Radboud University Nijmegen

Faculty of Science

Wang Tiles and Cubes

Aperiodic Sets

Thesis BSc Mathematics

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Chapter 1

Introduction

Wang's Hypothesis

In 1961, Hao Wang introduced square tiles with colored edges in computability theory, now known as Wang tiles. We may place these tiles next to each other only when the adjacent sides match their colors. This is called a tiling. In a tiling, we may use one tile any number of times. However, we are only allowed to use a finite number of different tiles.

With these constraints, some natural questions arise. For example: can we tile the plane? Can we tile the plane periodically: in such a way that the tiles create a repeating pattern? Can we tile the plane, but in a way that the pattern never repeats?

These questions are the foundation of this thesis. Wang himself already asked these questions. In his work, he even conjectured the following:

Wang's hypothesis: If the plane can be tiled by a finite set of Wang tiles, then it can also be tiled periodically by that same set of tiles [15].

Rotations and reflections

Though one may be tempted to think about tiles as physical objects, there are rules in place for Wang tiles wherein a subtlety becomes clear. In particular: we are not allowed to rotate tiles. Were we to allow rotations, the problem stated in the previous section would become trivial: with any one tile and its 180° rotation, we can tile the plane and even do so periodically. This is illustrated in Figure 1.1, with a random tile with 'colors' n, e, s and w on the top, right, bottom and left edges respectively. The 2-by-2 block of tiles can be repeated to fill the plane.

For the same reason, we do not allow tiles to be reflected: we can tile the plane periodically with any one tile and its three reflected variants (reflected horizontally, vertically and both horizontally as well as vertically), as shown in Figure 1.2. Alternatively, we could use only the original tile and the variant reflected both horizontally and vertically to get the situation of Figure 1.1 again.

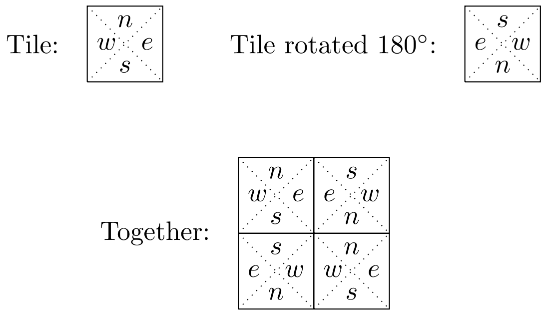


Figure 1.1: A random Wang Tile and its 180° rotated variant. Together they can form a 2-by-2 repeatable block.

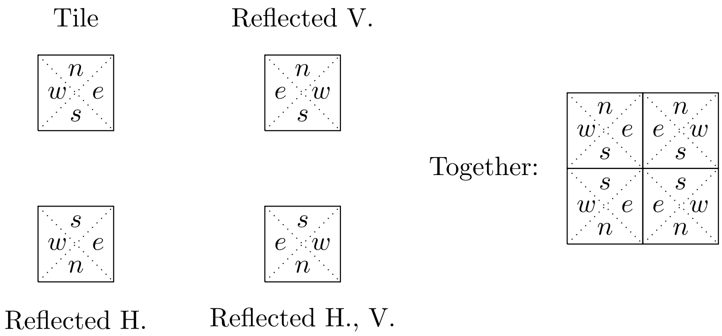


Figure 1.2: A Wang Tile and its reflected variants. Together they can form a 2-by-2 repeatable block. (H. and V. abbreviate 'Horizontally' resp. 'Vertically')

Dimension 1

To get a feel for working with Wang tiles, let us briefly consider a slightly simplified scenario: a 1-dimensional setting. In this setting, we are dealing with 'tiles' with 2 sides: dominoes. We will consider a domino as a unit interval and represent it with (w, e), if it has colors w and e on the left and right edge respectively.

We will (informally) prove that Wang's hypothesis is true in this setting.

Proposition 1.1. *Let D be a finite set of Wang dominoes. If we can tile the real line by D, then we can do so periodically.*

Proof. Let D be a finite set of Wang dominoes and assume there is a tiling of $\mathbb{Z}$ by D. As D is finite, there must be a tile $(w, e) \in D$ that appears at least twice, say on positions i and j (with $i < j$).

Note that the color of the right side of the domino at position $j - 1$ must be the same as the color on the left side of the domino at position j. Thus, the block of dominoes starting at i and ending at $j - 1$ is repeatable, as the color of the left side at i is the same as the color on the right side at $j - 1$: w. (Illustrated in Figure 1.3)

By repeating this block, we can obtain a periodic tiling of $\mathbb{Z}$ by D. $\square$

This construction of a repeatable block will prove to be useful and a similar, though more involved strategy will be applied in the proofs for Lemma's 2.9 and 6.8.

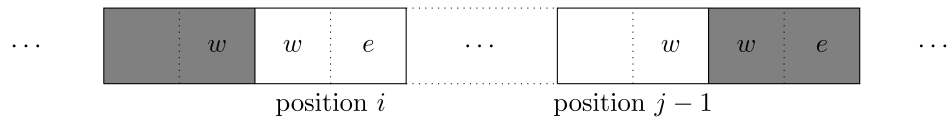


Figure 1.3: 1-dimensional tiling by some set of Wang dominoes D. Repeatable block highlighted. Unknown colors left blank.

Related Work

We have seen Wang's hypothesis is true for a 1-dimensional setting. However, this is not the case for the 2-dimensional setting. Already a few years after the formulation of the hypothesis, a counterexample was found by Wang's student Robert Berger. He constructed a set of 20,426 Wang tiles that could tile the plane, but could not do so periodically [2].

Berger's creation spawned a series of attempts to obtain smaller aperiodic sets. Berger himself reduced his set to only 104 tiles. Other examples include 56 tiles by Robinson [14] and (an adaptation of) 16 tiles by Ammann [8]. In 1995, Kari constructed an aperiodic set of 14 Wang tiles [11]. The same year, Culik constructed an aperiodic set of 13 tiles with a construction inspired by Kari's set [5]. In this thesis, this last set will be of special interest to us.

In 2011, Jeandel and Rao constructed an aperiodic set of 11 Wang tiles using 4 colors [10]. Using an exhaustive computer program, they prove there does not exist a smaller aperiodic set, meaning that no set of Wang tiles using fewer tiles or fewer colors will be aperiodic. Proofs that no set of tiles using 2 or 3 colors can be aperiodic were given by When-Guei Hu et al. in [9] and [3] respectively.

The construction of aperiodic sets is hardly the only interesting theoretical problem involving Wang tiles, though it is the one this thesis will focus on. Other problems may involve complexity. For example, the question whether a set of Wang tiles can tile the plane is proven to be undecidable [2]. Other problems again may involve the relation of Wang tiles to Turing machines.

Wang tiles see practical use too. For example, they are used in texture generation by computers [4]. Another example is modelling quasicrystals [12].

Thesis Overview

In this thesis, we will construct an aperiodic set of Wang tiles, inspired by the constructions of Kari and Culik [11, 5]. In Chapter 2, we will discuss basic definitions and formalize concepts already introduced in this introduction.

In Chapters 3 and 4, we will explore two constructions used to prove our set to be aperiodic: balanced number sequences and sequential machines.

In Chapter 5, we will construct a set of Wang tiles and prove it is aperiodic.

Finally, in Chapter 6, we will extend our definitions to a 3-dimensional setting, discuss some intricacies this extra dimension brings and create an aperiodic set of Wang cubes based on cellular automata theory and our aperiodic set of Wang tiles constructed in Chapter 5. This is again inspired by a construction of Kari and Culik in [7].

Chapter 2

Wang Tiles

In this chapter, we will introduce the concepts that are fundamental to this thesis. We will start with definitions for (Wang) tiles and (Wang) tilings, but will also explore some basic lemmas to familiarize ourselves with the introduced concepts.

By the end of this chapter, we will not only have gotten a feel for Wang tiles and their properties, but we will also be able to formulate the problem and goal that are leading in the bulk of this thesis.

2.1 Tiles and tilings

Definition 2.1. A *tile* is a unit square. A *Wang tile* is a tile where each side is marked with a color. If the left, top, bottom and right edges are colored w, n, s and e respectively, we will denote the tile as (w, n, s, e).

There are many definitions of tiles. We use one that suits our needs. Also, these tiles are not physical objects, so the behaviour of the borders is not terribly important to us. Still, for completion we say that only the borders on the left and bottom side belong to the tile, as well as the bottom-left corner. We also say the corners of a Wang tile remain colorless, as to not cause confusion when edges colored differently meet.

The notation (w, n, s, e) may seem unnatural. However, we will justify this seemingly arbitrary order in Chapter 4.

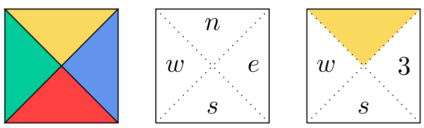


Figure 2.1: Three examples of Wang tiles.

In Figure 2.1, three Wang tiles are shown. Left, traditional colors are used, but in a mathematical sense, any marking can suffice as a color. For example, letters are used in the middle tile of Figure 2.1. Finally, the right Wang tile illustrates that we can use different 'types' of colors. We will often use numbers as colors to be able to make calculations using Wang tiles. Later in this thesis, we will explore how this can be achieved.

Definition 2.2. Let T be a finite set of tiles and $R \subseteq \mathbb{Z}^2$. A *tiling* of R by T is a function $f : R \to T$. If T consists of Wang tiles and f is a tiling by T such that the adjacent sides of all tiles match their color, f is called a *Wang-tiling*.

Remark 2.3. As a tiling is a function, it only assigns a tile to each point of its domain. As we would like these tiles to fill the whole plane, we must decide how a tile is placed on each point (x, y). We use the convention of the bottom-left corner being placed on (x, y).

As the definitions of tilings are used often, there is some conventional terminology: in this thesis, we will mostly concern ourselves with tilings of $\mathbb{Z}^2$, and we will simply say 'f is a tiling by T' if that is the case. Also, in the case of Wang-tilings, the requirement that adjacent sides must have the same color is often left implicit. If f is a Wang-tiling of a set of Wang tiles T, we simply refer to it as a tiling by T.

We can make the requirement that adjacent sides have the same color more formal. This approach is more technical then needed for the majority of this thesis, but we will use it on occasion. First, some notation:

Notation 2.4. Let T be a finite set of Wang tiles and f a tiling of some subset $R \subseteq \mathbb{Z}^2$ by T. Let C be the set of colors used in the tiles of T. We denote by f_n the function $R \to C$ that tells us the color of the top edge of the tile used on a point of R. Similarly we denote by f_w, f_e, f_s the color functions for the left, right and bottom edges.

With this notation, we can restate the requirement of matching colors. For a tiling of R by a set of Wang-tiles to be a Wang-tiling, the following must hold for any point $(x, y) \in R$:

- if $(x - 1, y) \in R$: $f_w(x, y) = f_e(x - 1, y)$
- if $(x, y + 1) \in R$: $f_n(x, y) = f_s(x, y + 1)$
- if $(x, y - 1) \in R$: $f_s(x, y) = f_n(x, y - 1)$
- if $(x + 1, y) \in R$: $f_e(x, y) = f_w(x + 1, y)$

The way these are stated, it is clear that diagonal adjacency is of no importance to us.

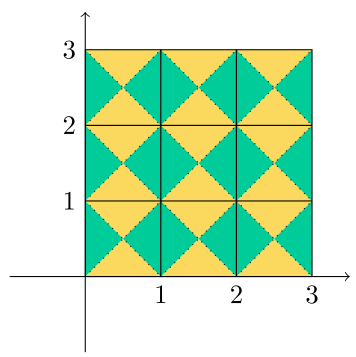


Figure 2.2: A tiling of a small portion of $\mathbb{Z}^2$

In Figure 2.2, we can see a tiling of a small portion of $\mathbb{Z}^2$ by one tile. Note that Definition 2.2 does not speak of rotations: we are not allowed to rotate tiles. One can easily see how the tiling of Figure 2.2 can expand to the whole of $\mathbb{Z}^2$: the tile can always be placed next to itself, as rotations are not allowed.

We can also see that the tiling creates a repeating pattern. This motivates to following definition:

Definition 2.5. Let T be a finite set of Wang tiles and f a tiling by T. f is called *periodic with period* $(a, b) \in \mathbb{Z}^2 \setminus \{(0, 0)\}$ if for all $(x, y) \in \mathbb{Z}^2$: $f(x, y) = f(x + a, y + b)$. f is called *aperiodic* if f is not periodic. In other words: we cannot find a and b such that the equality holds for all $(x, y) \in \mathbb{Z}^2$.

One can image a periodic tiling as being able to shift the tiles in such a way that the tiling remains the same. Of course, one should be cautious with the periods: if we're tiling a subset $R \subseteq \mathbb{Z}^2$, we must not escape R. In practice however, this is rarely an issue. In any case, we will concern ourselves with tilings of $\mathbb{Z}^2$ only, hence the omission of R in Definition 2.5.

An example of a periodic tiling is the tiling of Figure 2.2 expanded to $\mathbb{Z}^2$. It is easy to see that it is periodic with period $(1, 1)$, for example.

To help us get familiar with periodic tilings, we will look at the following small lemma. This lemma lets us use a positive horizontal or vertical period, without losing generality.

Lemma 2.6. *Let T be a finite set of Wang tiles and f a tiling by T with period (a, b). Then f is also periodic with period $(-a, -b)$.*

Proof. We know from assumption: $f(x, y) = f(x+a, y+b)$. Substituting $x-a$ and $y-b$ for x and y respectively, we see that $f(x - a, y - b) = f(x - a + a, y - b + b) = f(x, y)$ also holds for any $x, y \in \mathbb{Z}$. □

2.2 Doubly periodic tilings

By definition of periodic tilings, the horizontal and vertical period a and b come in a pairs. We will now introduce a special kind of periodic tilings that provide us with more information about the tiling without the need of additional requirements on the set of tiles. We will prove this claim after the definition.

