Some metrical observations on the approximation by continued fractions

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ABSTRACT

The following conjecture of H.W. Lenstra is proved. Denote by p_n/q_n, $n=1,2,\dots$ the sequence of continued fraction convergents of the irrational number x and define $\theta_n(x) := q_n|q_nx-p_n|$. Then for every z, $0\le z\le 1$, one has for almost all x

$$\lim_{n\to\infty}\frac{1}{n}\#\{j;\ j\le n, \theta_j(x)\le z\} = \begin{cases} \dfrac{z}{\log 2} & , 0\le z\le \frac{1}{2} \\ \dfrac{1}{\log 2}(-z+\log 2z+1), & \frac{1}{2}\le z\le 1 \end{cases}.$$

Similar results are proved for other functions connected with the regular continued fraction expansion, such as the quotient of

$$\left|x-\frac{p_{n-1}}{q_{n-1}}\right| \text{ and } \left|x-\frac{p_n}{q_n}\right|,$$

as well as for other type of expansions, such as the nearest integer and singular continued fractions. The main tool is the natural extension of the operator $x\mapsto(1/x)-[(1/x)]$, recently studied by Hitoshi Nakada.

1. INTRODUCTION

Let p_n/q_n, $n=1,2,\dots$ denote the convergents of the continued fraction expansion of the irrational number x, $0<x<1$. (The notation does not indicate the dependence of the integers p_n and q_n on x). An aspect of the rapid convergence of the rational numbers p_n/q_n to x as $n\to\infty$ is the well-known diophantine inequality

$$(1.1)\qquad \left|x-\frac{p_n}{q_n}\right|<\frac{1}{q_n^2}.$$

Thus, if the functions $\theta_n(x)$ are defined by

$$\left|x-\frac{p_n}{q_n}\right|=\frac{\theta_n(x)}{q_n^2},$$

then

$$0<\theta_n(x)<1.$$

The following conjecture regarding the distribution of the values of these functions $\theta_n(x)$ was made by H.W. Lenstra Jr. (oral communication):

For $0\le z\le 1$, let $A(n,x,z)$ denote the number of integers j with $1\le j\le n$ and $\theta_j(x)\le z$. Then for almost all x one has

$$(1.2)\qquad \lim_{n\to\infty}\frac{1}{n}A(n,x,z)=F(z)$$

where $F:[0,1]\to[0,1]$ is given by

$$(1.3)\qquad F(z)=\begin{cases}\dfrac{z}{\log 2} & 0\le z\le \frac{1}{2},\\[2ex] \dfrac{1}{\log 2}(-z+\log 2z+1) & \frac{1}{2}\le z\le 1.\end{cases}$$

An important first step towards proving Lenstra's conjecture was taken by D.E. Knuth who obtained the following result (not yet published when this paper was written):

If

$$E_n(z):=\{x;\ \theta_n(x)\le z\}$$

then

$$(1.4)\qquad m(E_n(z))=F(z)+0(g^n),\ g=\tfrac{1}{2}(\sqrt{5}-1)=0{,}61803\ldots,$$

where $m(E_n(z))$ is the Lebesgue measure of the set $E_n(z)$, and F is given by (1.3).

First we describe how Knuth's result (1.4) can be supplemented to obtain a complete proof of the conjecture: Let T be the operator defined by

$$Tx=\frac{1}{x}-\left[\frac{1}{x}\right],\quad 0<x<1,\ x\notin\mathbb{Q},$$

on which the whole machinery of the continued fraction expansion is based. Fundamental for the problem is the relation

$$(1.5)\qquad \theta_n(x)=\left(\frac{1}{T^n x}+\frac{q_{n-1}}{q_n}\right)^{-1}$$

see [7], p. 29, (11).

