ since all the other requirements remain valid. However since $a^2 \mid b^2 - p_0$ we also have that $a^2 \mid b^2 - p_0 + a^2(2b + a^2)$ therefore we have that $\frac{a}{b+a^2}$ has indeed an expansion of length n. □

2.5 Equivalence Classes

In the introduction of Section 2.2 we had the following examples:

$\frac{2}{3} = \sqrt{}[0; 2, 16]$

$\frac{2}{7} = \sqrt{}[0; 12, 16]$

And in Section 2.3 we looked at:

$\frac{3}{7} = \sqrt{}[0; 5, 5, 256]$

$\frac{3}{11} = \sqrt{}[0; 13, 5, 256]$

One of the properties that both of these examples have is that the fractions of length 2 and length 3 both have the same tail. The expansions are so similar that we could also say that they are equivalent, so that $\frac{2}{3} \sim \frac{2}{7}$ and $\frac{3}{7} \sim \frac{3}{11}$. This gives rise to an equivalence relation, if we say that two fractions are equivalent if they have the same tail and length.

Before we formally define the equivalence relation, let us first note that the case where the expansion has a length of 1 is considered a separate case. Because we do consider $\sqrt{}[0; 9] = \frac{1}{3} \sim \frac{1}{4} = \sqrt{}[0; 16]$ equivalent, but they do not end in the same number. We also consider the case where we are dealing with an infinite expansion separately, since the tails no longer need to start at the same coefficient.

Let $\frac{a}{b} = \sqrt{}[a_0; a_1, ..., a_n]$ and $\frac{c}{d} = \sqrt{}[c_0; c_1, c_2, ..., c_m]$ we say that

$$\frac{a}{b} \sim \frac{c}{d} \iff \begin{cases} a = c = 1 \\ \quad \text{or} \\ \text{if } n = m, \text{ then we can find } i \in \mathbb{Z}_{\geq 1} \\ \quad \text{such that } \phi_i^n(a_i, a_{i+1}, ..., a_n) = \phi_i^n(c_i, c_{i+1}, ..., c_n) \\ \quad \text{or} \\ \text{if } n = m = \infty, \text{ then we can find } i, j \in \mathbb{Z}_{\geq 1} \\ \quad \text{such that } \phi_i^\infty(a_i, a_{i+1}, ...) = \phi_j^\infty(c_i, c_{i+1}, ...) \end{cases}$$

It is clear that the defined equivalence relation is reflexive-, symmetric and transitive in all cases. Thus we have a proper equivalence relation.

Theorem 2.7. *Let* $\frac{a}{b} \in \mathbb{Q}$*, then* $[\frac{a}{b}]$ *contains an infinite amount of elements.*

Proof. One way to see this, is to notice the fact that when $\frac{a}{b} \in [\frac{a}{b}]$ then we also have that $\frac{a}{b} + n \in [\frac{a}{b}]$ for all $n \in \mathbb{Z}$.

Let $\frac{a}{b}$ have a finite expansion. If $a \neq 1$ we can use Theorem 2.6, it follows directly that $\frac{a}{b+ma^2} \in [\frac{a}{b}]$ for all $m \in \mathbb{Z}_{\geq 0}$. In the case that $a = 1$, we can use that $\frac{1}{k} \sim \frac{1}{l}$ for all $k, l \in \mathbb{Z} \setminus \{0\}$.

□

Chapter 3 Basic Continued Root Fractions

In the previous Chapter we discussed the property of a finite expansion, where we saw that we had certain similarities between continued fractions and continued root fractions. However, we also saw that not every property of a continued fractions holds for continued root fractions. Another basic property for a continued fraction is that the expansion exists of repeating coefficients if and only if the number we are expanding is a quadratic number. So let us first look at a simple example.

3.1 Golden Ratio

One of perhaps the most famous numbers is the golden ratio. This number is about the ratio between the sum of two numbers. One of the main reasons that it has become so famous, is because the ratio can be seen in many real world examples. It even has a nice property when we look at its expansion for our continued root fraction, which we will discuss now.

Proposition 3.1. *Let* $\phi = \frac{1+\sqrt{5}}{2}$ *be the golden ratio, then*

$$\phi = \surd[1; 2, 2, 2, ...]$$

Proof. Take

$$x = \surd[1; 2, 2, 2, ...] = 1 + \cfrac{1}{\sqrt{2 + \cfrac{1}{\sqrt{\cdots}}}} = 1 + \frac{1}{\sqrt{1+x}}$$

If we simplify this formula we get $(x-1)^2 = \frac{1}{x+1}$ which gives us the following polynomial: $x^3 - x^2 - x = 0$. Our first solution is given by $x_1 = 0$, which leaves us with $x^2 - x - 1 = 0$ for which we can use the abc-formula. This results in

$$x_2 = \frac{1+\sqrt{5}}{2}$$

and

$$x_3 = \frac{1-\sqrt{5}}{2}$$

Since we have that $a_0 = 1$, we know that $x > 1$. Therefore $x_1 = 0$ and $x_3 < 0$ are no valid solutions for x. This leaves us with $x_2 = \frac{1+\sqrt{5}}{2} = x$ as the correct solution, which is exactly the golden ratio. □

We can now also look at the expansion of the golden ratio for the continued fraction and continued root. If we were to take $y = [1; 1, 1, 1, 1, ...]^{CF}$ for the continued fraction expansion, we get $y = 1 + \frac{1}{y}$. Which gives us $(y-1)y - 1 = 0$ or as a polynomial $y^2 - y - 1 = 0$. Which is the same polynomial as with the continued root fraction, so we see that y must be the golden ratio. For the continued root we can take $z = \surd(0; 1, 1, 1, ..)^{CR}$, then $z = \sqrt{1+z}$. Or $z^2 - z - 1 = 0$, so z is also the golden ratio.