Definition 2.7. Let T be a finite set of Wang tiles, f a tiling by T and $a, b \in \mathbb{Z} \setminus \{0\}$. f is called *doubly periodic with horizontal period a and vertical period b* is for all $(x, y) \in \mathbb{Z}^2 : f(x, y) = f(x + a, y) = f(x, y + b)$.

Remark 2.8. By a simpler version of Lemma 2.6, we may assume both horizontal and vertical periods of a doubly periodic tiling to be positive.

In other words, a doubly periodic tiling is a tiling for which we would only need to shift the tiles either horizontally or vertically to get the same pattern. It is important to note that this is a property of the set of tiles, not the tiling: the doubly periodic tiling may bear no resemblance to the standard periodic tiling!

In Figure 2.3, we see (parts of) two tilings by the same set of three Wang tiles. The left tiling is clearly (a, a)-periodic: the pattern remains the same, were we to shift all tiles both a horizontally as well as a vertically. It is clear that this tiling is not doubly periodic: we cannot shift only horizontally or vertically.

However, the tiling to the right is clearly doubly periodic. This is a trivial example: all tiles are the same! Thus, we see that this same set of tiles can tile $\mathbb{Z}^2$ both periodically and doubly periodically.

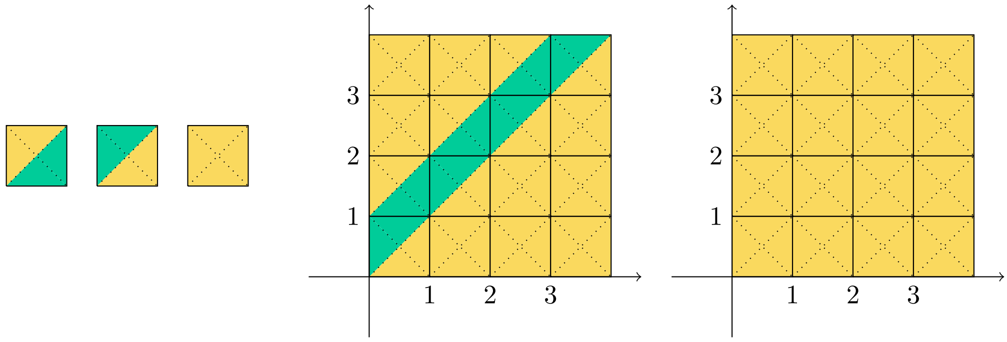


Figure 2.3: A periodic but not doubly periodic tilings and a doubly periodic tiling by the same set of 3 Wang tiles.

The example in Figure 2.3 raises the question whether there always exists a doubly periodic tilings by some set of tiles T if there also exists a periodic tiling by T and vice versa. We will now prove that this is indeed the case.

Lemma 2.9. *Let T be a a finite set of Wang tiles. Then there exists a periodic tiling of $\mathbb{Z}^2$ by T if and only if there exists a doubly periodic tiling by T.*

Proof. From right to left, the proof is trivial: say f is a doubly periodic tiling with horizontal period a and vertical period b, then it is clear that f is a also periodic tiling with, for example, period $(a, 0)$.

From left to right, the proof is not as straightforward. Let f be a tiling by T with period (a, b). Note that by definition, $(a, b) \neq (0, 0)$. We consider 3 cases:

i) $a \neq 0$ and $b = 0$.

ii) $a = 0$ and $b \neq 0$.

iii) $a \neq 0$ nor $b \neq 0$.

We prove case (i) first: by Lemma 2.6, we may assume $a > 0$. As T is finite, there are only finitely many ways to tile an a-by-1 segment. As a consequence, there must be a segment originating at some point (x_S, y_S) with an identically tiled segment above it: There exists $k \in \mathbb{N}_{>0}$ such that for all $i \in \{0, \ldots, a-1\}$ the following holds

$$f(x_S + i, y_S) = f(x_S + i, y_S + k)$$

We can consider the segments between the identically tiled segments with addition of the lower of those segments as an a-by-k block (see Figure 2.4) that can be repeated: the above equality shows that it can be repeated vertically with period k. By the $(a, 0)$-periodicity of f, the block can be repeated horizontally with period a. We can formalize this with a function:

$$g : \mathbb{Z}^2 \to T : (x, y) \mapsto f(x_S + (x \bmod a), y_S + (y \bmod k))$$

g is indeed a doubly periodic tiling. It has horizontal period a:

$$\begin{aligned} g(x, y) &= f(x_S + (x \bmod a), y_S + (y \bmod k)) \\ &= f(x_S + (x + a \bmod a), y_S + (y \bmod k)) \\ &= g(x + a, y) \end{aligned}$$

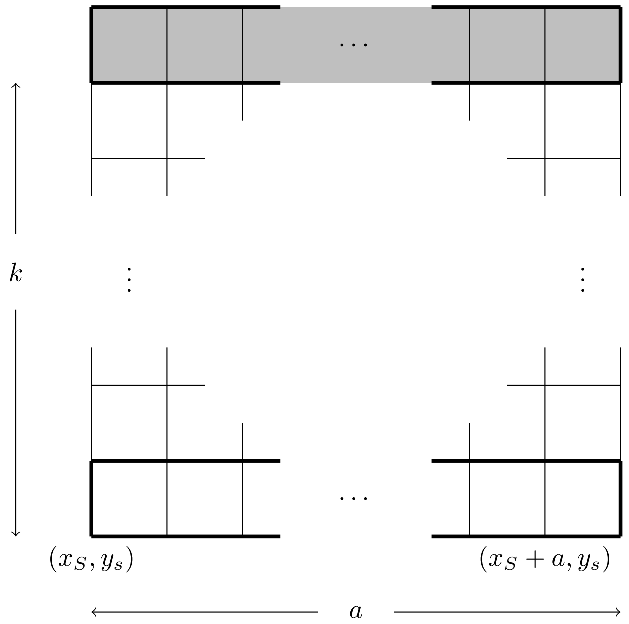


Figure 2.4: a-by-k block of tiles visualized. Outlined strips represent identically tiled segments. Gray tone used to color tiles outside of the a-by-k block.

And g has vertical period k:

$$\begin{aligned} g(x,y) &= f(x_S + (x \bmod a), y_S + (y \bmod k)) \\ &= f(x_S + (x \bmod a), y_S + (y + k \bmod k)) \\ &= g(x, y+k) \end{aligned}$$

As f is a tiling by T, all sides indeed match color. This proves case (i).

Case (ii) is proven analogously, but using Lemma 2.6 to get positive vertical period and by creating a block of size k'-by-b (for some $k' \in \mathbb{N}_{>0}$).

Now for case (iii). Let us assume $a > 0$. We will use a similar approach as used for case (i), but with a-by-$|b|$ segments instead of a a-by-1 strip (we must use $|b|$, as b can still be negative). As T is finite, there are only finitely many ways such a segment can be tiled, so there must be an a-by-$|b|$ segment originating at some point (x_S, y_S) with an identically tiled segment above it: there exists $k \in \mathbb{N}_{>0}$ such that for any $i \in \{0, \ldots, a-1\}, j \in \{0, \ldots, |b|-1\}$ the following holds

$$f(x_S, y_S) = f(x_S + i, y_S + j + bk)$$

This is illustrated in Figure 2.5. In this Figure, every square represents an a-by-b square. Squares numbered the same are tile identically. This is an immediate consequence of f being (a,b)-periodic.

Again, this gives us a block that ‘we can place next to itself’ to create a doubly periodic tiling. The block is of size ak-by-$|b|k$. We can again formalize this in a function g:

$$g : \mathbb{Z}^2 \to T : (x,y) \mapsto f(x_S + (x \bmod ak), y_S + (y \bmod bk))$$

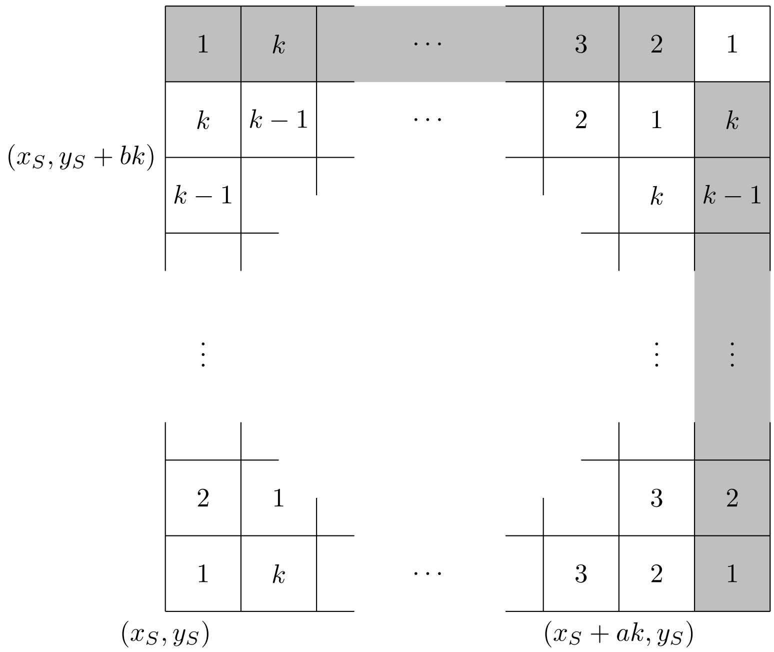


Figure 2.5: ak-by-bk block of tiles visualized (b positive in this example). Each square represents an a-by-b segment of tiles. Squares with the same number represent identically tiled segments. White and gray tones are used to distinguish blocks. Would b be negative, then the figure would be similar, just growing downwards.

It is again easy to check this function is doubly periodic:

$$\begin{aligned} g(x, y) &= f(x_S + (x \bmod ak), y_S + (y \bmod bk)) \\ &= f(x_S + (x + ak \bmod ak), y_S + (y \bmod bk)) \\ &= g(x + ak, y) \end{aligned}$$

$$\begin{aligned} g(x, y) &= f(x_S + (x \bmod ak), y_S + (y \bmod bk)) \\ &= f(x_S + (x \bmod a), y_S + (y + bk \bmod bk)) \\ &= g(x, y + bk) \end{aligned}$$

This proves case (iii). So, all cases are proven. This covers all possibilities for (a, b), so this proves the lemma. □

One may wonder why one would use the notion of periodicity at all, if we could just as well use doubly periodic tilings. There are some reasons. For starters, periodic and aperiodic tilings are nice counterparts. More importantly however, the notion of (doubly) periodic tilings extends nicely to higher dimensions, but the claim that the existence of a periodic tiling implies the existence of a multi-periodic tiling does not! We will see an example in a later chapter (Example 6.5 and Lemma 6.7). So, it is still relevant to think about periodic and multi-periodic tilings separately.

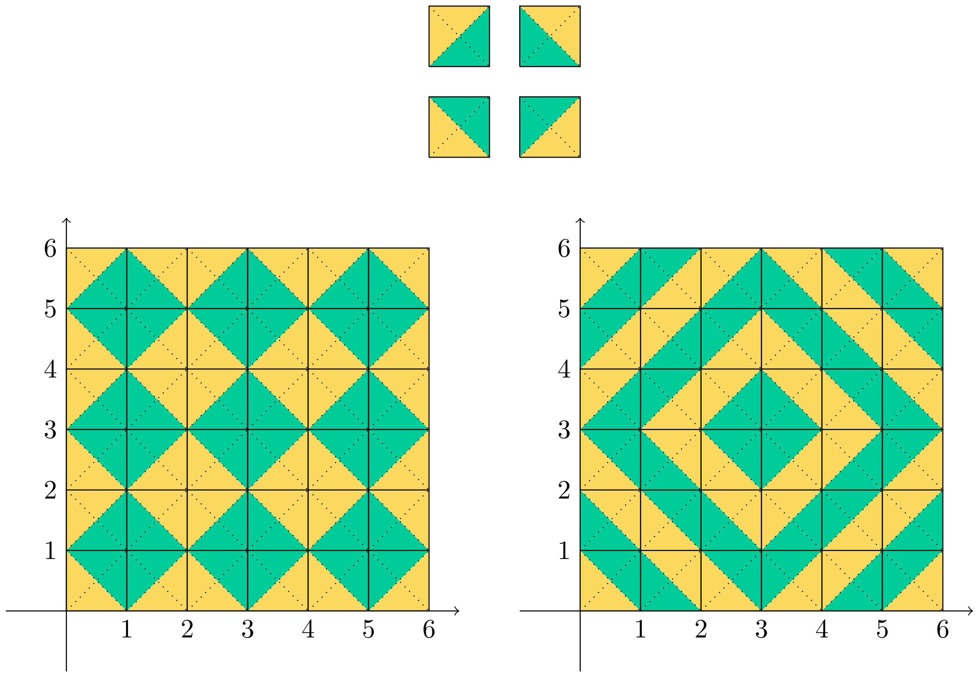


Figure 2.6: A tiling that is periodic (left) and not periodic (right) by the same set of 4 Wang tiles.

2.3 Aperiodic sets of Wang tiles

By paying special attention to periodic tilings, the reader may have intuitively figured that not all tilings need to be periodic. It is not hard to come up with an example. However, one must remember that only a finite number of different tiles may be used. We can now see why this requirement was imposed: were we to allow infinitely many colors, we can make tilings that are not periodic by constantly introducing new colors. Thus, we want to use a finite number of colors. This has the immediate consequence of the set of possible tiles being finite. Vice versa: a finite set of tiles only uses a finite number of colors.