Using the ergodicity of T with respect to the Gauss measure, Birkhof's ergodic theorem and (1.5) one can prove that there does indeed exist a function F, satisfying (1.2). Once the *existence* of such an F is established, it is easy to derive (1.3) from (1.4). To this end we introduce the characteristic function $\chi_{n,z}$ of $E_n(z)$. Clearly

$$A(n,x,z)=\sum_{j\leq n}\chi_{j,z}(x)$$

and with the aid of Lebesgue's theorem on dominated convergence we find that

$$F(z)=\lim_{n\to\infty}\frac{1}{n}A(n,x,z)=\int_0^1\lim_{n\to\infty}\frac{1}{n}A(n,x,z)dx=$$

$$=\lim_{n\to\infty}\frac{1}{n}\sum_{j\leq n}\int_0^1\chi_{j,z}(x)dx=\lim_{n\to\infty}\frac{1}{n}\sum_{j\leq n}m(E_j(z)).$$

Hence (1.3) follows from (1.4) and from Cesàro's theorem. Finally, we remark that one can prove the first part of (1.3), i.e. the form of F for $0\leq z\leq\frac{1}{2}$, not using (1.4) but Legendre's theorem which says that if

$$\left|x-\frac{p}{q}\right|<\frac{1}{2q^2},$$

then p/q is a convergent of x, together with Paul Lévy's theorem on the growth of the denominators q_n, see (3.1), and the configuration of the Ford circles.

We shall not give the details of the above sketched reasoning which establishes the existence of a function F satisfying (1.2) since we will presently prove the conjecture by a different method, see section 2. This method also enables us to answer the question of the distribution of values for other functions connected with the continued fraction expansion. As the most interesting example we mention here the function $r_n(x)$, defined by

$$r_n(x):=\left|x-\frac{p_n}{q_n}\right|\left|x-\frac{p_{n-1}}{q_{n-1}}\right|^{-1}.$$

The function F that describes the distribution of the values of $r_n(x)$ turns out to be

$$F(z)=\frac{1}{\log 2}\left(\log(1+z)-\frac{z}{1+z}\log z\right),\quad 0\leq z\leq 1.$$

For details see section 3. In section 4 we consider the analogue of Lenstra's question for a whole class of continued fraction expansions, recently studied by Hitoshi Nakada, [8]. To mention one result here in the introduction, the analogue of (1.2) and (1.3) for the nearest integer continued fraction as well as for Hurwitz's singular continued fraction yields the following function F:

$$(1.6)\qquad F(z)=\begin{cases}\dfrac{z}{\log G}, & 0\le z\le 1-g\\[2ex] \dfrac{1}{\log G}(z-G^2z+\log G^2z+1), & 1-g\le z\le g.\\[2ex] 1, & g\le z\le 1\end{cases}$$

where here as throughout the rest of the paper, g and G are the constants defined by

$$g:=\tfrac{1}{2}(\sqrt{5}-1) \text{ and } G:=\tfrac{1}{2}(\sqrt{5}+1).$$

In section 4 we also discuss the question which expansion from the whole class of expansions of Nakada is, from a certain numerical point of view, the best.

For a concise formulation of our results we shall work with the notion of the *limiting distribution* of an arithmetical function $f:\mathbb{N}\to[0,1]$, see [4]. Denote by $A(n,z)$ the number of integers j with $1\le j\le n$ and $f(j)\le z$ and write $\nu_n(z)= =(1/n)A(n,z)$. Then ν_n is a distribution function on $[0,1]$ which means that $\nu_n(0)=0$, $\nu_n(1)=1$, ν_n is monotonically non-decreasing and right-continuous. Now if there exists a distribution function F on $[0,1]$ to which ν_n converges weakly as $n\to\infty$, i.e. if

$$\lim_{n\to\infty}\nu_n(z)=F(z)$$

for each point z in which F is continuous, then one says that the arithmetical function f has limiting distribution F.

2. PROOF OF LENSTRA'S CONJECTURE

THEOREM 1. *Let for every irrational number x, $0<x<1$, the arithmetical function $\theta(x)$ be defined by $\theta(x):n\mapsto\theta_n(x)$, where $\theta_n(x)$ is given by*

$$\left|x-\frac{p_n}{q_n}\right|=\frac{\theta_n(x)}{q_n^2}.$$

Then for almost all x, in the Lebesgue sense, the function $\theta(x)$ has limiting distribution F, where F is the function described in (1.3).