This gives us the remarkable result that for our continued root fraction, the continued fraction and the continued root the expansion of the golden ratio has the form of a starting coefficient followed by a single repeating coefficient.

3.2 Basic root

Let us now look at a more simple example, the $\sqrt{2}$. For the continued fraction we can do something similar as before. So if we take $y = [1; 2, 2, 2, 2, 2]^{CF}$, then $y = 1 + \frac{1}{y+1}$, so $(y-1)(y+1)-1 = 0$. And we now get the polynomial $y^2 - 2 = 0$ or $(y-\sqrt{2})(y+\sqrt{2}) = 0$. So we can see that y must be $\sqrt{2}$. If we now look at the continued root, we can take $z = \sqrt{}(0; 0, 2, 2, ...)^{CR}$. So $z^2 = \sqrt{2 + \sqrt{...}}$ and $z^4 = 2 + \sqrt{2 + \sqrt{...}} = 2 + z^2$. Rewriting the last formula gives us $(z^2 - 2)(z^2 + 1) = 0$, therefore we have that z must be $\sqrt{2}$ since the other solutions are not valid.

So now we have seen two cases where both the continued fraction and continued root have a clear pattern for a quadratic number. It would be nice if we can draw a similar kind of conclusion for our continued root fraction, however this is not the case. Since we have the following expansion for out continued root fraction:

$$\sqrt{2} = \sqrt{}[1, 5, 1, 4, 1, 2, 2, 2, 101, 2, 69, ...]$$

So it is immediately clear that we do not have a similar kind of property for the continued root fraction. We shall now explore this further in the next part.

3.3 Repeating one coefficient

As shown in both examples, we are interested in the case where our expansions consists of the same repeating coefficient. So we could now look at the case where we have a continued root fraction of the form $x = \sqrt{}[a_0, a, a, a, a, ...]$. However since a_0 only shifts x over the real axis, we can take this to be 0.

So let $x = \sqrt{}[0; a, a, a, ...]$, we can rewrite x with a similar trick as before. This gives us:

$$x = \frac{1}{\sqrt{a + \frac{1}{\sqrt{...}}}} = \frac{1}{\sqrt{a + x}}$$

By rewriting this formula we get the polynomial equation $x^3 + ax^2 - 1 = 0$. To gain more insight in this polynomial we shall first look at its components and determinant [1]. For these we have that $p = -\frac{1}{3}a^2$, $q = \frac{2}{27}a^3 - 1$. Which results in the determinant being $D = q^2 + p^3 = (\frac{2}{27}a^3 - 1) + (-\frac{a^2}{3})^3 = -\frac{23}{729}a^6 - \frac{4}{27}a^3 + 1$. For which we have that $D > 0$ when $a < 3$ and $D < 0$ when $a \geq 3$. This gives us three different distinct cases, where $a = 1, 2$ or where a is larger than 2.

For the base case we have $a = 1$, which results in the equation $x^3 + ax^2 - 1 = 0$ for $x = \sqrt{}[0; 1, 1, 1, ...]$. The discriminant now takes the value of $\frac{1}{4} - \frac{1}{27} = \frac{23}{108} > 0$. The theory of cubic equations says now that our polynomial has 3 zeros, from which there is 1 real and 2 complex, we call these respectively x_1, x_2, x_3. Then $x_1 \cdot x_2 \cdot x_3 = x^3 + ax^2 - 1$, therefore we must have that $x_2 = \overline{x_3}$ the complex conjugate and we must have that $x_1 \cdot x_2 \cdot x_3 = -1$. Thus we have that $x_1^{-1} = -x_2 \cdot x_3$. From the standard solutions of cubic equations we get that

$$x_1 = -\frac{1}{3} + \frac{1}{3}\sqrt[3]{\frac{25}{2} - \frac{3\sqrt{69}}{2}} + \frac{1}{3}\sqrt[3]{\frac{25}{2} + \frac{3\sqrt{69}}{2}} \approx 0.75$$

Our second case is the special case where $a = 2$, for which we have already seen that $x = \sqrt{}[0; 2, 2, ...] = \phi - 1$.

Finally we have the case where $a > 2$, we still have that $p = -\frac{1}{3}a^2$ and $q = -\frac{1}{27}a^3 + \frac{1}{2}$. In this case our polynomial has three real solutions. These are given by

$$x_k = -\frac{a}{3} + \omega_k \sqrt[3]{-\frac{q}{2} + \sqrt{\frac{q^2}{4} + \frac{p^3}{27}}} + \omega_k^2 \sqrt[3]{-\frac{q}{2} - \sqrt{\frac{q^2}{4} + \frac{p^3}{27}}}$$

by Cordano's solutions [1]. Where $\omega_1 = 1, \omega_2 = -\frac{1}{2} + \frac{\sqrt{3}}{2}i, \omega_3 = -\frac{1}{2} - \frac{\sqrt{3}}{2}i$. It differs per case which ω_i gives the correct solution, but we do know that $0 < x < 1$. So we can rule out any ω_i that lies outside this interval.