An example of a tiling that is not periodic is shown in Figure 2.6 (right). This Figure also shows that the same 4 tiles can be used in a periodic tiling. One may wonder if this is the case for any set of tiles T: if T can tile the $\mathbb{Z}^2$, can it do so periodically? As seen in the introduction, Wang hypothesised that this is indeed the case.

The following definition is now motivated:

Definition 2.10. A finite set of Wang tiles T is aperiodic if there exists a tiling of $\mathbb{Z}^2$ by T, but no tiling is periodic.

Again, it is important to note that Definition 2.10 describes a property of the set of tiles. The set of 4 tiles in Figure 2.6 is *not* an example of an aperiodic set of tiles. Generally, it is very hard to give a convincing example of an aperiodic set of Wang tiles.

Aperiodic tilings are of special interest to us. Theoretically, they are interesting as their behaviour is inherently chaotic and thus gives rise to some hard to answer questions, like 'is there an aperiodic tileset that only uses 4 colors?'. Practically, this chaotic nature has seen several use cases, for example in texture generation. Though the practical side may be compelling, we will limit ourselves to the theoretical side in this thesis.

Another theoretical challenge is proving a set of Wang tiles T to be aperiodic. From the definition, it is clear a proof would consist of two parts:

- Prove there exists a tiling by T
- Prove no tiling can be periodic

We will do exactly that to prove in Chapter 5 a specific set of Wang tiles, T_{13}, is aperiodic. The next chapters will be dedicated to the tools needed for this proof.

Chapter 3

Balanced Numbers

To tile the plane, we would need to define a tiling function f on the whole of $\mathbb{Z}^2$. However, as this domain is rather large, it can be a difficult task to define f by making local constraints while still making it adhere to the requirement that adjacent sides of *all* tiles match in color.

One strategy to define f would be to first fix one row of tiles. Based on that row, we determine the rows above and below it. We repeat this process until we have tiled the plane. Of course, it is not as straightforward as simply choosing tiles at random, as the adjacent sides must match in color.

In this chapter, we will introduce balanced number representations. These are special sequences that will provide us with a nice way to fix a row of the tiling: we will use the numbers from such a sequence as the colors on the bottom and top edges of the tiles.

Definition 3.1. A *bi-infinite sequence* of elements of a set S is a function $x : \mathbb{Z} \to S$. Often, we denote $x(i)$ as x_i. The name 'bi-infinite sequence' explains itself by noticing that x can be viewed as the sequence $(x_i)_{i\in\mathbb{Z}}$.

Some examples:

$$\begin{array}{lll} x : \mathbb{Z} \to \mathbb{Z} : z \mapsto 2z & \text{defines the bi-infinite sequence} & \{\dots, -4, -2, 0, 2, 4, \dots\} \\ x : \mathbb{Z} \to \mathbb{Z} : z \mapsto z^2 & \text{defines the bi-infinite sequence} & \{\dots, 4, 1, 0, 1, 4, \dots\} \end{array}$$

In 1926, Samuel Beatty described a problem (Problem 3173) which led to the creation of a special kind of bi-infinite sequences: Beatty sequences [1]. We will first introduce some useful notation.

Definition 3.2. For a real number α, we denote by $\lfloor\alpha\rfloor$ the *integer part* of α: the greatest integer z such that $z \leq x$. We denote by $\{\alpha\}$ the *fractional part* of α: $\{\alpha\} = \alpha - \lfloor\alpha\rfloor$.

Again, some examples:

Real number α	integer part $\lfloor\alpha\rfloor$	fractional part $\{\alpha\}$
3	3	0
4.2	4	0.2
-4.2	-5	0.8

Definition 3.3. The *Beatty sequence* $A(\alpha)$ of a real number α is the bi-infinite sequence given by $A(\alpha)_i = \lfloor i \cdot \alpha \rfloor$.

As an example, we can look at $A(\pi)$:

$$\ldots, -16,\ -13,\ -10,\ -7,\ -4,\ 0,\ 3,\ 6,\ 9,\ 12,\ 15,\ 18,\ 21,\ 25\ldots$$

Beatty sequences let us relate real numbers to bi-infinite sequences. As $(\lfloor i \cdot \alpha \rfloor)_{i \in \mathbb{N}}$ is an unbounded sequence, there will definitely not be a finite number of entries in the Beatty sequence of any real number that is not 0. However, recall our goal: we want these sequences to determine a row of tiles. Thus, we want to use the entries from bi-infinite sequences as colors in our set of tiles. As by Definition 2.2, a tiling must be a function to a finite set of tiles. So, Beatty sequences do not provide us with the desired tools still!

However, using Beatty sequences, we can construct another type of bi-infinite sequences that will turn out to have the desired properties:

Definition 3.4. The *balanced representation* $B(\alpha)$ of a real number α is the bi-infinite sequence given by $B(\alpha)_{i+1} = A(\alpha)_{i+1} - A(\alpha)_i$.

We can again look at $\alpha = \pi$. $B(\alpha)$ is then the bi-infinite sequence:

$$\ldots,\ 3,\ 3,\ 3,\ 3,\ 4,\ 3,\ 3,\ 3,\ 3,\ 3,\ 3,\ 3,\ 4,\ldots$$

Let us look at another example, with each step explicitly written down. If we take $\alpha = \frac{2}{3}$. The bi-infinite sequence $(i \cdot \alpha)_{i \in \mathbb{Z}}$ looks like this:

$$\ldots,\ -4,\ -3\tfrac{1}{3},\ -2\tfrac{2}{3},\ -2,\ -1\tfrac{1}{3},\ -\tfrac{2}{3},\ 0,\ \tfrac{2}{3},\ 1\tfrac{1}{3},\ 2,\ ,\ 2\tfrac{2}{3},\ 3\tfrac{1}{3},\ 4,\ldots$$

Which means de Beatty sequence $A(\frac{2}{3})$ looks like:

$$\ldots,\ -4,\ -4,\ -3,\ -2,\ -2,\ -1,\ 0,\ 0,\ 1,\ 2,\ 2,\ 3,\ 4,\ldots$$

And thus, the following is the balanced sequence $B(\frac{2}{3})$:

$$\ldots,\ 0,\ 1,\ 1,\ 0,\ 1,\ 1,\ 0,\ 1,\ 1,\ 0,\ 1,\ 1,\ldots$$

We will now prove that for any α, the balanced representation $B(\alpha)$ contains a finite number of different entries, only 2 in fact, making them better suited for our setting.

Lemma 3.5. *For any real number α, the bi-infinite sequence $B(\alpha)$ contains at most 2 different numbers: if $k \leq \alpha < k+1$ for some $k \in \mathbb{Z}$, then $B(\alpha)_i \in \{k, k+1\}$ for all $i \in \mathbb{Z}$.*

Proof. First, we note that any entry of $A(\alpha)$ must be an integer, as each entry is the integer part of some number. Thus, any entry of $B(\alpha)$ must be an integer as well, as it is the difference of entries of $A(\alpha)$.

Moreover, note that if $k \leq \alpha < k+1$, then $k = \lfloor \alpha \rfloor$, by definition of the integer part.

Due to the previous remarks, it is sufficient to prove $\lfloor \alpha \rfloor \leq B(\alpha)_{i+1} \leq \lfloor \alpha \rfloor + 1$ for all $i \in \mathbb{Z}$. We will do this by proving the inequalities separately.

- $\lfloor \alpha \rfloor \leq B(\alpha)_{i+1}$

$$\begin{aligned}\lfloor \alpha \rfloor &= \lfloor i \cdot \alpha \rfloor + \lfloor \alpha \rfloor - \lfloor i \cdot \alpha \rfloor \\ &\leq \lfloor i \cdot \alpha + \alpha \rfloor - \lfloor i \cdot \alpha \rfloor \\ &= A(\alpha)_{i+1} - A(\alpha)_i \\ &= B(\alpha)_{i+1}\end{aligned}$$

- $B(\alpha)_{i+1} \leq \lfloor \alpha \rfloor + 1$

$$\begin{aligned}B(\alpha)_{i+1} &= A(\alpha)_{i+1} - A(\alpha)_i \\ &= \lfloor i \cdot \alpha + \alpha \rfloor - \lfloor i \cdot \alpha \rfloor \\ &\leq i \cdot \alpha + \lfloor \alpha \rfloor - \lfloor i \cdot \alpha \lfloor \\ &\leq i \cdot \alpha + \lfloor \alpha \rfloor - (i \cdot \alpha - 1) \\ &= \lfloor \alpha \rfloor + 1\end{aligned}$$

So indeed, for any $i \in \mathbb{Z}$ we have $\lfloor \alpha \rfloor \leq B(\alpha)_{i+1} \leq \lfloor \alpha \rfloor + 1$. Thus, we conclude that $B(\alpha)$ contain at most 2 different numbers. □

Balanced number representations have another nice property that lets us easily relate balanced numbers to real numbers: the average value of finite subsequences of $B(\alpha)$ converges to α. We will now formalize and prove this property:

Lemma 3.6. *Let α be a real number and i an integer. Then the average value of the subsequence $S_{i,n} = (B(\alpha)_{i+1}, \ldots, B(\alpha)_{i+n})$ of length $n \in \mathbb{N}$ converges to α as n increases.*

Proof. Let α be a real number and i an integer and n a natural number. Let $S_{i,n} = (B(\alpha)_{i+1}, \ldots, B(\alpha)_{i+n})$. Our approach will be as follows: we will calculate the average value $v_{i,n}$ of $S_{i,j}$ in terms of i and n and then show $\lim_{n \to \infty} v - \alpha = 0$.

We first calculate the sum of entries of $S_{i,n}$:

$$\begin{aligned}\sum_{m=1}^{n} B(\alpha)_{i+m} &= \sum_{m=1}^{n} B(\alpha)_{i+m} \\ &= \sum_{m=1}^{n} A(\alpha)_{i+m} - A(\alpha)_{i+m-1} \\ &= \sum_{m=1}^{n} A(\alpha)_{i+m} - \sum_{m=1}^{n} A(\alpha)_{i+m-1} \\ &= \sum_{m=1}^{n} A(\alpha)_{i+m} - \sum_{m=0}^{n-1} A(\alpha)_{i+m} \\ &= A(\alpha)_{i+n} - A(\alpha)_i\end{aligned}$$

So, the average value of entries of S is:

$$v_{i,n} = \frac{A(\alpha)_{i+n} - A(\alpha)_i}{n} = \frac{\lfloor (i+n)\alpha \rfloor - \lfloor i\alpha \rfloor}{n} = \frac{\lfloor (i+n)\alpha \rfloor}{n} - \frac{\lfloor i\alpha \rfloor}{n}$$

To prove $\lim_{n\to\infty} v_{i,n} - \alpha = 0$, we use the squeeze theorem. Notice that for n sufficiently large we have $(i+n)\alpha - 1 \leq \lfloor (i+n)\alpha \rfloor \leq (i+n)\alpha + 1$, so:

$$\frac{(i+n)\alpha - 1}{n} \leq \frac{\lfloor (i+n)\alpha \rfloor}{n} \leq \frac{(i+n)\alpha + 1}{n}$$

When the right-hand side is substituted, we see that there is indeed convergence to 0:

$$\begin{aligned} \lim_{n\to\infty} \frac{(i+n)\alpha+1}{n} - \frac{\lfloor i\alpha \rfloor}{n} - \alpha &= \lim_{n\to\infty} \frac{i\alpha}{n} + \frac{n\alpha}{n} + \frac{1}{n} - \frac{\lfloor i\alpha \rfloor}{n} - \alpha \\ &= \lim_{n\to\infty} \frac{i\alpha}{n} + \alpha + \frac{1}{n} - \frac{\lfloor i\alpha \rfloor}{n} - \alpha = 0 \end{aligned}$$

Similarly, for the left-hand side:

$$\lim_{n\to\infty} \frac{(i+n)\alpha-1}{n} - \frac{\lfloor i\alpha \rfloor}{n} - \alpha = \lim_{n\to\infty} \frac{i\alpha}{n} + \frac{n\alpha}{n} - \frac{1}{n} - \frac{\lfloor i\alpha \rfloor}{n} - \alpha = 0$$

By the squeeze theorem, we can conclude $\lim_{n\to\infty} v_{i,n} - \alpha = 0$, which proves the lemma.

□

In this chapter, we have introduced balanced number sequences. They are bi-infinite, and contain at most 2 different numbers. Our aim is to use them as 'input' to the tiling: we let a balanced number sequence determine the first row of our tiling while using a set of tiles that has the numbers contained in this sequence as colors on the bottom edge.

To make a proper Wang tile, we would need the tile to have colors on the left, right and top edges as well. In the next chapter, we will introduce sequential machines, a tool that can help us represent Wang tiles another way. In particular, we can determine suitable colors for the left and right edges. Sequential machines will also enable us to make multiplications on bi-infinite sequences, which will help us determining a tiling from a fixed row of tiles.

Chapter 4

Sequential Machines

In the previous chapter, we have introduced special bi-infinite sequences. In this chapter, we will introduce sequential machines, machines that take such sequences as input and produce another bi-infinite sequence as output. Recall that we will use balanced number sequences to fix a row of tiles. We will use sequential machines to determine the rows above and below it.

Finite-state machines are models of computation consisting of a finite number of states and transitions between them. If the finite-state machine uses an input and output tape, it is called a finite-state transducer. They are divided into two categories: deterministic machines and non-deterministic machines. If at most one transition is possible, given a state and an input symbol, then the machine is said to be deterministic. If not, it is non-deterministic. We will use non-deterministic machines.

The machine just described is formalized in Definition 4.1. As not to defer from the convention in the literature (e.g., [5]), we refer to this machine as a sequential machine.

