The Matrix Tree Theorem

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1 Introduction

In a connected graph G, it is (usually) easy to find a tree that contains all the vertices and some edges of G; such a subgraph is called a a *spanning tree.* And maybe one can find two, or three such trees. But how many spanning trees does that graph contain?
That is what Gustav Robert Kirchhoff (1824-1887) was wondering. Kirchhoff was a German physicist, who contributed to the fundamental understanding of electrical circuits, spectroscopy and radiation.
Kirchhoff found an answer to this question, which is formulated in the Matrix Tree Theorem. By means of this theorem, solutions to (among others) linear resistive electrical network problems can be expressed much easier.

To formulate the Matrix Tree Theorem, we first have to define a matrix A_G.

Definition 1.1 *Let G be a connected graph with n vertices and m edges (numbered arbitrarily). We orient each edge random. The* incidence matrix *of G is the $n \times m$ matrix $A_G = [a_{ij}]$ with*

$$a_{ij} = \begin{cases} +1 & \textit{if the } j^{th} \textit{ edge is oriented to the } i^{th} \textit{ vertex} \\ -1 & \textit{if the } j^{th} \textit{ edge is oriented away from the } i^{th} \textit{ vertex} \\ 0 & \textit{otherwise} \end{cases}$$

If you number the vertices or edges in an other way, the rows or columns of the matrix A_G will be permuted. If you orient an edge otherwise, the $+1$ and -1 will be swapped. So the matrix A_G will not really be different.

From this matrix, we can make a *reduced incidence matrix* $\widetilde{A_G}$ of G. This matrix we make by deleting the nth row from the matrix A_G.

Example:

The matrices of the graph G in Figure 1 (in which we have numbered the vertices already, and oriented and numbered the edges) are:

$$A_G = \begin{pmatrix} -1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 1 & 0 & -1 & 0 & -1 \\ 0 & -1 & 0 & -1 & 1 \end{pmatrix} \quad \text{and} \quad \widetilde{A_G} = \begin{pmatrix} -1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 1 & 0 & -1 & 0 & -1 \end{pmatrix}$$

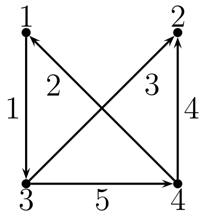


Figure 1: Graph G of the example.

With this matrix $\widetilde{A_G}$ we can calculate the number of spanning trees in G.

Theorem 1.2 (Matrix Tree Theorem) *If $\widetilde{A_G}$ is a reduced incidence matrix of the connected graph G, then the number of spanning trees of G equals the determinant of $\widetilde{A_G} \cdot \widetilde{A_G}^T$.*

In the next sections, we will prove this theorem.

2 Properties

Before we prove the Matrix Tree Theorem, we prove some properties of A_G and $\widetilde{A_G}$.

2.1 Rank

Lemma 2.1 *Let G be a graph with n vertices. Then G is connected $\Longleftrightarrow$ the rank of its incidence matrix A_G is $n-1$.*

Proof:

- $\Rightarrow$) Let $r < n$. The sum of any r rows must contain at least one non-zero entry. This is because otherwise there would be no edge connecting any of these r points to any point outside the set of r points, which would contradict the connectedness of G.
  Because of this, no r rows are linearly dependent if $r < n$.
  The sum of the n rows is 0, so the rank is $n-1$.
- $\Leftarrow$) If $\operatorname{rank}(A_G) = n-1$, there are no $r < n$ rows which add up to the all-zero row. So there are no r points which are not connected to the other $n-r$ points. Hence G is connected.

$\square$

If the j^{th} edge goes from vertex i_1 to i_2, then $a_{i_1,j} = -1$ and $a_{i_2,j} = 1$ and the other column elements are 0. So every column of A_G contains exactly one $+1$ and exactly one -1.
Hence, the sum of all the rows is the zero-row, so the last row of A_G is -1 times the sum of all the other rows. So this row is dependent of all the rows of $\widetilde{A_G}$. Now, it is clear that $\operatorname{rank}(A_G) = \operatorname{rank}(\widetilde{A_G})$, so a consequence of this lemma is:

$$G \text{ is connected} \Longleftrightarrow \operatorname{rank}(\widetilde{A_G}) = n-1$$

2.2 Determinants

The non-singular square submatrices of A_G have a special property if we calculate the determinant.