3.4 Alternate between two coefficients

Instead of only repeating one coefficient, we could also alternate between several coefficients. Let us look at the simplest case where we only have 2 alternating coefficients, so let us take $a, b \in \mathbb{Z}_{>0}$. For which we will look at $x = \sqrt{}[0; a, b, a, b, a, b, a, ...]$. We can rewrite this again with the same trick as before, giving us

$$x = 0 + \cfrac{1}{\sqrt{a + \cfrac{1}{\sqrt{b + x}}}}$$

so we have

$$x^2 = \cfrac{1}{a + \cfrac{1}{\sqrt{b + x}}}$$

which gives us

$$\frac{1}{x^2} = a + \frac{1}{\sqrt{b + x}}$$

subtracting a and squaring both sides gives

$$(\frac{1}{x^2} - a)^2 = \frac{1}{b + x}$$

or

$$\frac{(1 - ax^2)^2}{x^4} = \frac{1}{b + x}$$

All that is left now is a cross multiplication and some simple addition and multiplication. For which we get

$$\begin{aligned} x^4 &= (1 - ax^2)^2(b + x) \\ &= (a^2x^4 - 2ax^2 + 1)(b + x) \end{aligned}$$

which results in the final polynomial

$$0 = a^2x^5 + (a^2b - 1)x^4 - 2ax^3 - 2abx^2 + x + b = F$$

So here we can see that we get a polynomial of degree 5. Since our possible x values are not limited to rational numbers, we know that we can find $a_1, ..., a_5 \in \mathbb{C}$, which do not all need to be complex, such that $F = (x - a_1) \cdot ... \cdot (x - a_5)$. There is not much that we can say for all these a_i, but we do know that $a_1 \cdot ... \cdot a_5 = \frac{b}{a^2} \in \mathbb{R}$. So for every complex a_i there must also be a complex conjugate $\overline{a_i}$ and since the degree of F is 5, we must have at least one real solution.

For the case that we had one alternating coefficient, we had a polynomial with a degree of 3. And when we had two alternating coefficients we had a polynomial with a degree of 5. If we were to expand this to the case where we are alternating between n coefficients, we would expect a degree of $2n + 1$. This wouldn't be surprising, from the fact that we would have n roots.

With a degree of $2n + 1$ we have $2n + 1$ possible complex solution. However just as before we need a complex conjugate for every solution of the polynomial. And since the degree is $2n + 1$ we will always have a real solution.

3.5 Transcendental numbers

Instead of looking at algebraic numbers, we can also look at irrational numbers. Even although we lose the repeating property for continued fractions, we can still have a nice looking sequence of coefficients.

For this we look at the exponential constant $e \approx 2.71628$. And if we look at its continued fraction expansion we get $e = [2; 1, 2, 1, 1, 4, 1, 1, 6, 1, 1, 8, ...]^{CRF}$. Which does have a clear pattern, since we constantly have two consecutive ones followed by a coefficient of the form $2n$. This can also be defined recursively in the following way:

Let $a_0 = 2, a_1 = 1$ and $a_2 = 2$ and for $n > 2$ we have

$$a_n = \begin{cases} a_{n-3} + 2 & \text{if n } \equiv 2 \, mod \, 3 \\ 1 & \text{else} \end{cases}$$

In the case of our continued root fraction however we have

$$e = \sqrt{}[2, 1, 1, 54, 96, 4, 36, 487, 1, 1, 23, ...],$$

so unfortunately some properties are lost to us since there is no clear pattern to be found here.

Chapter 4 Statistical Analysis

For the expansion of the exponential, $e = \sqrt{}[2, 1, 1, 54, 96, 4, 36, 487, 1, 1, 23, ...]$, one thing that stands out is the high amount coefficients with the value 1. Which gives us the question whether or not this is always case or that it is just a mere coincidence.

This problem has also been looked at for continued fraction's by Carl Friedrich Gauss, who derived a distribution for it around 1800. For which the bound was later given my Rodion Kuzmin, which gave the distribution the name "Gauss-Kuzmin distribution" [3]. It gives us that the probability that coefficient k_i has the value k can be given by:

$$\lim_{n\to\infty} \mathbb{P}\{k_i = k\} = -\log_2(1 - \frac{1}{(k+1)^2})$$

We are now interested in the distributed for our continued root fraction. In order for us to get a general idea first, we can use a simple experiment to determine the distribution. The already established distribution for the continued fraction also gives us the possibility to see how accurate our experiment will be. Let us first set the experiment up.

4.1 Experiment set-up

We first need a testing environment, for this we will use the program called *Magma*. After we have gathered all the data we use *Python* to create some simple graphs for us.

There are several methods how we can set up the experiment, one of them being that we first chose a number beforehand, like π and then calculate its expansion up to a certain length. Another one would be to pick numbers randomly and then calculate a shorter expansion but with more random numbers.

For this experiment we have chosen the latter, since this gives us a form of randomness and calculating shorter expansions is a less demanding process for the computer. For this we use a random number generator which gives us numbers between 0 and 1 with a certain amount of decimals. After we have generated a random number we calculate its expansion and then look at the value of the coefficients. This process is repeated several times in order for us to get a good impression of the distribution.

Note that this experiment set-up allows us to easily switch between continued fraction expansions and continued root expansions. All that is needed for this is a change in the calculation of the expansion.

There are a few things that we need to keep in mind, which might cause the experiment to deviate from the actual result:

- The length of the expanded continued fraction is limited.
- The amount of decimals is limited, so this means that we cannot calculate the expansion of any numbers that are truly irrational.

However by using enough numbers and by having enough possible decimals we can reduce the interference of these possible problems as much as possible.

4.2 Experiment results

For the first experiment we compared the expansion of our continued fraction against the known Guass-Kuzmin formula. We used random numbers which can have up to 50 decimals, where we tested a total of 2000 numbers which we all expanded to a continued fraction of length 13.

The results of the first coefficients can be seen in Table 1, by noting that our found values of our experiment are indeed very close compared to the Gauss-Kuzmin values. We can conclude that the experiment does work correctly.

k	Experiment result (%)	Gauss-Kuzmin (%)
1	42.15	41.50
2	17.28	16.99
3	9.288	9.311
4	5.983	5.889
5	4.062	4.064
6	3.058	2.975

Table 1: Results for the continued fraction against Gauss-Kuzmin

4.3 Continued root fractions versus continued fractions

We can now use the experiment set-up for our continued root fraction, to get a general idea of the distribution of our coefficient values. If we look at the alternative idea of a proof from Chapter 1 we can already make a prediction based on the intervals stated there. Since we have that the higher the value of a coefficient the smaller its interval will be. Thus we would expect that lower valued coefficient will appear more often.