Definition 4.1. A *sequential machine* M is a non-deterministic finite-state transducer where the output of a transition depends on a combination of state and input. Mathematically, M is a 4-tuple (S, Σ, Λ, T) with:

- S the set of states
- Σ the set of input characters
- Λ the set of output characters
- $T \subseteq S \times \Sigma \times \Lambda \times S$ the set of transitions. Each $t = (w, n, s, e) \in T$ is a transition from state w to state e, with input n and output s.

We deviate slightly from the traditional definition: as we work on bi-infinite sequences, there is no need for initial states and final states. Convention is to let $w_0 = 0$ if $(w_i)_{i \in \mathbb{Z}}$ is a bi-infinite sequence of states and 0 is a state.

A finite-state machine can be represented as a state diagram. Circles represent states and arrows transitions. For sequential machines, the transitions are labeled in the following manner: $n \mid s$, n being the input and s the output. An example of a sequential machine is shown in Figure 4.1

We are now ready to examine the connection between sequential machines and Wang tiles. We have actually formulated our sequential machines in such a way that the Wang

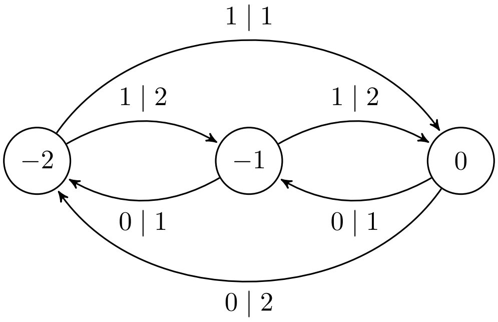


Figure 4.1: The state diagram of the sequential machine M_3

tiles are easily distilled: any transition $(w, n, s, e) \in T$ corresponds to a tile with colors w, n, s and e on the left, top, bottom and right edge respectively. This is why the order (w, n, s, e) was chosen in Definition 2.1: it is the natural way to interpret a transition $w \xrightarrow{n|s} e$.

An example of a sequential machine and the corresponding set of tiles is shown in Figure 4.2.

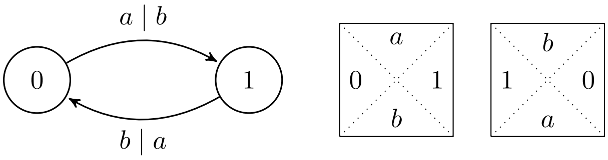


Figure 4.2: A sequential machine (left) and the corresponding set of tiles (right)

Finite-state machines can be used to compute relations between bi-infinite sequences of characters from the input and output alphabet. This is done by running over the input and recording the output. In our case, we say that bi-infinite sequences $(n_i)_{i\in\mathbb{Z}}$ and $(s_i)_{i\in\mathbb{Z}}$ are in the relationship $\rho(M)$ of a sequential machine M if and only if there is a bi-infinite sequence of states $(w_i)_{i\in\mathbb{Z}}$ such that for all i: $(w_{i-1}, n_i, s_i, w_i) \in T$. In other words: for any $i \in \mathbb{Z}$, there is a transition from w_{i-1} to w_i labeled $n_i \mid s_i$. We will use these relation to multiply balanced numbers.

4.1 Construction of M_q

Now that we have discussed the definitions concerning sequential machines and their relation to Wang tiles, we will look at the manner in which specific sequential machines can be constructed: machines that multiply. In the following, we assume all colors are numbers, so that it is sensible to talk about multiplication.

Let $q = \frac{x}{y}$ be a positive rational number, $x, y \in \mathbb{Z}$ and $y \neq 0$. We say a Wang tile (thus, a transition) (w, n, s, e) multiplies by q if and only if $w + qn = s + e$. We say a sequential machine multiplies by q if every transition multiplies by q and denote it with M_q (though it is not unique in general, it is unique for chosen input and output alphabets). We have already seen one such machine: M_3 in Figure 4.1, which can multiply by 3. In Proposition 4.5, we will see that the term 'multiplication' is justified.

We will now examine how such a machine M_q can be constructed. This is relevant to us, as it shows there are sets of tiles that multiply a balanced number by q.

- The states of M_q represent the possible values $q\lfloor r\rfloor - \lfloor qr\rfloor$ for $r \in \mathbb{R}$. Why? This represents the difference between $q\lfloor i\alpha\rfloor$ (direct multiplication of input) and $\lfloor qi\alpha\rfloor$ (what we expect to see as output). These values are necessary to ensure multiplication of balanced numbers, which is what we are after.

  We must verify there are only finitely many such values, as a sequential machine is a finite-state machine.

  Notice the following inequality:

  $$q\lfloor r\rfloor - 1 \leq qr - 1 < \lfloor qr\rfloor \leq qr < q(\lfloor r\rfloor + 1)$$

  Multiply by -1 and we get:

  $$-q\lfloor r\rfloor + 1 > -\lfloor qr\rfloor > -q(\lfloor r\rfloor + 1)$$

  Add $q\lfloor r\rfloor$:

  $$1 > q\lfloor r\rfloor - \lfloor qr\rfloor > -q$$

  As $q = \frac{x}{y}$, $1 = \frac{y}{y}$ and $q\lfloor r\rfloor - \lfloor qr\rfloor$ must be a multiple of $\frac{1}{y}$ (as $\lfloor r\rfloor$ and $\lfloor qr\rfloor$ are both integers), we see that the set of possible states S is the following:

  $$S = \left\{-\frac{x-1}{y}, -\frac{x-2}{y}, \ldots, \frac{y-2}{y}, \frac{y-1}{y}\right\}$$

- We want any transition (w, n, s, e) to adhere to the equality $w + qn = s + e$. So, we say there is a transition from w to $w + qn - s$ exactly when such a state exists.

  There is a subtlety: for any states w and e, there will be an infinite number of pairs n and s such that $w + qn = s + e$ is satisfied. Thus, the transitions of M_q are determined by the input and output alphabets. These are generally chosen so suit a certain domain. In the case of M_3 (Figure 4.1), we've chosen input alphabet $\{0, 1\}$ and output alphabet $\{1, 2\}$. Due to Lemma 3.5, this means we can multiply any number α by 3 using M_3 if $0 \leq \alpha \leq 1$ and $1 \leq 3\alpha \leq 2$. In other words, if $\alpha \in [\frac{1}{3}, \frac{2}{3}]$.

We will now look at a lemma that may look daunting, but is actually quite simple. It states that the sum of colors on the top edges equals the sum of colors on the bottom edges in a strip of length a of a tiling by tiles generated by M_q. This is exactly the property we have been looking for in our introduction of sequential machines. This lemma will be essential in the proofs of the next chapter.

In the lemma, we use a 'periodic tiling on $\mathbb{Z} \times \{i\}$'. In Definition 2.5, we have only defined 2-dimensional periodic tilings on the whole of $\mathbb{Z}^2$. However, for one row of tiles, we can easily the top and bottom edges are of no importance to the color-matching requirement. Thus, the periodicity of a tiling of a single row is intuitively inherited from the 1-dimensional setting.

Lemma 4.2. *For $q \in \mathbb{Q}$, let T_q denote the set of Wang-tiles generated by M_q and let f be a tiling by T_q of $\mathbb{Z} \times \{i\}$ with horizontal period a ($i, a \in \mathbb{Z}, a \neq 0$). Then for any $x \in \mathbb{Z}$:*

$$q \cdot \sum_{k=0}^{a-1} f_n(x+k, i) = \sum_{k=0}^{a-1} f_s(x+k, i)$$

(Here, f_n and f_s are used as introduced in Notation 2.4)

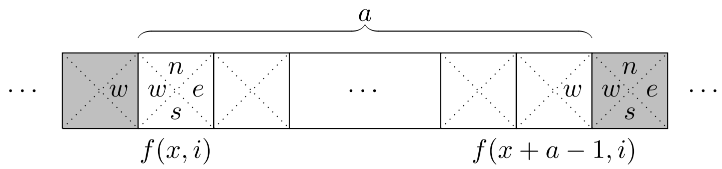


Figure 4.3: A strip of length a in a a-periodic tiling. Unknown colors left blank.

Proof. We may assume a to be positive (a simpler version of Lemma 2.6). As each tile is generated by M_q, we know the following equality must hold for any x:

$$f_w(x,i) + q \cdot f_n(x,i) = f_s(x,i) + f_e(x,i) \tag{4.1}$$

We know $f_e(x,i) = f_w(x+1,i)$. In particular, as f has horizontal period a, $f_w(x,i) = f_e(x+a-1,i)$ for any x (see Figure 4.3). So we see:

$$\begin{aligned}
\sum_{k=0}^{a-1} f_w(x+k,i) &= f_w(x,i) + \sum_{k=1}^{a-1} f_w(x+k,i) \\
&= f_w(x,i) + \sum_{k=0}^{a-2} f_w(x+k+1,i) \\
&= f_e(x+a-1,i) + \sum_{k=0}^{a-2} f_e(x+k,i) \\
&= \sum_{k=0}^{a-1} f_e(x+k,i)
\end{aligned}$$

Thus, using equation (4.1) again, we see:

$$\begin{aligned}
\sum_{k=0}^{a-1} (f_w(x+k,i) + q \cdot f_n(x+k,i)) &= \sum_{k=0}^{a-1} (f_s(x,i) + f_e(x+k,i)) \\
\sum_{k=0}^{a-1} f_w(x+k,i) + q \cdot \sum_{k=0}^{a-1} f_n(x+k,i) &= \sum_{k=0}^{a-1} f_s(x,i) + \sum_{k=0}^{a-1} f_e(x+k,i) \\
q \cdot \sum_{k=0}^{a-1} f_n(x+k,i) &= \sum_{k=0}^{a-1} f_s(x+k,i)
\end{aligned}$$

□

We will now prove a similar result but for the whole sequence: it states that we can indeed multiply balanced numbers with sequential machines. This is a nice justification for the use of these machines.

Proposition 4.3. *Let $\alpha \in \mathbb{R}$ and let $M_q = (S, \Sigma, \Lambda, T)$ be a sequential machine as described ealier ($q \in \mathbb{Q}$). If $\alpha \in \mathbb{R}$ such that for all $i \in \mathbb{Z}$: $B(\alpha)_i \in \Sigma$ and $B(q\alpha)_i \in \Lambda$, then M_q can calculate $B(q\alpha)$ as output when given $B(\alpha)$ as input.*

Proof. Let α and M_q be as stated in the lemma and $i \in \mathbb{Z}$. We make the following observation:

$$(q\lfloor(i-1)\alpha\rfloor - \lfloor(i-1)q\alpha\rfloor) + q(\lfloor i\alpha\rfloor - \lfloor(i-1)\alpha\rfloor) - (q\lfloor i\alpha\rfloor - \lfloor iq\alpha\rfloor) = \lfloor iq\alpha\rfloor - \lfloor(i-1)q\alpha\rfloor$$

In other words:

$$(q\lfloor(i-1)\alpha\rfloor - \lfloor(i-1)q\alpha\rfloor) + qB(\alpha)_i - (q\lfloor i\alpha\rfloor - \lfloor iq\alpha\rfloor) = B(q\alpha)_i$$

The terms between brackets are exactly of the form of the states of M_q (recall M_q's construction). This lets us define $w_i = q\lfloor(i-1)\alpha\rfloor - \lfloor(i-1)q\alpha\rfloor$ and $e_i = q\lfloor i\alpha\rfloor - \lfloor iq\alpha\rfloor$. By substituting $i+1$ for i, we see $w_{i+1} = e_i$.

As $B(\alpha)_i$ and $B(q\alpha)_i$ are elements of the input and output alphabets respectively, and because w_i and e_i are states, there is a transition $w_i \xrightarrow{B(\alpha)_i|B(q\alpha)_i} e_i$ present in M_q.

Because $w_{i+1} = e_i$, we can chain these transitions for any i, thus reading input sequence $B(\alpha)$ and producing output sequence $B(q\alpha)$. ☐

Remark 4.4. There is a caveat to Proposition 4.5: it does not prove M_q must multiply by q. This is not needed for our goal of constructing an aperiodic set of Wang tiles, but will be a subtle problem in Chapter 6.

However, proving M_q must multiply by q is not trivial at all. An example is given in Figure 4.4. Here, the input sequence $B(\frac{1}{2})$ is used. The top figure illustrates the transitions (in tile form) as described in the proof of Proposition 4.5. The bottom figure illustrates that the same input can be accepted using entirely different transitions.

In this example, the output sequence of the bottom figure is just a shifted version of $B(q\alpha)$, which would not be problematic in any case (the 0-index of a bi-infinite sequence is arbitrary). However, it is not obvious these are the only possible outputs when reading $B(\alpha)$.

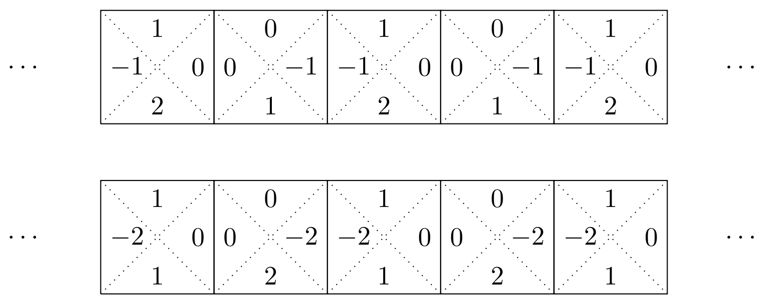


Figure 4.4: Two ways M_3 can read input $B(\frac{1}{2})$

We finish this chapter by another Lemma that is another desirable property sequential machine M_q regarding multiplication: when given input sequence $B(\alpha)$, the average value of the output sequence does converge to $q\alpha$. This still is not enough to counter Remark 4.4 however.