PROOF. Let $M\subset\mathbb{R}^2$ be defined by

$$M:=([0,1]-\mathbb{Q})\times[0,1]$$

and the operator $\mathcal{T}:M\to M$ by

$$\mathcal{T}(x,y)=\left(Tx,\frac{1}{a+y}\right)$$

where a denotes the first partial quotient of x, i.e. $a=[(1/x)]$. Hence, if x has the continued fraction expansion $x=[0;\, a_1, a_2, \dots]$, then

$$(2.1)\qquad \mathscr{T}^n(x,y)=(T^n x, [0;\, a_n, a_{n-1}, \dots, a_2, a_1+y]),\quad 0\le y\le 1,\ n=1,2,\dots.$$

Note that in particular

$$(2.2)\qquad \mathscr{T}^n(x,0)=\left(T^n x,\ \frac{q_{n-1}}{q_n}\right).$$

Finally, M is provided with the measure μ, defined by

$$\mu(E):=\frac{1}{\log 2}\iint_E \frac{dxdy}{(1+xy)^2},\quad E\in\mathscr{B},$$

where $\mathscr{B}$ is the notation for the class of all Borel subsets of M. For a proof of the fundamental result that

(2.3) $(M, \mathscr{B}, \mu, \mathscr{T})$ *is an ergodic system*

the reader is referred to [8] or to [3], p. 241.

Denote by $\Omega(c)$, $c\ge 1$, that part of M which lies on or above the hyperbola $(1/x)+y=c$. In view of (1.5) and (2.2), the condition

$$\theta_n(x)\le z,\quad 0<z\le 1,$$

is equivalent to

$$\mathscr{T}^n(x,0)\in\Omega\left(\frac{1}{z}\right).$$

Comparing (2.1) and (2.2) we see that for every $\varepsilon>0$ there exists an $n_0(\varepsilon)$ such that for all $n\ge n_0(\varepsilon)$ and all $y\in[0,1]$ we have

$$\mathscr{T}^n(x,y)\in\Omega\left(\frac{1}{z}+\varepsilon\right)\Rightarrow \mathscr{T}^n(x,0)\in\Omega\left(\frac{1}{z}\right)$$

and

$$\mathscr{T}^n(x,0)\in\Omega\left(\frac{1}{z}\right)\Rightarrow \mathscr{T}^n(x,y)\in\Omega\left(\frac{1}{z}-\varepsilon\right),$$

from which we infer that for almost all pairs (x,y), in the μ sense,

$$\lim_{n\to\infty}\frac{1}{n}\#\left\{j;\, j\le n,\, \mathscr{T}^j(x,y)\in\Omega\left(\frac{1}{z}+\varepsilon\right)\right\}\le \liminf_{n\to\infty}\frac{1}{n}\#\left\{j;\, j\le n,\, \mathscr{T}^j(x,0)\in\Omega\left(\frac{1}{z}\right)\right\}\le$$

$$\le \limsup_{n\to\infty}\frac{1}{n}\#\left\{j;\, j\le n,\, \mathscr{T}^j(x,0)\in\Omega\left(\frac{1}{z}\right)\right\}\le \lim_{n\to\infty}\frac{1}{n}\#\left\{j;\, j\le n,\, \mathscr{T}^j(x,y)\in\Omega\left(\frac{1}{z}-\varepsilon\right)\right\}$$

Both limits exist in view of (2.3) and the individual ergodic theorem and indeed, this last theorem says that they are respectively $\mu(\Omega((1/z)+\varepsilon))$ and $\mu(\Omega((1/z)-\varepsilon))$. Since this holds for every $\varepsilon>0$, the limit in (1.2) does exist for

almost all x and equals $\mu(\Omega(1/z))$. It is a straightforward calculation to show that $\mu(\Omega(1/z))$ equals the $F(z)$ from (1.3). This proves the theorem.
The convergents of the nearest integer continued fraction expansion of an irrational number x form a subsequence of the sequence p_n/q_n, $n=1,2,\ldots$ of convergents of the regular continued fraction expansion of x with, for almost all x, a density $(\log G/\log 2)=0{,}69424\ \ldots$. This was proved by Adams, [1], see also [6]. The corresponding question for Minkowski's diagonal expansion is now also answered. The convergents of this expansion also form a subsequence of the sequence p_n/q_n, $n=1,2,\ldots$, and they are characterised by $\theta_n(x)<\frac{1}{2}$, see [9], p. 177. Hence, by theorem 1 with $z=\frac{1}{2}$ we have

COROLLARY 1. *The convergents of Minkowski's diagonal expansion of an irrational number x form a subsequence of the sequence of convergents of the regular continued fraction expansion of x with, for almost all x, a density*

$$\frac{1}{2\log 2}=0{,}72134\ \ldots.$$

COROLLARY 2. *For almost all x one has*

$$\lim_{n\to\infty}\frac{1}{n}\sum_{j\leq n}\theta_j(x)=\frac{1}{4\log 2}=0{,}360673\ \ldots.$$