Lemma 2.2 *If B is a non-singular square submatrix of A_G, then the determinant of B is ± 1.*

To prove this lemma, we have to introduce a new definition:

Definition 2.3 *An* $r \times s$ *matrix* M *is* totally unimodular (TU) *if every* $k \times k$ *submatrix has determinant equal to* $-1, 1$ *or* 0 *(*$1 \leq k \leq \min\{r, s\}$*).*

Lemma 2.4 *Let* M *be a matrix with entries* $-1, 1$ *or* 0*, such that each column contains at most one* -1 *and at most one* 1*. Then* M *is TU.*

Proof: We show by induction on k that every $k \times k$ submatrix of M has the right determinant. When $k = 1$ it is trivial.
Assume $k \geq 2$ and let M_k be a $k \times k$ submatrix of M. If M_k has a column with exactly one non-zero entry, we can develop $\det(M_k)$ with respect to that column. This gives that $\det(M_k) = \pm \det(M_{k-1})$, where M_{k-1} is a $(k-1) \times (k-1)$ submatrix of M_k, and $\det(M_{k-1}) = -1, 1$ or 0 by the induction hypothesis. So $\det(M_k) = -1, 1$ or 0.
If M_k has no column with exactly one non-zero entry, then the rows of M_k add up to the all-zero row. Then $\det(M_k) = 0$.
So $\det(M_k) = -1, 1$ or 0 for all k. □

Now we can prove lemma 2.2:
Proof: Let B be a non-singular square (say $k \times k$) submatrix of A_G.
In the matrix A_G, every column contains exactly one -1 and exactly one 1. The other entries are all 0. Because of lemma 2.4, A_G is TU.
The definition of TU says that every $k \times k$ submatrix has determinant $-1, 1$ or 0. So $\det(B) = -1, 1$ or 0. Because B is non-singular, $\det(B) \neq 0$. Hence $\det(B) = \pm 1$. □

2.3 Matrices and trees

But what do these matrices have to do with spanning trees? This, we will see in the next lemma:

Lemma 2.5 *If* B *is a square submatrix of order* $n - 1$ *of* A_G*, then:*
B *is non-singular* $\Longleftrightarrow$ *the edges corresponding to the columns of* B *determine a spanning subtree of* G*.*

Proof: Let H be the subgraph of G determined by B.
Because B is of order $n - 1$, H contains $n - 1$ edges. Hence H is a tree $\Longleftrightarrow$ H is connected. But by lemma 2.1, we have that H is connected $\Longleftrightarrow$

$\mathrm{rank}(B) = n-1$. But $\mathrm{rank}(B) = n-1 \iff B$ is non-singular, because B is a $(n-1)\times(n-1)$ matrix.
So H is a tree $\iff B$ is non-singular. □

2.4 Binet-Cauchy

The theorem of Binet and Cauchy says: let R be a $p \times q$ matrix and S be a $q \times p$ matrix, where $p \leq q$. Then

$$\det(RS) = \sum \det(B) \cdot \det(C)$$

where the sum is over all $p \times p$ submatrices B of R and C of S where B is made out of R in the same way as C is made out of S. And 'made in the same way' means: the numbers of the columns deleted from R to get B are the same as the numbers of the rows deleted from S to get C.

If we write the k^{th} column of R as R_k, the theorem is:

Theorem 2.6 (Binet-Cauchy) *Let R be a $p \times q$ matrix and S be a $q \times p$ matrix, where $p \leq q$. Then*

$$\det(RS) = \sum_{1 \leq k_1 < k_2 < \cdots < k_p \leq q} \det(R_{k_1}, R_{k_2}, \ldots, R_{k_p}) \cdot \det(S^T_{k_1}, S^T_{k_2}, \ldots, S^T_{k_p})$$

.