Proposition 4.5. *Let $\alpha \in \mathbb{R}$ and let $M_q = (S, \Sigma, \Lambda, T)$ be a sequential machine as described earlier ($q \in \mathbb{Q}$). If $\alpha \in \mathbb{R}$ such that for all $i \in \mathbb{Z}$: $B(\alpha)_i \in \Sigma$, then the average value of entries of the output sequence of M_q given $B(\alpha)$ as input converges to $q\alpha$.*

Proof. We can view this calculation as a single row of tiles generated by M_q, with $B(\alpha)$ used as input. We will denote the tile at step $i \in \mathbb{Z}$ with (w_i, n_i, s_i, e_i). Thus, we aim to prove the bi-infinite sequence $(s_i)_{i\in\mathbb{Z}}$ is the same sequence as $B(q\alpha)$, modulo indexing (i.e., there exists $j \in N$ such that $s_i = B(q\alpha)_{i+j}$ for all $i \in \mathbb{Z}$).

We will first look at a subsequence $\sum_{m=0}^{N} s_{i+N}$ and prove the average converges to $q\alpha$. We apply a similar strategy as in the proof given for Lemma 4.2. Recall the property that was given to each transition:

$$w_i + qn_i = s_i + e_i$$

We can use that in the following manner:

$$s_i = w_i - e_i + qn_i$$

Using $e_i = w_{i+1}$, we can now easily calculate the summation:

$$\begin{aligned}\sum_{m=0}^{N} s_{i+m} &= \sum_{m=0}^{N} (w_i - e_i + qn_i) \\ &= w_i - e_{i+N} + q \sum_{m=0}^{N} n_{i+m}\end{aligned}$$

Now, we can calculate the average value as N approaches infinity by using that the average value of n_i converges to α (as per Lemma 3.6)

$$\begin{aligned}\lim_{N\to\infty} \frac{1}{N} \sum_{m=0}^{N} s_{i+m} &= \lim_{N\to\infty} \frac{1}{N} \left(w_i - e_{i+N} + q \sum_{m=0}^{N} n_{i+m} \right) \\ &= \lim_{N\to\infty} \frac{w_i}{N} - \frac{e_{i+N}}{N} + \frac{q}{N} \sum_{m=0}^{N} n_{i+m} \\ &= q\alpha\end{aligned}$$

So indeed, the average value of entries of the output sequence converges to $q\alpha$, which proves the lemma.

□

In this chapter, we have introduced sequential machines and explored the connection to Wang-tiles. We examined a was of constructing sequential machines in such a way that we can use balanced number sequences as input and have the machine produce a multiple of the input as output.

In the next chapter, we will use balanced numbers and the construction from this chapter to generate a specific tileset that we will prove to be aperiodic using the acquired tools.

Chapter 5

An aperiodic set of Wang Tiles

In the previous chapters, we have introduced Wang-tiles, Wang-tilings, balanced number sequences and sequential machines. In this chapter, we will focus on a specific tileset, T_{13}, and prove it is aperiodic.

Recall that by Definition 2.10, a set of tiles is aperiodic if two conditions are met: there exists a tiling of $\mathbb{Z}^2$ by the set, but no tiling is periodic. After we have generated the tileset, we will do exactly that in two separate propositions.

We will define the tileset T_{13}. The way we introduced the concepts of earlier chapters reflect our approach with this set of tiles: we first generate tiles using sequential machines that multiply bi-infinite sequences. We then fix a row of tiles by means of a balanced number sequence. We will then determine all other rows by multiplication.

5.1 Construction of T_{13}

We start by generating the tilesets T_3 and $T_{\frac{1}{2}}$ by sequential machines M_3 and $M_{\frac{1}{2}}$, see Figures 5.1 and 5.2 respectively.

There is one implicit choice already made before the generation of these tiles: the input and output alphabets used in the construction for the machines. The importance of this step cannot be overstated, choosing alphabets that are unfit could result in different results. For example, were the output of one machine totally different from the input for the other, we would not be able to alternate between them at all!

For M_3, we have chosen input alphabet $\{0, 1\}$ and output alphabet $\{1, 2\}$, allowing us to use input sequences $B(\alpha)$ with $\alpha \in [\frac{1}{3}, \frac{2}{3}]$.

For $M_{\frac{1}{2}}$, we have chosen input alphabet $\{0, 1, 2\}$ and output alphabet $\{0, 1\}$, allowing us to use input sequences $B(\alpha)$ with $\alpha \in [0, 2]$.

We will see that these intervals suit us nicely. We can take the interval $[\frac{1}{3}, 2]$ and note that there is some overlap with the previous intervals (it is not an intersection, so that both multiplication with 3 and division by 2 are still possible within this interval). Moreover, upon further inspection we can see that when $\alpha \in [\frac{1}{3}, \frac{2}{3}]$ we can multiply by 3 so that 3α is in the interval associated with $M_{\frac{1}{2}}$, while we can multiply by $\frac{1}{2}$ when

$\alpha \in [1,2]$ to get $\frac{1}{2}\alpha$ to be in the interval associated with M_3. This will enable us to repeatedly multiply by $\frac{1}{2}$ and 3.

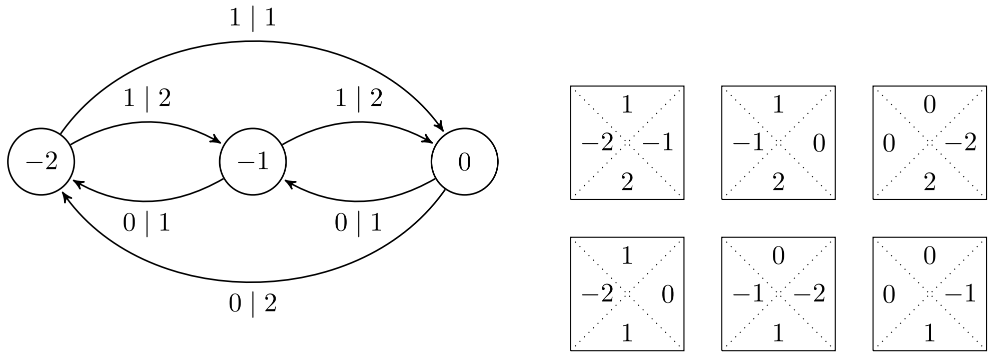


Figure 5.1: State diagram of machine M_3 (left) and the tileset T_3 it generates (right).

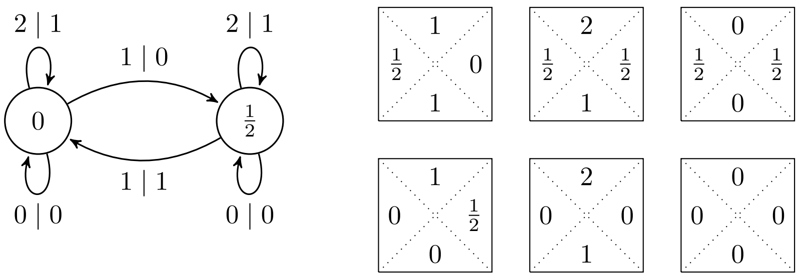


Figure 5.2: State diagram of machine $M_{\frac{1}{2}}$ (left) and the tileset $T_{\frac{1}{2}}$ it generates (right).

We will want to iterate rows with tiles from T_3 and rows with tiles from $T_{\frac{1}{2}}$ in a way that no tiling can have a vertical period $b < 3$. We can already see rows with tiles from T_3 can't be stacked on top of each other. To prevent us using tiles from two machines in the same row and to prevent us stacking more that 2 rows of tiles from $T_{\frac{1}{2}}$ on top of each other, we will adapt this machine slightly, as shown in Figure 5.3. We call the set of tiles created by this machine $T'_{\frac{1}{2}}$. We calculate with $0'$ in the same manner we would calculate with 0.

Remark 5.1. Notice that for input in the interval $[1,2]$, we must use tiles from the 5 tiles of $T'_{\frac{1}{2}}$ with either a 1 or a 2 on the top edge (a consequence of Lemma 3.5). Of these tiles, there is only one tile with 0' on the bottom edge, $(0', 1, 0', \frac{1}{2})$ and one with 0 on the bottom edge, $(0', 1, 0, \frac{1}{2})$. As they have the same colors on all other edges, we may 'choose' whether we want an output sequence using 0 or 0' when the input sequence is in $[1,2]$.

Definition 5.2. We define the sequential machine M to be the union of machines M_3 and $M'_{\frac{1}{2}}$ (by simply joining the both sets of states, alphabets and transitions). We define T_{13} to the tileset generated by M: the union of tiles from T_3 and $T'_{\frac{1}{2}}$.

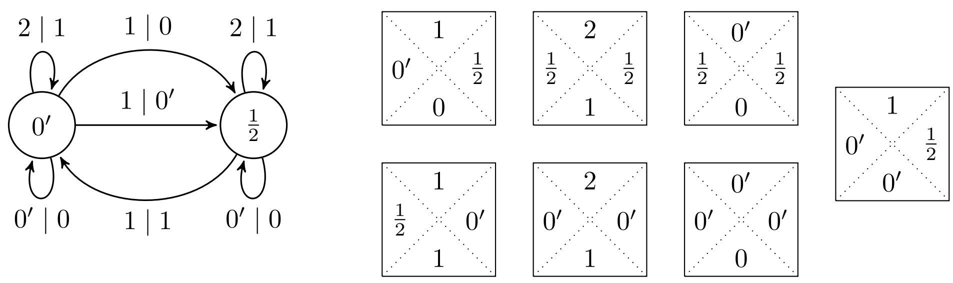


Figure 5.3: State diagram of machine $M'_{\frac{1}{2}}$ (left) and the tileset $T'_{\frac{1}{2}}$ it generates (right).

5.2 Proving aperiodicity

We will now separately prove the two requirements for T_{13} to be aperiodic: the existence of a tiling by T_{13} and the aperiodicity of any tiling by T_{13}.

Proposition 5.3. *There exists a tiling of* $\mathbb{Z}^2$ *by* T_{13}

Proof. Let $\alpha_0 \in [\frac{1}{3}, 2]$. Then $B(\alpha_0)$ is a valid input for M, and M can calculate $B(3\alpha)$ if $\alpha \in [\frac{1}{3}, \frac{2}{3}]$ and $B(\frac{\alpha}{2})$ if $\alpha \in [\frac{2}{3}, 2]$ (Proposition 4.5). We may assume that our initial choice of α_0 uses 0 entries and no $0'$ entries in $B(\alpha)$.

By remark 5.1, we can 'choose' whether we create an output sequence using 0 or $0'$ when using tiles from $T'_{\frac{1}{2}}$ when the input is in the interval $[1, 2]$. Choosing $0'$ however, forces us to multiply by $\frac{1}{2}$ again, as the color $0'$ is not present on the top edges of tiles from T_3.

To prove there is a tiling possible, we will show that M can be iterated in both ways. That is: we will show the output of the current row can be used as input for the next row as well as the input of the current row being the output of some previous row. We will denote with α_i the input of row i and with α_{i-1} the output of row i. Forward use of M thus means growth downwards.

- For forward use of M, we make a case distinction on α_i:
  - If $\alpha_i \in [\frac{1}{3}, \frac{2}{3}]$, we can multiply $B(\alpha_i)$ by 3 by using tiles from M_3. The output sequence is $B(3\alpha_i)$, so $\alpha_{i-1} = 3\alpha_i$ resides in the interval $[1, 2] \subset [\frac{1}{2}, 2]$.
  - If $\alpha_i \in (\frac{2}{3}, 1\frac{1}{3})$, we can multiply $B(\alpha_i)$ by $\frac{1}{2}$ by using tiles from $M'_{\frac{1}{2}}$. By the remark above, we can choose tiles so that the output sequence has no 0' entries. The output sequence is $B(\frac{1}{2}\alpha_i)$, so $\alpha_{i-1} = \frac{1}{2}\alpha_i$ resides in the interval $(\frac{1}{3}, \frac{2}{3}) \subset [\frac{1}{3}, 2]$.
  - If $\alpha \in [1\frac{1}{3}, 2]$, we can multiply $B(\alpha_i)$ by $\frac{1}{2}$ by using tiles from $M'_{\frac{1}{2}}$. By the remark above, we can choose tiles so that the output sequence has no 0 entries. The output sequence $B(\frac{1}{2}\alpha_i)$ so $\alpha_{i-1} = \frac{1}{2}\alpha_i$ resides in the interval $[\frac{2}{3}, 1]$. Thus, we can multiply by $\frac{1}{2}$ again by using tiles from $M'_{\frac{1}{2}}$, this time choosing an output with no 0' entries. The output sequence is $B(\frac{1}{4}\alpha_i)$, so $\alpha_{i-2} = \frac{1}{4}\alpha_i$ and it resides in the interval $[\frac{1}{3}, \frac{1}{2}] \subset [\frac{1}{3}, 2]$.

  These cases cover all options for the interval $[\frac{1}{3}, 2]$. By using these instructions, we can start with our chosen α_0 to tile a first row and determine all rows below it.

- For the backward use of M, we work in a similar fashion, but reversed. Again, a case distinction on α_i:
  - If $B(\alpha_i)$ contains $0'$ entries, then $\alpha_i \in [\frac{1}{3}, 1]$. Thus, $2\alpha_i \in [\frac{2}{3}, 2] \subset [\frac{1}{3}, 2]$. So $M'_{\frac{1}{2}}$ applied to $B(2\alpha_i)$ has $B(\alpha_i)$ as possible output (Remark 5.1) and we can use $\alpha_{i+1} = 2\alpha_i$.
  - If $B(\alpha_i)$ does not contain 0 entries and $\alpha_i \in [\frac{1}{3}, 1]$, then $2\alpha_i \in [\frac{2}{3}, 2] \subset [\frac{1}{3}, 2]$. So, $M'_{\frac{1}{2}}$ applied to $B(2\alpha_i)$ has $B(\alpha_i)$ as possible output (Remark 5.1) and we can use $\alpha_{i+1} = 2\alpha_i$.
  - If α_i does not contain 0 entries and $\alpha_i \in (1, 2]$, then $\frac{1}{3}\alpha_i \in (\frac{1}{3}, \frac{2}{3}] \subset [\frac{1}{3}, 2]$. So, M_3 applied to $B(\frac{1}{3}\alpha_i)$ has $B(\alpha_i)$ as output and we can use $\alpha_{i+1} = \frac{1}{3}\alpha_i$.