For this experiment we use the same variables as before. So we take random numbers up to 50 decimals, we experiment a total of 2000 numbers and we expand them all to a length of 13. The results can be seen in Figure 2.

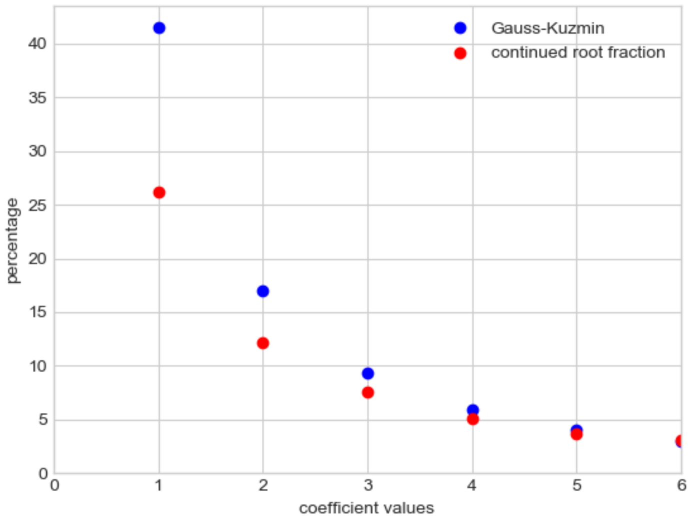

Figure 2: Continued root fraction vs Gauss-Kuzmin

One thing that is clear immediately is that the odds for lower coefficient occurring is smaller for our continued root fraction.

This was also to be expected if we look at the alternative proof in Chapter 1 once more. The starter intervals for the continued fraction are of the form $[\frac{1}{n+1}, \frac{1}{n}]$ whilst for the continued root fraction they are of the form $[\frac{1}{\sqrt{n+1}}, \frac{1}{\sqrt{n}}]$. So this means that early on there will be a big difference between the values, but this difference becomes smaller as n becomes larger.

Chapter 5 Generalization

So far we have seen three different ways how we can rewrite numbers as discussed in the introduction. Namely as a continued fraction, a continued root or a continued root fraction. However, there are even more different ways to rewrite any number. We could for example take a continued root fraction of a 3rd degree, or a 4th, or even a qth degree root.

For this new generalization of a continued fraction we shall use a new notation discussed by D. Jones [2]. Here we put the power on the left side instead of on the right to make clear where it belongs to. So let us take $p \in \mathbb{R}$, then we have

$$\begin{aligned}\mathop{\mathbf{K}}_{i=0}^{\infty} a_i^p &:= \lim_{i\to\infty} a_0 + (a_1 + (a_2 + (...)^p ...)^p)^p \\ &= a_0 + {}^p(a_1 + {}^p(a_2 + {}^p(...)...))\end{aligned}$$

and we have

$$\begin{aligned}\mathop{\mathbf{K}}_{i=0}^{n} a_i^p &:= a_0 + (a_1 + (a_2 + (... + (a_n)^p ...)^p)^p \\ &= a_0 + {}^p(a_1 + {}^p(a_2 + {}^p(... + {}^p(a_n)...))\end{aligned}$$

Note that this limit doesn't necessarily exist, which we saw for the following continued root example:

$$x = \sqrt{}(2; 2^3, 2^6, 2^9, ...)^{CR} = 2 + {}^{1/2}(2^3 + {}^{1/2}(2^6 + {}^{1/2}(2^9 + ...)))$$

5.1 Distribution for different p values

In Chapter 4 we used a experiment to get an idea what the distribution of the coefficients is for our continued root fraction. We could now expand this experiment by adapting it such that it works for all p values. This will give us not only a general idea for the distribution, but also for the convergence for different p values.

We will now use the same experiment as before, the only difference is that we test 100 different p values ranging from -1 to 1 and for the p values larger than 0 we take an expansion length of 50. This is done because the convergence, if we have any, is a lot slower than in the case where p is smaller than 0. Another note is that for $p > 0$ we can also have coefficient values that are equal to 0. The results can be seen in Figure 3 and Table 2.

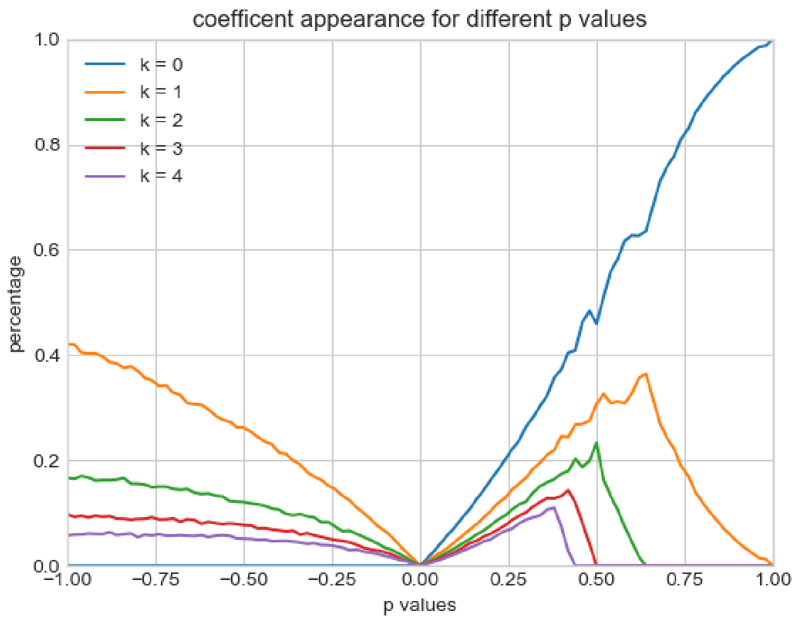

Figure 3: Coefficient appearance rate percentage

p	$k = 1$ (%)	$k = 2$ (%)	$k = 3$ (%)
-1	42.70	16.76	9.243
-4/5	36.17	15.87	8.736
-3/5	29.66	13.52	8.293
-1/2	25.76	12.60	8.203
-2/5	21.39	10.66	6.421
-1/5	12.10	6.656	4.197
0	0.0	0.0	0.0
1/5	12.14	9.100	7.098
2/5	38.20	21.17	20.98
1/2	55.38	42.21	0.0
3/5	84.41	12.10	0.0
4/5	90.08	0.0	0.0
1	0.0	0.0	0.0