Proof: The element in row i and column j of RS is $\sum_{k=1}^{q} r_{ik}s_{kj}$, so the j^{th} column is $(RS)_j = \sum_{k=1}^{q} R_k s_{kj}$. So:

$$\begin{aligned}
\det(RS) &= \det(RS_1, RS_2, \ldots, RS_p) \\
&= \det(\sum_{k_1=1}^{q} R_{k_1}s_{k_1 1}, \sum_{k_2=1}^{q} R_{k_2}s_{k_2 2}, \ldots, \sum_{k_p=1}^{q} R_{k_p}s_{k_p p}) \\
&= \sum_{k_1=1}^{q} s_{k_1 1} \det(R_{k_1}, \sum_{k_2=1}^{q} R_{k_2}s_{k_2 2}, \ldots, \sum_{k_p=1}^{q} R_{k_p}s_{k_p p}) \qquad \text{(multilinearity)} \\
&= \sum_{k_1=1}^{q}\sum_{k_2=1}^{q} \cdots \sum_{k_p=1}^{q} s_{k_1 1}s_{k_2 2}\cdots s_{k_p p} \det(R_{k_1}, R_{k_2}, \ldots R_{k_p})
\end{aligned}$$

It is clear that $\det(R_{k_1}, R_{k_2}, \ldots R_{k_p}) = 0$ when $k_i = k_j$ for some $i \neq j$. Hence, we only have to sum over all different k_i's.

We get all choices by first choosing p numbers $k_1, \dots, k_p$ from $1, \dots, q$ with $k_1 < k_2 < \cdots < k_p$ and then permute these numbers by a permutation $\sigma \in S_p$. The sign of the permutation is $\operatorname{sgn}(\sigma)$. In general, we have

$$\det(B_{\sigma(1)}, B_{\sigma(2)}, \dots, B_{\sigma(n)}) = \operatorname{sgn}(\sigma) \det(B_1, B_2, \dots, B_n)$$

and (the Leibniz formula)

$$\begin{aligned}
\det(B) &= \det(B_1, \dots, B_n) \\
&= \det(b_{11}\mathbf{e}_1 + \cdots + b_{1n}\mathbf{e}_n, \dots, b_{n1}\mathbf{e}_1 + \cdots + b_{nn}\mathbf{e}_n) \\
&= \sum_{\sigma} b_{1\sigma(1)} \cdots b_{n\sigma(n)} \det(\mathbf{e}_{\sigma(1)}, \dots, \mathbf{e}_{\sigma(n)}) \\
&= \sum_{\sigma} \operatorname{sgn}(\sigma) b_{1\sigma(1)} \cdots b_{n\sigma(n)}
\end{aligned}$$

So, in our case, we get:

$$\begin{aligned}
\det(RS) &= \sum_{1 \le k_1 < k_2 < \cdots < k_p \le q} \sum_{\sigma \in S_p} s_{k_{\sigma(1)}1} s_{k_{\sigma(2)}2} \cdots s_{k_{\sigma(p)}p} \det(R_{k_{\sigma(1)}}, R_{k_{\sigma(2)}}, \dots, R_{k_{\sigma(p)}}) \\
&= \sum_{1 \le k_1 < k_2 < \cdots < k_p \le q} \sum_{\sigma \in S_p} s_{k_{\sigma(1)}1} s_{k_{\sigma(2)}2} \cdots s_{k_{\sigma(p)}p} \operatorname{sgn}(\sigma) \det(R_{k_1}, R_{k_2}, \dots, R_{k_p}) \\
&= \sum_{1 \le k_1 < k_2 < \cdots < k_p \le q} \det(R_{k_1}, R_{k_2}, \dots, R_{k_p}) \sum_{\sigma \in S_p} \operatorname{sgn}(\sigma) s_{k_{\sigma(1)}1} s_{k_{\sigma(2)}2} \cdots s_{k_{\sigma(p)}p} \\
&= \sum_{1 \le k_1 < k_2 < \cdots < k_p \le q} \det(R_{k_1}, R_{k_2}, \dots, R_{k_p}) \cdot \det(S^T_{k_1}, S^T_{k_2}, \dots, S^T_{k_p})
\end{aligned}$$

$\square$

3 Proof of the Matrix Tree Theorem

Now we have proved all the lemmas and theorems of section 2, the proof of the Matrix Tree Theorem is rather easy.