  These cases are exhaustive. By using these instructions, we can start with our chosen α_0 to tile a first row and determine all rows above it.

So, we can choose $\alpha_0 \in [\frac{1}{3}, 2]$ for the input sequence of the row 0 and iterate M forwardly and backwardly to obtain a tiling of $\mathbb{Z}^2$ by tiles from T_{13}. □

Proposition 5.4. *There is no periodic tiling of $\mathbb{Z}^2$ by T_{13}.*

Proof. We will give a proof by contradiction: assume there is a periodic tiling by T_{13}. By Lemma 2.9, there exists a doubly periodic tiling f too, say with horizontal period a and vertical period b. We may assume them to be positive (Remark 2.8).

Note that $b \geq 3$ by construction (we can verify by eye that $b = 1$ or $b = 2$ is not possible). Also note that we cannot have tiles from T_3 and $T'_{\frac{1}{2}}$ in the same row. Without loss of generality, we may assume the first row (that of $(0, 0)$) is tiled by tiles from T_3.

Define n_i as the sum of colors on the top edges of a strip of tiles of length a, starting at $(0, i)$:

$$n_i = \sum_{k=0}^{a-1} f_n(k, i)$$

Using Lemma 4.2, we make the following observation:

$$n_{i-1} = q_i n_i \quad \text{with } q_i = \begin{cases} 3 & \text{if row } i \text{ consists of tiles from } T_3 \\ \frac{1}{2} & \text{if row } i \text{ consists of tiles from } T'_{\frac{1}{2}} \end{cases}$$

(Of course we could do this with strips starting at any (x, i), but starting at $(0, i)$ will be sufficient)

Now, as b is the vertical period of f, we see that $n_0 = n_b = q_{b-1}q_{b-2}\ldots q_0 \cdot n_0$. As the first row is tiled by T_3, $n_0 \neq 0$ (it is clear we cannot have repeated zeroes on the top edges). Thus we must have: $q_{b-1}q_{b-2}\ldots q_0 = 1$. However, no non-empty product of 3 and $\frac{1}{2}$ is 1, so $b = 1$. But we already saw $b \geq 3$. This is a contradiction, so a periodic tiling of $\mathbb{Z}^2$ by T_{13} cannot exist.

□

Theorem 5.5. *T_{13} is an aperiodic set of Wang tiles.*

Proof. There are two conditions to be met for a set of Wang tiles to be aperiodic: there needs to exists a tiling and no tiling can be periodic. These conditions are proven for T_{13} in Proposition 5.3 and Proposition 5.4 respectively. □

Corollary 5.6. *There exists an aperiodic set of 13 Wang tiles using 5 colors.*

Proof. Recall we may not rotate Wang tiles. Thus, we can use the same set of colors for the horizontal and vertical edges. We see that there are 4 colors used for the horizontal edges of T_{13} ($0', 0, 1$ and 2) and 5 for the vertical edges ($-2, -1, 0, 0'$ and $\frac{1}{2}$). So we can easily use the set of 5 colors for the horizontal edges too, for example by replacing each instance of 1 with -1 and 2 with -2. □

An example of a different coloring of the described tileset using 5 colors is shown in Figure 5.4

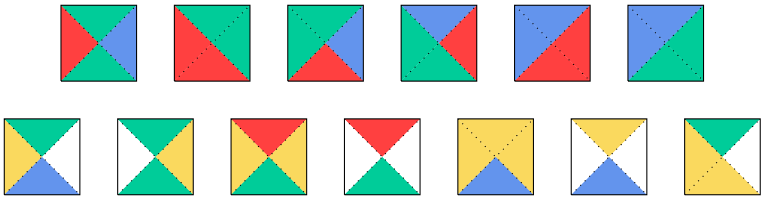


Figure 5.4: An aperiodic tileset of 13 tiles using 5 colors

5.3 Minimal sets and Wang's Hypothesis

The result of Theorem 5.5 is quite nice. Still, it is a natural question to ask whether we actually need all tiles present in T_{13}. Such a question motivates the following definition.

Definition 5.7. A finite set of Wang tiles T is called *minimal* if there exists a tiling of $\mathbb{Z}^2$ by T, but there does not exists a tiling of $\mathbb{Z}^2$ by any strict subset of T.

The question whether T_{13} is minimal has yet to be answered. However, there is an example of minimal set: the one with 11 tiles using 4 colors shown in [10]. We refer to it by T_{11}. This tileset was proven to be aperiodic using an extensive computer program too elaborate to discuss here.

Not only is T_{11} minimal in the sense that no subset of it is an aperiodic set of Wang tiles, it is also minimal in a more broad sense: it is impossible for a set with fewer tiles or using fewer colors to be aperiodic.

Proofs that sets of Wang tiles using only 2 or 3 colors cannot be aperiodic require different strategies than discussed in this thesis. We refer to [9] and [3] for the proofs for 2 and 3 colors respectively.

An alternate way of formulating this would be: Wang's hypothesis is true for 2 and 3 colors. For 4 colors, it is false, as was also proven by us (Corollary 5.6).

Chapter 6

Wang Cubes

Until now, we have worked in a 2-dimensional setting: we have tiled portions of $\mathbb{Z}^2$. In this chapter, we will expand to a 3-dimensional setting and use the construction of T_{13} to create a set of aperiodic Wang Cubes.

Adding a third dimension to Wang tiles is a quite natural desire. Not only does it raise interesting questions of theoretical nature (are aperiodic tilings possible? Do the results from the 2-dimensional setting generalize to a 3-dimensional setting?), but it is also useful on a practical level. For example, Wang cubes have been used in volume illustration [13].

6.1 Extension of familiar concepts

Definition 6.1. A *Wang cube* is a unit cube where each side is marked with a color. If the left, back, front, right, top and bottom faces are colored w, n, s, e, t and b respectively, we will denote the tile as (w, n, s, e, t, b).

We make a remark for completeness: similar to Definition 2.1, we say that only the borders that meet in the front-left-bottom corner are part of the cube. However, this is of little importance to us.

One should pay special attention to the notation of a Wang cube and the relation it has to a Wang tile. In extending tiles to cubes, the vertical faces of the cubes will be used to 'represent' the 2 original dimensions, while the horizontal faces are 'new'. This is illustrated in Figure 6.1.

We will now adapt some other basic definitions to a three-dimensional setting.

Definition 6.2. Let C be a finite set of Wang cubes and $R \subseteq \mathbb{Z}^3$. A Wang-tiling of R by C is a function $f : R \to C$ such that adjacent faces of all cubes match.

Recall that we were able to make the requirement for matching colors more formal by introducing color functions in Notation 2.4. We can naturally extend this formalism by adding color functions f_t and f_b for the top and bottom colors respectively. (And, of course, using functions on $\mathbb{Z}^3$) Again, it should be clear we are not allowed to rotate or reflect cubes.

As we will only be dealing with Wang-tilings, we will often abbreviate this simply to 'tiling'. By omitting R and just saying f is a tiling by C, we mean to say the subset $R \subseteq \mathbb{Z}^3$ is the whole of $\mathbb{Z}^3$.

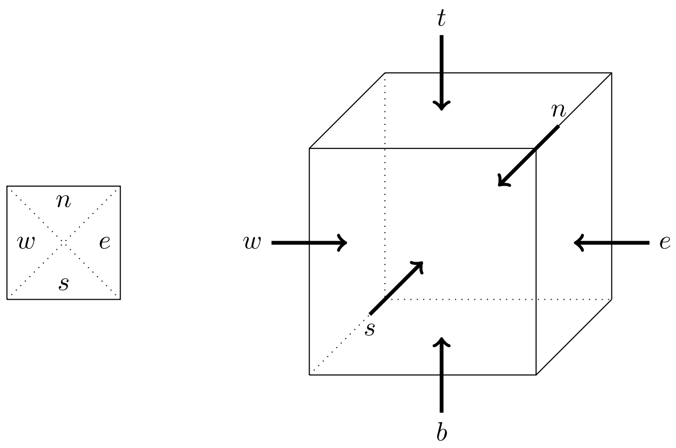


Figure 6.1: A Wang Tile (w, n, s, e) (left) and a Wang cube (w, n, s, e, t, b) (right)

Definition 6.3. Let C be a finite set of Wang cubes and f a tiling by C. f is called *periodic with* period (a, b, c) with $(a, b, c) \in \mathbb{Z}^3 \setminus \{(0, 0, 0)\}$ if for all $(x, y, z) \in \mathbb{Z}^3$: $f(x, y, z) = f(x+a, y+b, z+c)$. f is called *aperiodic* if it is not periodic for any period.

It is obvious that there exist periodic tilings on $\mathbb{Z}^3$. For example, $\mathbb{Z}^3$ can be tiled periodically by the singleton set containing the tile (w, n, n, w, t, t) (shown in Figure 6.2), as opposing sides have the same color. This tiling would have, for example, period $(1, 1, 1)$.

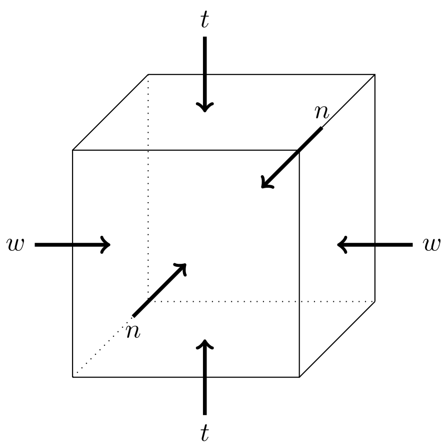


Figure 6.2: A single Wang cube that can be used to periodically tile $\mathbb{Z}^3$

Definition 6.4. A finite set of Wang cubes C is aperiodic if there exists a tiling of $\mathbb{Z}^3$ by C, but no tiling is periodic.

As with the 2-dimensional case, it is not at all obvious such an aperiodic set of cubes exists. One may intuitively try to stack aperiodic 2-dimensional tilings on top of each other, but it is not clear how this should be done to force aperiodicity.

Example 6.5. We could construct the following set of Wang cubes:

$$C_{13} = \{(w, n, s, e, w, w) \mid (w, n, s, e) \in T_{13}\}$$

Every layer of this tiling is aperiodic (as it necessarily corresponds to a 2-dimensional tiling by T_{13}), but as we can stack the same layer on top of itself, we could easily create a tiling with period $(0, 0, 1)$. Moreover, we can see the only way a tiling can be created is by repetition of one layer.

6.2 Triply periodic tilings

Before we will turn our attention to aperiodic sets of Wang cubes, we will explore a subtle difference between tilings in 2 and 3 dimensions. The goal is to illustrate that the transition is not seamless and emphasize we must carefully construct a set of cubes for it to be aperiodic.

In the 2-dimensional setting, we introduced a variant to periodic tilings: doubly periodic tilings (Definition 2.7). Though we have proven that a set of Wang tiles having a doubly periodic tiling is not a stronger property than regular periodicity (Lemma 2.9), it has still been a useful tool to us.

The notion of double periodicity can be extended to the 3-dimensional setting: a tiling is periodic over 2 axes. There is also a 3-dimensional 'equivalent', concerning all 3 axes:

Definition 6.6. Let C be a finite set of Wang cubes, f a tiling by C and $a, b, c \in \mathbb{Z}\setminus\{0\}$. f is called *triply periodic* with first period a, second period b and third period c if for all $(x, y, z) \in \mathbb{Z}^2 : f(x, y, z) = f(x + a, y, z) = f(x, y + b, z) = f(x, y, z + c)$

From the definition, a triply periodic tiling may seem like the 3-dimensional variant to doubly periodic tilings and thus sharing the same properties. However, there is a difference in behaviour. Recall that in the 2-dimensional setting, the existence of a periodic tiling implied the existence of a doubly periodic tiling. This property is not carried over to the 3-dimensional equivalent: the existence of a periodic tiling *does not* imply the existence of a triply periodic tiling. The following lemma is a reformulation of this claim, which we will prove with a counterexample.

Lemma 6.7. *Let C be a finite set of Wang cubes and f a periodic tiling of $\mathbb{Z}^3$ by C, then there need not exist a triply periodic tiling of $\mathbb{Z}^3$ by C.*

Proof. Recall the example given in Example 6.5. As stated, there exist periodic tilings by this set. Assume there also exists a triply periodic tiling f with first, second and third periods a, b and c: for all $(x, y, z) \in \mathbb{Z}$ holds $f(x, y, z) = f(x + a, y, z) = f(x, y + b, z) = f(x, y, z + c)$.