Table 2: Appearance rate coefficients

When $p < 0$ we can see a clear pattern where the frequency of lower valued coefficient increases as p decreases. We also know that we always have convergence for $p = -1$ and $p = -0.5$. So we would expect this to be also the case for the p values between -1 and -0.5. And in fact as we will see in the next section we will have convergence for all p values less than 0.

In the case that we have $p > 0$ things become more complicated. There seems to be a clear switching point for every coefficient value, after which they become less frequent. One possible explanation for this is that the convergence is slower. Which would mean that we would get higher coefficients less often.

5.2 Convergence of the p-th expansion

Let us now first look at convergence when $p < 0$, we have already said something about its convergence. Which we will now look at in more detail in the next theorem.

Theorem 5.1. *If* $-1 \leq p \leq 0$ *then* $\mathop{\mathbf{K}}_{i=0}^{\infty} a_i^p$ *converges.*

Proof. We first look at the case where $p = 0$, since we have $a^0 = 1$ for $a \geq 1$ we find that

$$\mathop{\mathbf{K}}_{i=0}^{\infty} a_i^p = a_0 + 1$$

we see that we always have convergence. However we only have convergence to any number x if $x \in \mathbb{Z}$.

Let us now take $-1 < p < 0$, from Theorem 1 in the article "Continued reciprocal roots" by D. J. Jones [2] we get the result that

$$\mathop{\mathbf{K}}_{i=0}^{\infty} a_i^p \text{ diverges} \iff \limsup_{i \to \infty} a_i^{-p^i} < 1$$

Since we have the requirement that $a_i \geq 1$ for $i \geq 1$, we have that $a_i^{-p^i} < 1$ for all $i \geq 1$. So we can see that by the Theorem that we always have convergence.

In the case that $p = -1$ we are dealing with the standard Continued Fraction, for which we already know that it converges [4].

□

Conjecture 5.2. *If* $p \leq -1$ *then* $\mathop{\mathbf{K}}_{i=0}^{\infty} a_i^p$ *converges.*

In the case that $p < -1$ we expect convergence from the fact that

$$\mathop{\mathbf{K}}_{i=0}^{\infty} a_i^p < \mathop{\mathbf{K}}_{i=0}^{\infty} a_i^{-1}$$

However we cannot immediately derive convergence from this fact, since we also have a slower convergence. This can be seen by adapting the "alternative idea for a proof of convergence" from Chapter 1. For example if we would take $a_1 = 1$, we have that in the case that $p = -1$ our expansion must be inside the interval $[\frac{1}{2}, 1]$. If we on the other hand take $p = -2$, the interval in which the expansion must be inside is $[\frac{1}{4}, 1]$ which is clearly larger.

We also cannot adapt our main proof from Chapter 1 since the inequality from Lemma 1.5 no longer holds.

We shall therefore leave this part as an open problem.

For the case that $p > 0$ we can look at the article *Continued Roots* by Walter S. Sizer [5] where a proof is given for the convergence of a normal continued root. We will instead this proof and generalize it, and make a few changes to say something about the convergence in the generalized form.

Proposition 5.3. *For all $q > 1$ there is a large enough $B > 0$ such that $B \leq B^q - B$*

Proof. Note that by rewriting the equation, we see that this is the same as $B^{q-1} \cdot B = B^q \geq 2 \cdot B$. So we need $B^{q-1} > 2$, or $B > \sqrt[q-1]{2} > 0$. Since we have that $q > 1$, we see that we can always find a B large enough to satisfy this equation. □

Now let us first introduce $K_n^p := \mathop{\mathbf{K}}_{i=0}^{n} a_i$ as a shorter way to write our new expression. We will now see that we have a conditional convergence for $0 < p < 1$.

Proposition 5.4. *Let $0 < p < 1$, if $\{a_i\}_{i \geq 0}$ is bounded then K_i^p converges*

Proof. Because we assumed that all a_i's are bounded, we know that there is a B such that $a_i < B$ for all $i > 0$. If we take $q = \frac{1}{p}$ we can find a B such that $B^q - B \geq B$ by Proposition 5.3. This gives us that $a_i \leq B \leq B^q - B = B(B^{q-1} - 1)$. So we have that

$$\begin{aligned} K_n^p &= a_0 + \sqrt[q]{a_1 + \sqrt[q]{... \sqrt[q]{a_n}}} \\ &= \sqrt[q]{/}(a_0; a_1, ..., a_n) \\ &\leq \sqrt[q]{/}(a_0; B^q - B, B^q - B, ..., B^q - B) \end{aligned}$$

We shall now use induction to show that $K_n^p \leq a_0 + B$ for all $n \geq 0$. When $n = 0$ it is clear, since $K_0^p = a_0 \leq a_0 + B$. Now let $n = 1$, then we have that

$$\begin{aligned} K_1^p &= a_0 + \sqrt[q]{a_1} \\ &\leq a_0 + \sqrt[q]{B^q - B} \\ &\leq a_0 + \sqrt[q]{B^q} \\ &= a_0 + B \end{aligned}$$