Theorem 3.1 (Matrix Tree Theorem) *If $\widetilde{A_G}$ is a reduced incidence matrix of the connected graph G, then the number of spanning trees of G equals the determinant of $\widetilde{A_G} \cdot \widetilde{A_G}^T$.*

Proof:

$$
\begin{aligned}
\det(\widetilde{A_G} \cdot \widetilde{A_G}^T) &= \sum \det(B) \cdot \det(B^T) && \text{(Binet-Cauchy)} \\
&= \sum \det(B)^2 \\
&= \sum_{\text{B non-singular}} \det(B)^2 + \sum_{\text{B singular}} \det(B)^2 \\
&= \sum_{\text{B non-singular}} \det(B)^2 \\
&= \sum_{\text{B non-singular}} 1 && \text{(Lemma 2.2.)} \\
&= \#\ \text{non-singular } (n-1) \times (n-1) \text{ submatrices } B \text{ of } \widetilde{A_G} \\
&= \#\ \text{spanning trees of } G. && \text{(Lemma 2.5.)}
\end{aligned}
$$

□

4 Implementation in magma

We can also let magma calculate the number of trees in a graph G:

```
numberoftrees := function(G)
    V := Vertices(G);
    E := Edges(G);
    v := #V;
    e := #E;
    A := Matrix(IntegerRing(), v, e, []);
    k := 1;
    for i in V do
        N := Neighbours(V!i);
        for j in N do
            A[Index(V,i), k] := -1;
            A[Index(V,j), k] := 1;
            k := k+1;
            RemoveEdge(~G,i,j);
        end for;
    end for;
    A2 := Submatrix(A,1,1,v-1,e);
    B := Transpose(A2);
    C := A2*B;
    n := Determinant(C);
    return n;
end function;
```

In this algorithm, A is the matrix A_G. We make this matrix by taking a vertex and then the edges that are adjacent to that vertex. We put those edges in the matrix (which is containing all zeros in the beginning) and then delete them from the graph. The direction is from the vertex we are looking at to the neighbour. If we deleted all the edges that are adjacent to the vertex, we take the next vertex.

5 Special formulas

There are some graphs for which there are direct formulas for the number of spanning subtrees. We will not always prove the formulas, because sometimes they are difficult and long.

5.1 Complete graph

Let G be K_n, the complete graph with n vertices. Then we get the $(n-1) \times (n-1)$-matrix

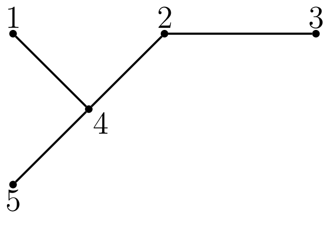
$$\widetilde{A_G} \cdot \widetilde{A_G}^T = \begin{pmatrix} n-1 & -1 & \dots & -1 \\ -1 & n-1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & -1 \\ -1 & \dots & -1 & n-1 \end{pmatrix}$$

The number of spanning subtrees of G is n^{n-2}.

The proof of this formula is easy and very surprising.

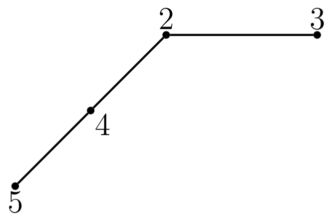
We all know how many different sequences of length $n-2$ whose elements are in $\{1, 2, \dots, n\}$ we can make: n^{n-2}.
With such a sequence we can make a tree, and vice versa. This, we will show with an example (in which we take $n = 5$).