Now, by fixing a value on the third axis, we can construct a 2-dimensional tiling $g : \mathbb{Z} \to T_{13}$ with the following definition:

$$g(x, y) = (w, n, s, e) \iff f(x, y, 0) = (w, n, s, e, s, s)$$

By the triple periodicity of f, we see that for g holds: $g(x, y) = g(x + a, y) = g(x, y + b)$, making g a doubly periodic tiling. In particular, g is periodic. However, g is a tiling by T_{13}, contradicting Theorem 5.5. By contradiction, we have proven the claim. $\square$

From this result, it should be clear that the 3-dimensional setting is more intricate than the 2-dimensional setting we've explored so far. However, there is nice property that 3-dimensional tilings do have:

Lemma 6.8. *Let C be a finite set of Wang cubes. If there exists a doubly periodic tiling of $\mathbb{Z}^3$ by C, then there also exists a triply periodic tiling of $\mathbb{Z}^3$ by C.*

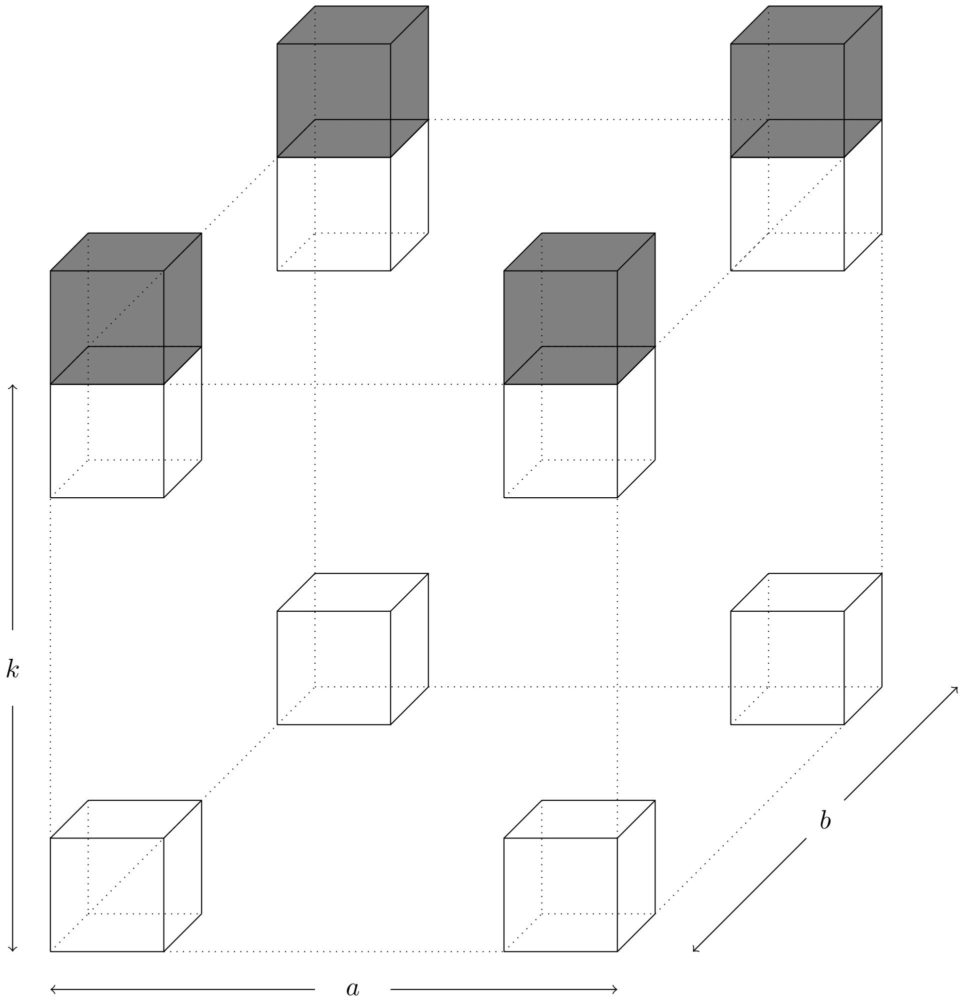


Figure 6.3: An a-by-b-by-k repeatable block(white). a-by-b layer of gray colored blocks is the same as bottom a-by-b layer.

Proof. Let C be a finite set of Wang cubes and f a doubly periodic tiling of $\mathbb{Z}^3$ by C, with periods a and b. We may assume f is periodic in the first and second axes, the strategy will be analogous otherwise: for all $(x, y, z) \in \mathbb{Z}^3$ holds $f(x, y, z) = f(x + a, y, z) = f(x, y + b, z)$.

We may also assume $a, b > 0$ (if not, we can mirror our approach in the appropriate axis)

Similar to the proof of Lemma 2.9, we will construct a block of several cubes that can be placed next to and stacked on top of itself. Because C is finite, there are but finite ways to tile a section of a-by-b cubes. Thus, there must be a section of a-by-b cubes (starting at some $(x_S, y_S, z_S) \in \mathbb{Z}^3$) with a copy of the section lying some $k \in \mathbb{N}$ above it (see Figure 6.3). This lets us create a repeatable a-by-b-by-k block.

We finish the proof by constructing an explicit triply periodic tiling g from f:

$$g : \mathbb{Z}^3 \to C : (x, y, z) \mapsto f(x_S + (x \bmod a), y_S + (y \bmod b), z_S + (z \bmod k))$$

We can now see that g is indeed triply periodic:

$$\begin{aligned} g(x, y, z) &= f(x_S + (x \bmod a), y_S + (y \bmod b), z_S + (z \bmod k)) \\ &= f(x_S + (x + a \bmod a), y_S + (y \bmod b), z_S + (z \bmod k)) \\ &= g(x + a, y, z) \end{aligned}$$

The other arguments are analogous. We conclude that indeed, if there exists a doubly periodic tiling of $\mathbb{Z}^3$ by C, there also exists a triply periodic tiling of $\mathbb{Z}^3$ by C. $\square$

6.3 The XOR-automaton

We will now take a brief look at cellular automata. Though there is a vast body of literature regarding cellular automata theory, we will only state that which is relevant to us, even limit ourselves to one automaton. For a more precise and in-depth discussion of this automaton, we refer to Culik and Dube's 'Fractal and Recurrent Behaviour of Cellular Automata' [6].

Cellular automata are used to model complex systems with local interactions. They consist of a grid of cells, a set of possible states and local rules. Each cell is assigned a state. Using the cell's neighbourhood, its current state and the rules, a new state can be assigned to a cell. This is simultaneously done to all other cells. This process can be repeated, each iteration called a 'step'.

Step 1:						1					
Step 2:					1		1				
Step 3:				1		0		1			
Step 4:			1		1		1		1		
Step 5:		1		0		0		0		1	
Step 6:	1		1		0		0		1		1

Figure 6.4: An XOR pattern

Figure 6.4 shows a pattern that may be familiar. It is often used to describe a XOR or modulo 2 pattern. We will now describe the XOR-automaton, that aims to capture this behaviour in a formal way.

- The set $S = \{0, 1\}$ represent the possible states
- The grid of cells is a bi-infinite strip. This can be viewed as a function $g : \mathbb{Z} \to S$ which assigns a state to each position.
- Let $g_i : \mathbb{Z} \to S$ denote the grid at step i ($i \in \mathbb{N}$). The interaction rule is as follows:

$$g_{i+1}(z) = g_i(z) \oplus g_i(z + 1)$$

  Here, the symbol $\oplus$ is the XOR symbol, which in this case behaves like mod2.

By placing the consecutive steps below each other, a pattern can be spotted. This is illustrated in Figure 6.5. The relation to Figure 6.4 can be seen by 'tilting' the pattern and 'ignoring' excess 0s.

In these figures, input 1 is used, meaning that one random cell was in state 1 in the first step. Other input sequences (words of 0s and 1s) can be used too. The XOR-automaton has an important property related to its input, which is formulated in Lemma 6.9.

Step 1:	...	0	0	0	0	0	0	0	0	1	0	...
Step 2:	...	0	0	0	0	0	0	0	1	1	0	...
Step 3:	...	0	0	0	0	0	0	1	0	1	0	...
Step 4:	...	0	0	0	0	0	1	1	1	1	0	...
Step 5:	...	0	0	0	0	1	0	0	0	1	0	...
Step 6:	...	0	0	0	1	1	0	0	1	1	0	...
Step 7:	...	0	0	1	0	1	0	1	0	1	0	...
Step 8:	...	0	1	1	1	1	1	1	1	1	0	...

Figure 6.5: First steps of the XOR-automaton with input 1

Lemma 6.9. *Let w be a finite sequence of 0s and 1s, and let n be the smallest power of 2 greater than or equal to the length of w. Let w_n denote the sequence of length n that is w with 0s appended to the left. Given input sequence $\ldots 00w_n00\ldots$, the XOR-automaton produces the sequence $\ldots 00w_nw_n00\ldots$ in no fewer steps than n.*

We will not prove this property, but refer to Culik and Dube's work in [6]. We can verify the property for the example shown in 6.5: input $w_n = 1$ is repeated after 1 step, input $w_n = 0101$ is repeated after 4 steps, etc. (We use w_n so that any sequence w can be used)

One should note: the row of cells from the XOR-automaton is a bi-infinite sequence as introduced in Definition 3.1. Were we to put the rows of cells from the XOR-automaton below each other, the resemblance to $\mathbb{Z}^2$ becomes clear. This motivates a new question: can we capture the behaviour of this automaton using Wang tiles? Just like we did using balanced number sequences, the sequences of cell-states can encode the colors of the top and bottom edges of tiles. The colors on the left and right edges can be chosen to our liking.

We claim the set of 4 Wang tiles $X = \{(0,0,0,0), (0,0,1,1), (1,1,1,0), (1,1,0,1)\}$, shown in Figure 6.6, captures the behaviour of the XOR-automaton.

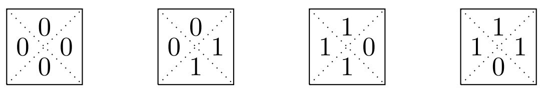


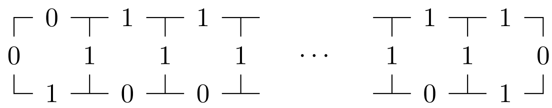
Figure 6.6: The tiles of the set of Wang tiles X

To show X captures this behaviour, we must show that given any sequence of 0s and 1s, we can choose tiles such that this sequence can be read on the top edges and the next step can be read on the bottom edges. It is sufficient to show we can capture the behaviour for a sequence of 1s of arbitrary length bordered by a 0 on the left and show these segments of tiles can be placed next to each other, because any configuration of the XOR-automaton is a sequence of such sequences.

Our approach is as follows: let 01^n denote the sequence starting with a 0 followed by $n \in \mathbb{N}$ 1s.

- If $n = 0$: use tile $(0,0,0,0)$
- If $n > 0$: use tile $(0,0,1,1)$, followed by $n-1$ $(1,1,0,1)$ tiles, followed by a $(1,1,1,0)$ tile. This is illustrated in Figure 6.7.

It is easy to verify that this is the only possible way these inputs can be converted to tiles (for each combination of top and bottom color, we have but one tile). Additionally, Step i: 011...11



Step $i+i$: 100...01

Figure 6.7: The sequence 01^n converted to tiles. A different visualization of Wang tiles is used to emphasize the relation the input sequences.

as each strip has a 0 on the leftmost and rightmost edges, these strips of tiles can be placed next to each other.

This shows that X is indeed a suitable choice: any input for the XOR-automaton determines a unique tiling.

It must be noted that we can apply X backwards in analogous manner to obtain 'negative steps' of the automaton.

6.4 Construction of C_{21}

In Example 6.5, we saw the following set of Wang cubes:

$$C_{13} = \{(w, n, s, e, w, w) \mid (w, n, s, e) \in T_{13}\}$$

We could tile $\mathbb{Z}^3$ periodically using these cubes, but periodicity would only be possible in one direction (the vertical direction). We will adapt this set in a way that forces aperiodicity along the vertical axis.

In Figure 6.1, the relation between Wang tiles and cubes was illustrated: the vertical dimension is 'added' to the tiles. However, we could also see a Wang cube as a combination of two (or more) 2-dimensional tiles. For a Wang cube (w, n, s, e, t, b), we refer with 'the horizontal cut' to the tile from original two dimensions: (w, n, s, e). With the 'vertical cut' we refer to (w, t, b, e). This is illustrated in 6.8.

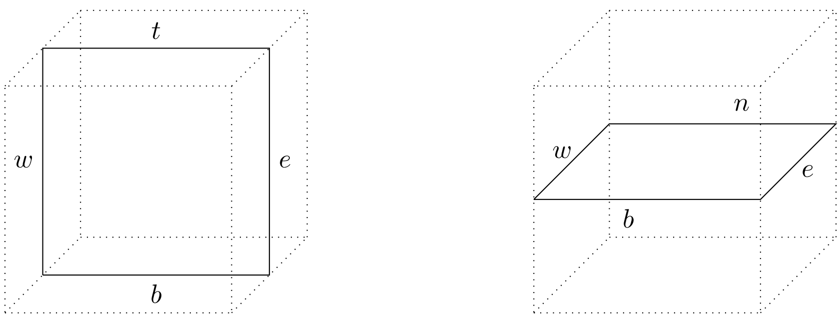


Figure 6.8: The vertical (left) and horizontal (right) cuts of a Wang cube (w, n, s, e, t, b)

So, when expanding the set C_{13} from Example 6.5 so that aperiodicity along the vertical axis is forced, we will need to adapt the vertical cut of the cubes. We will achieve this by 'merging' the tiles related to the XOR-automaton (seen in Figure 6.6) to C. Merging can be achieved by converting the colors on the faces of the vertical cut to pairs of colors. We will do this in such way that the tiles from T_{13} with a 2 on the top edge (i.e. cubes with a 2 on the back face) simulate the XOR-automaton. We will see in the next section why these are suited to do this.

We will now construct the set of Wang Cubes C_{21}, which we will prove to be aperiodic in the following section.

Let T_9 denote the set $T_{13} \setminus \{(0', 2, 1, 0'), (\frac{1}{2}, 1, 1, 0'), (\frac{1}{2}, 2, 1, \frac{1}{2}), (0', 1, 0, \frac{1}{2})\}$. Now, define $C_{21} := C_T \cup C_X \cup C_Z$, with:

- $C_T = \{((w, 1), n, s, (e, 1), (1, 1), (1, 1)) \mid (w, n, s, e) \in T_9\}$

  This set contains 9 cubes that simulate the regular behaviour of (the associated 9 tiles of) T_{13} along the horizontal cuts. Again, the pairs of colors are a result of the merging of two sets of tiles. The second bits of the cubes of C_T are all 1, which results in them not being restricted by the vertical cut, only the horizontal cut.