Let us now assume that for n the following induction hypothesis holds

$$\begin{aligned} K_n^p &\leq \sqrt[q]{B^q - B + \sqrt[q]{... + \sqrt[q]{B^q - B}}} \\ &\leq \sqrt[q]{B^q - B + \sqrt[q]{... + \sqrt[q]{B^q}}} \\ &\leq a_0 + B \end{aligned}$$

then for $n+1$ we have

$$\begin{aligned}
K^p_{n+1} &= \sqrt[q]{}(a_0; a_1, ..., a_n) \\
&= a_0 + \sqrt[q]{a_1 + \sqrt[q]{... + \sqrt[q]{a_n + \sqrt[q]{a_{n+1}}}}} && (n{+}1 \text{ terms}) \\
&\leq a_0 + \sqrt[q]{B^q - B + \sqrt[q]{... + \sqrt[q]{B^q - B + \sqrt[q]{B^q - B}}}} && (n{+}1 \text{ terms}) \\
&\leq a_0 + \sqrt[q]{B^q - B + \sqrt[q]{... + \sqrt[q]{B^q - B + \sqrt[q]{B^q}}}} && (n{+}1 \text{ terms}) \\
&\leq a_0 + \sqrt[q]{B^q - B + \sqrt[q]{... + \sqrt[q]{B^q - B + B}}} && (n \text{ terms}) \\
&\leq a_0 + \sqrt[q]{B^q - B + \sqrt[q]{... + \sqrt[q]{B^q}}} && (n \text{ terms}) \\
&\leq a_0 + B
\end{aligned}$$

So we now have that K^p_n is bounded, note that it is also increasing since every a_i is positive. We can thus conclude that it converges. □

We have now seen that in the case where $\{a_i\}$ is bounded and where $0 < p < 1$ we have convergence for $\mathop{\mathbf{K}}_{i=0}^{\infty} a_i^p$. However as we can see in Figure 3, this does not need to hold when $p \geq 1$. Which we shall explore further in the following theorem.

Theorem 5.5. *Let $p > 0$ then we that have the following holds for $\mathop{\boldsymbol{K}}_{i=0}^{\infty} a_i^p$*

(i) If $p < 1$, we have convergence if and only if the set $S = \{a_i^{p^i} : i \geq 1\}$ is bounded

(ii) If $p \geq 1$, we have divergence

Proof. (*i*) First let $q = \frac{1}{p}$, we can have that $K^q_n = \mathop{\mathbf{K}}_{i=0}^{n} a_i^p$ so we only have to proof the statement for K^q_n.

($\Rightarrow$) Suppose that K^q_n converges but that $S = \{a_i^{p^i} : i \geq 1\}$ is not bounded. Then for any $B > 0$ there is an $N > 0$ such that $a_N^{p^N} > B$. However since we have that $K^q_n \geq a_N^{p^N} B$ for all $B > 0$ we have divergence for K^q_n. But this is a contradiction with our assumption, thus S must be bounded.

($\Leftarrow$) Assume that the set $S = \{a_i^{p^i} : i \geq 1\}$ is bounded, and let it be bounded by a $B > 0$.

First we will rewrite K_n^p for any $D > 0$, we can do so by:

$$\begin{aligned}
D \cdot K_n^p &= D \cdot (a_0 + \sqrt[q]{a_1 + \sqrt[q]{a_2 + \sqrt[q]{... + \sqrt[q]{a_n}}}}) \\
&= D \cdot a_0 + D \cdot \sqrt[q]{a_1 + \sqrt[q]{a_2 + \sqrt[q]{... + \sqrt[q]{a_n}}}} \\
&= D \cdot a_0 + \sqrt[q]{D^q \cdot a_1 + D^q \cdot \sqrt[q]{a_2 + \sqrt[q]{... + \sqrt[q]{a_n}}}} \\
&= D \cdot a_0 + \sqrt[q]{D^q \cdot a_1 + \sqrt[q]{D^{q^2} \cdot a_2 + D^{q^2} \cdot \sqrt[q]{... + \sqrt[q]{a_n}}}} \\
&= D \cdot a_0 + \sqrt[q]{D^q \cdot a_1 + \sqrt[q]{D^{q^2} \cdot a_2 + \sqrt[q]{... + \sqrt[q]{D^{q^n} \cdot a_n}}}}
\end{aligned}$$

In a similar way we get $K_n^p = \sqrt[q]{}(a_0, a_1, ..., a_n) = B * \sqrt[q]{}(\widetilde{a_0}, \widetilde{a_1}, ..., \widetilde{a_n})$ where

$$\widetilde{a_i} = \frac{a_i}{B^{q^i}}$$

Since we have that S is bounded by B we now have $0 \leq \widetilde{a_i} \leq 1$ for all $i \geq 0$. So we know that $\sqrt[q]{}(\widetilde{a_0}, \widetilde{a_1}, ..., \widetilde{a_n})$ converges for n by proposition 5.4. Because B is fixed, we must now also have that $K_n^p = \sqrt[q]{}(a_0, a_1, ..., a_n)$ converges.

(ii) Since we have that $a_i \geq 1$ for all $i \geq 1$ we get

$$\mathop{\mathbf{K}}_{i=1}^{\infty} a_i^p \geq \sum_{i=1}^{\infty} 1 = \infty$$

so we always have divergence for $p \geq 1$ □

W. S. Sizer shortly mentions a way to rewrite any number n in a continued root form with the following formula [5]:

$$n = \sqrt{}(0; n(n-1), n(n-1), n(n-1), ...)$$

By changing a small part we can also do the same for the qth root.