From tree to sequence:
We take the tree



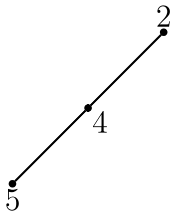
We look at the vertex with degree 1 and the smallest label. The first number of the sequence is the number of the neighbour of that vertex. So, in our tree, the first number of the sequence is the neighbour of vertex 1, which is 4.

Then we delete our vertex from the tree.

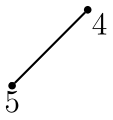


(4,

We repeat this steps until there are just 2 vertices left:



(4, 2



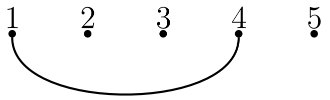
$(4, 2, 4)$

So our tree can be associated with the sequence $(4, 2, 4)$.

We will see that it is not necessary to put the last number, in our case a 5, in our sequence. With this sequence, the tree is defined.

From sequence to tree:
We found the sequence $(4, 2, 4)$. Now we will construct the tree.
Look at the vertex which number is the smallest number in $V = \{1, \ldots, 5\}$ that does not appear in our sequence. We connect this vertex with the first number of our sequence. Then we will remove the vertex from V and the first number of our sequence. We get:

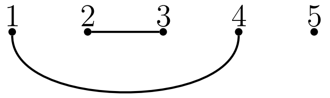
$\{\cancel{1}, 2, 3, 4, 5\}$



$(\cancel{4}, 2, 4)$

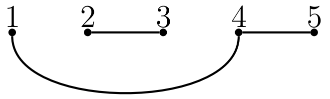
We repeat this steps until our sequence is empty.

$\{2, \cancel{3}, 4, 5\}$



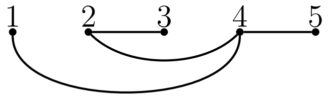
$\cancel{2}, 4)$

$\{2,4,\cancel{5}\}$



$\cancel{4})$

Finally, we connect the numbers left in V:



We get the same tree as we started with, so there is a one-to-one correspondence between the sequences and the trees.

As we have said before, we can make n^{n-2} different sequences. So we also can make n^{n-2} different trees.

This formula is called *Cayley's formula* and was found by Cayley in 1889.

5.2 Complete bipartite graph

Let G be $K_{m,n}$, the complete bipartite graph with $m+n$ vertices.

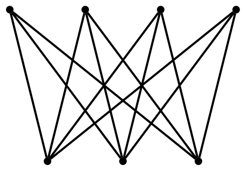


Figure 2: The complete bipartite graph $K_{4,3}$.

Then, if you number the vertices on the upper side 1 to m and the vertices on the lower side $m+1$ to $m+n$, we get

$$\widetilde{A_G}\cdot\widetilde{A_G}^T = \begin{pmatrix} nI_m & -1 \\ -1 & mI_{n-1} \end{pmatrix}$$

The number of spanning subtrees of G is $m^{n-1}n^{m-1}$ (which we won't prove).

5.3 Wheels

A wheel with n spokes, W_n, consists of an n-cycle and one additional vertex that is connected to all vertices of the cycle. For example:

Figure 3: W_4 and W_5.

Then, if $G = W_n$ and vertex nr. 1 is the point in the middle of the wheel, we get

$$\widetilde{A_G} \cdot \widetilde{A_G}^T = \begin{pmatrix} n-1 & -1 & -1 & \dots & \dots & -1 \\ -1 & 3 & -1 & 0 & \dots & 0 \\ -1 & -1 & 3 & \ddots & \ddots & \vdots \\ \vdots & 0 & \ddots & \ddots & -1 & 0 \\ \vdots & \vdots & \ddots & -1 & 3 & -1 \\ -1 & 0 & \dots & 0 & -1 & 3 \end{pmatrix}$$

The number of spanning trees in W_n is

$$\Big(\frac{3+\sqrt{5}}{2}\Big)^n + \Big(\frac{3-\sqrt{5}}{2}\Big)^n - 2.$$

(This formula we also won't prove.) So, for example, the number of trees in W_4 is 45 and in W_5 it is 121.

6 References

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