- $C_X = \{((w, x), 2, 1, (w, y), (1, x), (1, x \oplus y)) \mid w \in \{0', \frac{1}{2}\}, x, y \in \{0, 1\}\}$

  This set contains 8 cubes, representing the tiles from T_{13} with a 2 on the top edge (4 cubes for each tile). The second bits of the faces of the vertical cut exactly represent the tiles from the set X shown in Figure 6.6 (one copy of X for each T_{13} with a 2 on the top edge).

- $C_Z = \{((\frac{1}{2}, 1), 1, 1, (0', x), (0, 1), (0, 1)), ((0', 1), 1, 0, (\frac{1}{2}, x), (0, 1), (0, 1)) \mid x \in \{0, 1\}\}$

  This set contains 4 cubes. Note that these are the only cubes with 0s as first bit on the top and bottom faces, with the result that cubes from C_Z can be stacked on top of each other. As a consequence, every horizontal layer must have the same horizontal cut. This justifies a change we've made compared to C_{13} (from Example 6.5): we do not need the w-colored horizontal faces anymore to force the tiling to have the same horizontal cuts on every level. This is a desirable change, as it reduces the number of colors used.

  Of course, we must ensure these cubes are actually used in the tiling to use this property. We will see this is the case in Remark 6.11.

6.5 Proving aperiodicity

In proving C_{21} is aperiodic, we need a yet unproven property of T_{13}. The following lemma formulates this property: it states that in any tiling by T_{13}, there are sequences of tiles with 2's as color on the top edge of arbitrary length.

Lemma 6.10. *Let $f : \mathbb{Z}^2 \to T_{13}$ be a tiling and $N \in \mathbb{N}$ a natural number. Then there exists a sequence of N tiles $f(x, y), f(x+1, y), \ldots, f(x+N-1, y)$ $(x, y \in \mathbb{Z}^2)$ all with 2 as color of the top edge.*

Proof. In this proof, we closely follow the one given for the same lemma in [7].

Let $f : \mathbb{Z}^2 \to T_{13}$ be a tiling of $\mathbb{Z}^2$ by T_{13}. In this proof, we assume every tiling by T_{13} is indeed of the form described in the proof of Proposition 5.3. More on this assumption after this proof.

For $i \in \mathbb{Z}$ and $N \in \mathbb{Z}$, let $n_{N,i}$ denote the sum of colors on the upper edges of the sequence $f(1, i), f(2, i), \ldots, f(N, i)$:

$$n_{N,i} = \sum_{k=1}^{N} f_n(k, i)$$

(Here we use the notation introduced in Notation 2.4) Moreover, let $q_i = 3$ when row i is tiled with tiles from T_3 and let $q_i = \frac{1}{2}$ when row i is tiled with tiles from $T'_{\frac{1}{2}}$. We first make three observations:

- Observation 1: $|n_{N,i} - q_i n_{N,i+1}| \leq 2$

  This is due to the following: for every $k \in \mathbb{Z}$:

  $$f_w(k,i) + q_i f_n(k,i) = f_e(k,i) + f_s(k,i)$$

  Summation gives us:

  $$\sum_{k=1}^{N} f_w(k,i) + q_i f_n(k,i) = \sum_{k=1}^{N} f_e(k,i) + f_s(k,i)$$

  Using $f_w(k,i) = f_e(k+1,i)$, we see:

  $$f_w(1,i) + q_i \sum_{k=1}^{N} f_n(k,i) = f_e(N,i) + \sum_{k=1}^{N} f_s(k,i)$$

  As the differences of vertical colors in T_{13} are at most 2, we see:

  $$|q_i \sum_{k=1}^{N} f_n(k,i) - \sum_{k=1}^{N} f_s(k,i)| = |f_e(N,i) - f_w(1,i)| \leq 2$$

  So using $f_n(k,i) = f_s(k,i-1)$ and the definition of $n_{N,i}$, we see this observation holds:

  $$|n_{N,i} - q_i n_{N,i+1}| \leq 2$$

- Observation 2: as the bound of the interval are 2 and $\frac{1}{3}$, the factor between two numbers in that interval is at most $2/\frac{1}{3} = 6$. Thus, any product of q_i's is bounded by 6: for any $i, j \in \mathbb{Z}$:

  $$q_i q_{i+1} \dots q_{i+j} \leq 6$$

- Observation 3: Combining Observations 1 and 2, we see that for any $N, m \in \mathbb{N}_{>0}$:

  $$|n_{N,0} - q_0 q_1 \dots q_{m-1} n_{N,m}| \leq 2(1 + q_0 + q_0 q_1 + \dots + q_0 q_1 \dots q_{m-1}) \leq 2(m \cdot 6) = 12m$$

Let $\varepsilon \in \mathbb{R}_{>0}$ be an arbitrarily small number. We will show that for N sufficiently large, there exists m such that $\frac{n_{N,m}}{N} > 2 - 14\varepsilon$. This would be sufficient to prove this Lemma.

First, another observation:

- Observation 4: $\log_2 3$ is an irrational number. As a consequence, the set $\{m \log_2 3 \bmod 1 \mid m \in \mathbb{Z}\}$ is dense in the interval $[0,1]$.

  For a subinterval $I \subset [0,1]$, we then see there exists $M \in \mathbb{N}$ such that for any $x \in \mathbb{R}$ there exists $m \in \mathbb{Z}$ with $0 < m < M$ such that $x + m \log_2 3 \bmod 1 \in I$.

  In other words: 'for any x, we need fewer than M steps of size $\log_2 3$ to arrive from $x \bmod 1$ in I'

  (This observation is a consequence of the equidistribution theorem, for example proven in [16])

Now, we will choose I, N, M and x.

Let us use $I = [\log_2(2-13\varepsilon), \log_2(2-12\varepsilon)]$ and M as described above. Let $N \in \mathbb{N}$ with $N > 12M(1+\log_2 3)/\varepsilon$ and N large enough such that $N > \frac{1}{3} - \varepsilon$. This is possible as there can never occur three consecutive 0's as top colors of tiles from T_{13} ($\frac{1}{3}$ is the minimum input).

For x, we use $x = \log_2\left(\frac{n_{N,0}}{N}\right)$. Then using Observation 4, there is $m \in \mathbb{Z}, 0 < m < M$ such that $x + m\log_2 3 - k \in I$ (with $k \in \mathbb{Z}_{>0}$ and $k < M\log_2 3$). We will want m and k to represent the number of times we multiply with 3 resp. 2

Due to the bound of our interval I, we thus see:

$$2 - 13\varepsilon \leq \frac{n_{N,0}}{N} \cdot \frac{3^m}{2^k} \leq 2 - 12\varepsilon$$

It must be that $q_0 q_1 \ldots q_{m+k-1} = \frac{3^m}{2^k}$. We will show this by contradiction:

- Assume it is not, then the product would be between $\frac{3^{m-1}}{2^{k+1}}$ (division by 6) and $\frac{3^{m+1}}{2^{k-1}}$ (multiplicaition by 6) as we must alternate between multiplying with 3 or (twice) $\frac{1}{2}$. This would mean $\frac{n_{N,0}}{N} q_0 q_1 \ldots q_{m+k-1}$ is smaller than $\frac{1}{3} - 2\varepsilon$ or greater than $12 - 78\varepsilon$.

  Because $m < M$ and $k < M(1+\log_2 3)$, we see $m + k < M(1+\log_2 3)$. Combined with Observation 3, we know that:

  $$\left|\frac{n_{N,m+k}}{N} - q_0 q_1 \ldots q_{m+k-1}\frac{n_{N,0}}{N}\right| \leq \frac{12(m+k)}{N} < \varepsilon$$

  So, we would have one of two options:

  - Either $\frac{n_{N,m+k}}{N} < \frac{1}{3} - \varepsilon$, a contradiction
  - Or $\frac{n_{N,m+k}}{N} < 12 - 79\varepsilon$, a contradiction if ε is small.

We conclude that, indeed, $q_0 q_1 \ldots q_{m+k-1} = \frac{3^m}{2^k}$. Thus, we conclude the Lemma with the consequence:

$$2 - 13\varepsilon \leq \frac{n_{N,0}}{N} \cdot \frac{3^m}{2^k} = \frac{n_{N,m+k}}{N} \leq 2 - 12\varepsilon$$

□

Remark 6.11. There are but two tiles from T_{13} with 2 on the top edge: $(\frac{1}{2}, 2, 1, \frac{1}{2})$ and $0, 2, 1, 0$. As these clearly cannot be placed side by side, each strip of tiles with 2's on the top edge is a strip of either one of these tiles. Note that these are exactly the tiles used for the generation of the cubes from set C_X.

By the this remark, cubes from C_X must be used in tilings by C_{21}, which in turn means cubes from C_Z must be used, as the cubes from C_T cannot have a 0 as second bit on the right face.

Remark 6.12. As stated in the proof, we have assumed any tiling by T_{13} to be one that multiplies balanced number sequences in the interval $[\frac{1}{3}, 2]$. However, it is not obvious this is the case. There are two subtleties:

- As noted in Remark 4.4, we have not proven a sequential machine M_q (and thus, tiles T_q) *must* multiply by q.

- Moreover, we have not proven that any input sequence that is not a balanced number from the the interval $[\frac{1}{3}, 2]$ cannot produce a valid tiling. Sequential machines are not restricted to reading balanced numbers as input only (for example, $M'_{\frac{1}{2}}$ can read the sequence $\dots 00200 \dots$)

Remark 6.12 describes a subtle, though fundamental gap in the proof. At this time, we have not been able to give the proofs necessary to fill this gap. In the following, we will use Lemma 6.10, but one should bear in mind this assumption.

We will now prove C_{21} is aperiodic. Again, we need to prove the existence of a tiling and prove no tiling can be periodic. This time, we will prove these not in separate lemmas, but in one theorem. For convenience, we denote the cubes of C_{21} again:

- $C_T = \{((w,1), n, s, (e,1), (1,1), (1,1)) \mid (w,n,s,e) \in T_9\}$
- $C_X = \{((w,x), 2, 1, (w,y), (1,x), (1, x \oplus y)) \mid w \in \{0', \frac{1}{2}\}, x, y \in \{0,1\}\}$
- $C_Z = \{((\frac{1}{2},1), 1, 1, (0',x), (0,1), (0,1)), ((0',1), 1, 0, (\frac{1}{2},x), (0,1), (0,1)) \mid x \in \{0,1\}\}$

Theorem 6.13. *C_{21} is an aperiodic set of Wang cubes.*

Proof. Again, we closely follow the proof given for the same theorem in [7]. We also work with the assumptions of Remark 6.12.

For the existence of a tiling, we note the following: recall the set of cubes C_{13} from Example 6.5: a horizontal layer can obviously be tiled as we would tile the 2-dimensional plane with T_{13}. As noted before, each layer must have the same horizontal cut. As the horizontal faces encode an input sequence for the XOR-automaton, which can accept any input, a tiling is possible.

Now we prove no tiling can be periodic. We have seen that in any tiling by T_{13}, there are sequences of arbitrary length of tiles with 2s on the top edge (Lemma 6.10). As each horizontal layer in tilings by C_{21} is a tiling of $\mathbb{Z}^2$ by T_{13} (via the horizontal cuts), there are also sequences of 2s (on the back faces) present in any tiling by C_{21}. The cubes with 2s must all come from C_X. On the left side of such sequences, there must be a cube from C_Z, as it is impossible to place C_T tiles there due to those cubes not having 0s as second bits on the left faces.

By construction of C_{21}, the XOR-automaton is simulated on the sequences with 2s on top of each other, bordered on the left by a vertical strip of cubes from C_Z (as noted, these can only be stacked on top of each other), forcing all layers to have the same horizontal cut.

Lemma 6.9 states that the XOR-automaton always repeats its input in no fewer steps than the length of the input. We can see that it would be impossible for all layers to have a sequence of only 0s as second bits on the top side, so there is a layer that can serve as step 1. The length of the input sequence is determined by the length of sequence of 2s. We have seem there are sequences of 2s of arbitrary length, meaning no period will work. This ensures aperiodicity along the vertical axis. (Note that the behaviour of the XOR-automaton upwards from the row with the input sequence is irrelevant to this conclusion)

Aperiodicity along the horizontal axes are ensured as the horizontal cut of a layer is associated with a tiling by T_{13}. This proves a tiling of $\mathbb{Z}^3$ by C_{21} cannot be periodic.

In conclusion: C_{21} is an aperiodic set of Wang cubes. $\square$

Corollary 6.14. *There exists an aperiodic set of 21 Wang cubes using 7 colors.*

Proof. Since T_{13} uses 4 colors on the top and bottom edges of its tiles, C_{21} uses 4 colors on its front and back faces.

For the top and bottom faces, we use only 3 colors: $(1,1)$, $(1,0)$ and $(0,1)$.

For the left and right faces, we use 7 colors: the original 5 colors from T_{13} (originating from the 5 states of M_3 and $M'_{\frac{1}{2}}$), 2 of which have been split into 2 ($0'$ and $\frac{1}{2}$) with the addition of the second bit.

As cubes may not be rotated or reflected, we can use the largest of these numbers, 7, for the other faces too. This creates an aperiodic set of 21 Wang cubes using 7 colors.

□

The results of Theorem 6.13 and Corollary 6.14 disprove Wang's hypothesis for 3 dimensions, when working with the assumptions formulated in Remark 6.12.

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