PROOF. If $\theta(x)$ has the limiting distribution F, the limit

$$\lim_{n\to\infty}\frac{1}{n}\sum_{j\leq n}\theta_j(x),$$

i.e. the first moment of $\theta(x)$, exists and equals $\int_0^1 z\,dF(z)$. This integral, with the F from (1.3) has the value

$$\frac{1}{4\log 2}.$$

3. THE LIMITING DISTRIBUTION OF SOME OTHER FUNCTIONS

In this section we shall see how our method for proving Lenstra's conjecture can also be used for finding the limiting distribution of other functions connected with the continued fraction expansion. We begin with considering the coordinates $T^n x$ and q_{n-1}/q_n in (2.2) separately. The method of section 2 yields at once that the $T^n x$ are for almost all x distributed according to the distribution law $\log(1+z)/\log 2$ but this is nothing else than the celebrated Gauss-Kuzmin-Lévy theorem, albeit without an error term. Because of the symmetry of the measure μ, the q_{n-1}/q_n are distributed in the same way. Hence

THEOREM 2. *Let for every irrational x,* $0<x<1$, *the arithmetical function* $Q(x)$ *be defined by*

$$Q(x): n\mapsto\frac{q_{n-1}(x)}{q_n(x)}.$$

Then for almost all x the function $Q(x)$ has limiting distribution F, where

$$F(z)=\frac{\log(1+z)}{\log 2}, \quad 0\le z\le 1.$$

Theorem 2 discloses the background of Paul Lévy's classical theorem on the growth of the denominators q_n, viz.

$$\lim_{n\to\infty}\frac{1}{n}\log q_n(x)=\frac{\pi^2}{12\log 2}, \text{ a.e.,} \tag{3.1}$$

to which we referred already in the introduction. For it follows that $\log Q(x)$ has the limiting distribution

$$\frac{\log(1+e^z)}{\log 2}, \quad -\infty<z\le 0,$$

and then, in turning to the first moment one finds

$$-\lim_{n\to\infty}\frac{1}{n}\log q_n(x)=\int_{-\infty}^{0} z d\,\frac{\log(1+e^z)}{\log 2}=-\frac{\pi^2}{12\log 2}, \text{ a.e.}$$

Borel introduced the notion of *adjacent fractions*, and showed that of the two pairs

$$\left(\frac{p_{n-1}}{q_{n-1}},\frac{p_n}{q_n}\right) \text{ and } \left(\frac{p_n}{q_n},\frac{p_{n+1}}{q_{n+1}}\right)$$

at least one is formed by two adjacent fractions. For details the reader is referred to [7], p. 33, 34. The fractions p_n/q_n and p_{n+1}/q_{n+1} are adjacent if and only if

$$\left(\frac{q_n}{q_{n-1}}+\frac{q_{n-1}}{q_n}\right)^{-1}<\frac{1}{\sqrt{5}},$$

that is, if and only if $(q_{n-1}/q_n)<g$. Thus theorem 2 gives us for almost all x the exact proportion of adjacent pairs in the sequence of convergents of x:

COROLLARY 3. *Let $A(n,x)$ denote the number of integers j with $1\le j\le n$ and such, that p_j/q_j and p_{j+1}/q_{j+1} are adjacent. Then*

$$\lim_{n\to\infty}\frac{1}{n}A(n,x)=\frac{\log G}{\log 2}, \text{ a.e.}$$

In the same way as theorem 2 lies behind (3.1), the next theorem can be regarded as the background of another well-known result from the metrical theory of continued fractions, viz.

$$\lim_{n\to\infty}\frac{1}{n}\log\left|x-\frac{p_n}{q_n}\right|=-\frac{\pi^2}{6\log 2} \text{ a.e.} \tag{3.2}$$

THEOREM 3. *Let for every irrational number x, $0<x<1$, the arithmetical function $r(x)$ be defined by*

$$r(x):n\mapsto \left|x-\frac{p_n}{q_n}\right|\left|x-\frac{p_{n-1}}{q_{n-1}}\right|^{-1}.$$