Let $n > 0$, we can rewrite n as follows: $n = \sqrt[q]{n^q} = \sqrt[q]{n^k - n + n}$. We can now use this process to create a possible continued qth root expansion of n. So let us take $0 < n < 1$ and $q > 1$, then

$$\begin{aligned}
n &= \sqrt[q]{n^q - n + n} = \sqrt[q]{n(n^{q-1} - 1) + n} \\
&= \sqrt[q]{n(n^{q-1} - 1) + \sqrt[q]{n^q - n + n}} \\
&= \sqrt[q]{n(n^{q-1} - 1) + \sqrt[q]{n(n^{q-1} - 1) + n}} \\
&= \sqrt[q]{n(n^{q-1} - 1) + \sqrt[q]{n(n^{q-1} - 1) + \sqrt[q]{n^q - n + n}}} \\
&= ...
\end{aligned}$$

Therefore we have that

$$n = \sqrt[q]{}(0; n(n^{q-1} - 1), n(n^{q-1} - 1), n(n^{q-1} - 1), ...)$$

Chapter 6 Generalized Experiments

6.1 Rate of convergence

In addition to convergence, we are also interested in the speed of convergence. Since this might indicate if a p value is preferable over another p value. It is clear that not every p-value gives the same rate of convergence, since the convergence speed is, amongst others determined by:

- The value of the coefficients.
- The value of p.

To see how this works in practical way, we have simulated an experiment where we estimate the value of π. We first take a p value and then calculate the expansion of π up to a fixed length. After this we reverse the process and estimate π by using our p-th continued fraction. This gives us an approximation of the value of π

The result can be seen in Figure 4. Here we calculated 50 coefficients before reversing the process.

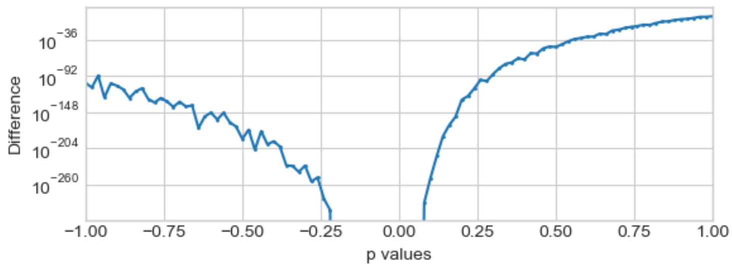

Figure 4: Approximation of π

As we can see here, the closer our value of p is to 0, the closer our estimation is to π.

This was also to be expected for $p < 0$ if we take a look back at the alternative idea for a proof in Chapter 1. Here we discussed that our approximation is limited by intervals which are constantly decreasing in size. And these speed in which they decrease is faster when p is closer to 0.

If $p > 0$ we can look at Figure 3. Here we see that the closer we are to 0 the higher our coefficients are expected to be. Even although a lower p value also slows our convergence speed, it seems that the process that the influence from higher coefficients is bigger.

6.2 Optimal p value

From the previous section the question might arise that if our expansion is so much more accurate when our p value is small, then why don't we just take our p value to be as small as possible?

One possible complaint for a chosen p value might be that we make our expansions unnecessary long with over complicated coefficient values. Let us for example take $x = \frac{1}{10}$ and let us calculate the expansion for some different p values. Then we have the following expansions:

p value	expansion
-1	[0; 10]
-0.5	[0;100]
-0.01	[0; 1E100]
0.01	[-1; 13779; 5.6E20, 5.5E22, 346, ...]
0.5	[-1; 0, 0, 1, 0, 0, 1, 2, 1, 2, 1, ...]

Table 3: Different expansions of $x = \frac{1}{10}$

We can see that for even a simple number as $\frac{1}{10}$ we sometimes get enormous coefficients. Does this mean however that taking smaller p values is always worthless? No, it mostly depends on what expansion we are calculating. Let us now take

$$x = -\frac{1}{3} + \frac{1}{3}\sqrt[3]{\frac{25}{2} - \frac{3\sqrt{69}}{2}} + \frac{1}{3}\sqrt[3]{\frac{25}{2} + \frac{3\sqrt{69}}{2}}$$

In Chapter 3 we have already seen that the expansion for $p = -0.5$ is $\sqrt{}[0; 1, 1, 1, 1, 1, ...]$ and for the other p values we have:

p value	expansion
-1	[0; 1, 3, 12, 1, 1, 3, 2, 3, 2, ...]
-0.01	[0; 16, 82, 81, 20857, 201, 694, ...]
0.01	[-1; 2.7E24; 0, 0, 0, 0, 0, 1, 2.2E24, ...]
0.5	[-1; 2, 0, 0, 0, 2, 0, 2, 2, 1, 1, ...]

Table 4: Different expansions of $x \approx 0.75$

So it is clear that in this case the choice $p = -0.5$ is the best choice. So we could say that indeed there is no value for p that is always the best, but instead it differs from case to case.

6.3 Expected coefficient value

In Chapter 4 & 5 we discussed the appearance rate of coefficients, even although this gives us more insight in the behaviour of our expansion. It is more interesting if we can predict which coefficient value we will get. However, just looking at the expected value will not give us any information, since the expected value for $p = -1$ is already infinite by the Gauss-Kuzmin distribution [3]. And since our other expansions, for $p < 0$, only have higher coefficients they will also have an expected value of infinite.

This does not mean that we cannot get any useful information here, because we can still see how the expected value grows for different p values. For this we set up another experiment. Here we took the same experiment set-up as in Chapter 5, but now we calculated the expected value up to a certain coefficient value. Where the results can be seen in Figure 5.

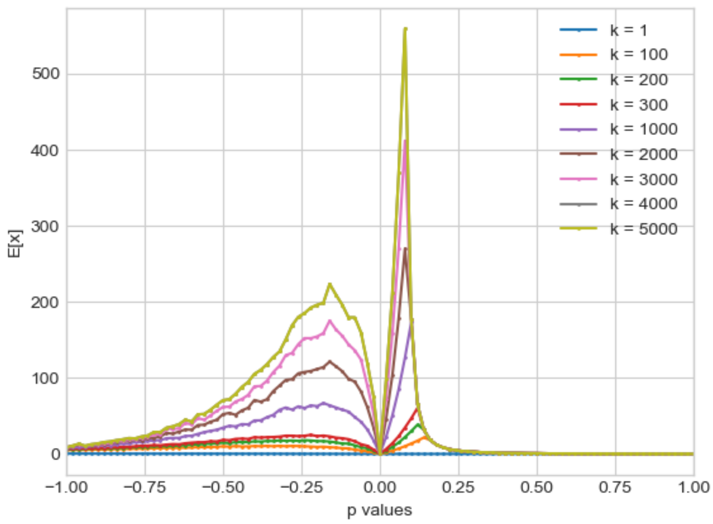

Figure 5: Expected coefficient value

As we can see in the figure, it is clear to see that the expected value of our coefficients are larger the closer we are to 0. As we already mentioned in the previous section. One interesting point here is the expected value around $p = 0$. Since it actually decreases the closer we are to 0. So this might mean that a p value of around $p = -0.2$ or $p = 0.08$ might actually better in terms of convergence speed.

We also see that when p is great enough, we no longer have an infinite expected value. This is also to be expected by looking at Figure 5. Because at some point higher coefficient values no longer occur for a high enough p value.

Chapter 7 Expanded Continued Root Fractions

So far we have worked with a limitation on continued fractions and continued root fraction where the coefficients can only be whole numbers greater or equal to 1, except for the first coefficient. If we were to partially remove this limitation for continued root fraction, such that we would also allow negative coefficients. We can also get complex numbers.

In this chapter we will look at some basic definitions for this idea, however it is meant meant as an introduction to a new open problem for any possible further research. Therefore only some basic properties will be looked upon. Now let us now first define our revised continued root fraction.

Definition 7.1. Let

$$C_n = \sqrt{}[a_0; a_1, a_2, ..., a_n] = a_0 + \cfrac{1}{\sqrt{a_1 + \cfrac{1}{\sqrt{\cdots}}}}$$

be the already established continued root fraction. Then we now have that $a_0 \in \mathbb{Z} \times \mathbb{Z}i$ and for all $i \geq 1$ we have $a_i \in \mathbb{Z} \setminus \{0\}$.

And

$$a_j + \frac{1}{\sqrt{\phi_{j+1}^n}} \neq 0 \quad \text{for all } j \geq 1$$

First we shall look at the new role of a_0, recall that before a_0 simply moved us along the real axis, where we could take $x = \lfloor x \rfloor$. Now we are moving over the complex plane. So we can take $a_0 = \lfloor Re(x) \rfloor + \lfloor Im(x) \rfloor \cdot i$.

We will now look at the different kind of complex numbers that we can get.

Proposition 7.2. *Let* $x = \sqrt{}[0; a_1, a_2, ...]$*, we can differentiate between the following cases:*

(i) If $a_i \geq 1$ *for all* $i \geq 1$ *we have* $x \in \mathbb{R}$

(ii) If $a_1 \leq -1$ *and* $a_i \geq 1$ *for all* $i \geq 2$ *we have* $x \in \mathbb{R}i$

(iii) If $a_i \leq -1$ *for any* $i \geq 2$ *we have* $x \in \mathbb{C}$*, where* x *is of the form* $a + bi$

Proof. (i) This is our normal continued root fraction form for which we know that it is real.

(ii) Let $x = \sqrt{}[0; a_1, a_2, ...] = \sqrt{}[0; a_1, \phi_2^\infty]$. Then $c = \phi_2^\infty \in \mathbb{R}$, since $a_i \geq 1$ for all $i \geq 2$. So

$$x = \frac{1}{\sqrt{a_1 + \frac{1}{\sqrt{c}}}} = \frac{1}{\sqrt{-A}}$$

for $A = -(a_1 + \frac{1}{\sqrt{c}}) > 0$, because $a_1 \leq -1$ and $0 < \frac{1}{\sqrt{c}} < 1$. Therefore we have

$$x = \frac{1}{\sqrt{A}\sqrt{-1}} = \frac{1}{Ai} = -\frac{1}{\sqrt{A}}i \in \mathbb{R}i$$

(*iii*) We will look at the case where $a_2 \leq -1$, where the case that $a_n \leq -1$ could be proved by induction.

So let $a_2 \leq -1$, then we can write x as before by:

$$x = \frac{1}{\sqrt{a_1 + \frac{1}{\sqrt{a_2 + \frac{1}{\sqrt{\phi_3^\infty}}}}}} = \frac{1}{\sqrt{a_1 + \frac{1}{\sqrt{-A}}}}$$

We do know not know if A is complex or real. But if we rewrite this further we would get

$$\begin{aligned} x &= \frac{1}{\sqrt{a_1 - \frac{i}{\sqrt{A}}}} = \frac{1}{\sqrt{\frac{a_1\sqrt{A} - i}{\sqrt{A}}}} = \frac{\sqrt[4]{A}}{\sqrt{a_1\sqrt{A} - i}} \\ &= \frac{\sqrt[4]{A}\sqrt{a_1\sqrt{A} + i}}{\sqrt{a_1^2 A - 1}} = \frac{\sqrt{a_1 A + i\sqrt{A}}}{\sqrt{a_1^2 - 1}} = \sqrt{\frac{a_1 A}{a_1^2 - 1} + \frac{\sqrt{A}}{a_1^2 - 1} i} \end{aligned}$$

We can see that x is of the form $\sqrt{a + bi}$, therefore we have that $x \in \mathbb{C}$ of the wanted form. $\square$

Some problems that still remain are now convergence and uniqueness. Both of these properties are expected to still hold. If we, for example, were the look at the uniqueness for the second case from Proposition 7.2, where we are looking at $x = \sqrt{}[0; a_1, a_2, ...] = \sqrt{}[0; a_1, \phi_2^\infty]$ for $a_1 \leq -1$. We have that ϕ_2^∞ is still unique by Chapter 1. Which can then be used to show that we also have uniqueness for x. However, the formal proof for this is left as an open problem.

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