Then for almost all x the function $r(x)$ has limiting distribution F, where

$$F(z)=\frac{1}{\log 2}\left(\log (1+z)-\frac{z}{1+z}\log z\right),\quad 0\leq z\leq 1. \tag{3.3}$$

PROOF. It follows from

$$x-\frac{p_n}{q_n}=\frac{(-1)^n T^n x}{q_n(q_n+q_{n-1}T^n x)}, \tag{3.4}$$

see [2], p. 41 and 42, and from

$$\frac{1}{T^{n-1}x}=a_n+T^n x \tag{3.5}$$

that

$$r_n(x)=\frac{q_{n-1}}{q_n}\,T^n x.$$

Reasoning as in section 2 we see that the limiting distribution F exists for almost all x and that $F(z)$ equals the μ-measure of that part of M which lies under the hyperbola $xy=z$, $0<z\leq 1$. A simple calculation then shows that F is given by (3.3).

The limit in (3.2) equals the first moment of $\log r_n(x)$, an arithmetical function with limiting distribution

$$\frac{1}{\log 2}\left(\log (1+e^z)-\frac{z}{1+e^{-z}}\right),\quad -\infty<z\leq 0.$$

Hence

$$\lim_{n\to\infty}\frac{1}{n}\log\left|x-\frac{p_n}{q_n}\right|=\frac{1}{\log 2}\int_{-\infty}^{0} z\,d\left(\log (1+e^z)-\frac{z}{1+e^{-z}}\right)=-\frac{\pi^2}{6\log 2},\ \text{a.e.}$$

COROLLARY 4. *For almost all x one has*

$$\lim_{n\to\infty}\frac{1}{n}\sum_{j\leq n} r_j(x)=\frac{\pi^2}{12\log 2}-1=0{,}18656\ldots.$$

PROOF. The limit equals $\int_0^1 z\,dF(z)$, with the F from theorem 3.

Just as theorem 1 is based upon the inequality (1.1), our next theorem is based upon another well-known and useful inequality, viz.

$$\left|x-\frac{p_n}{q_n}\right|<\frac{1}{q_n q_{n+1}}.$$

THEOREM 4. *Let for every irrational number x, $0<x<1$, the arithmetical function $d(x)$ be defined by $d(x): n \mapsto d_n(x)$, where $d_n(x)$ is determined by*

$$\left|x-\frac{p_n}{q_n}\right|=\frac{d_n(x)}{q_n q_{n+1}}.$$

Then for almost all x the function $d(x)$ has limiting distribution F, with

$$F(z)=\begin{cases} 0 & , \quad 0\le z\le \frac{1}{2} \\ \frac{1}{\log 2}\,(z\log z+(1-z)\log(1-z)+\log 2), & \frac{1}{2}\le z\le 1 \end{cases}.$$

PROOF. From (3.4) and (3.5) with in the latter n replaced by $n+1$ it follows that

$$d_n(x)=\left(1+\frac{q_n}{q_{n+1}}\,T^{n+1}x\right)^{-1}.$$

Hence in this case $F(z)$ equals the μ-measure of that part of M which lies above the hyperbola $xy=(1/z)-1$. Note that for $z<\frac{1}{2}$ this is an empty set.

The last theorem of this section concerns the inequality

$$\left|x-\frac{p_n}{q_n}\right|<\frac{1}{2q_{n-1}q_n}. \tag{3.6}$$

This is a sharper estimate than (1.1) if and only if $a_n=1$, which happens for almost all x with a probability

$$2-\frac{\log 3}{\log 2}=0{,}41503\ldots,$$

see [2], p. 45. It is not ruled out beforehand that *in the mean* (3.6) is a better estimate than (1.1) but we shall presently see from corollary 5 that this is not the case.

THEOREM 5. *Let for every irrational number x, $0<x<1$, the arithmetical function $D(x)$ be defined by $n\mapsto D_n(x)$, where $D_n(x)$ is determined by*

$$\left|x-\frac{p_n}{q_n}\right|=\frac{D_n(x)}{2q_{n-1}q_n}.$$

Then for almost all x the function $D(x)$ has limiting distribution F, with

$$F(z)=\frac{1}{\log 2}\left(\log 2-\frac{z}{2}\log z-\frac{2-z}{2}\log(2-z)\right),\quad 0\le z\le 1.$$

PROOF. One has

$$D_n(x)=2\left(\frac{q_n}{q_{n-1}}\cdot\frac{1}{T^n x}+1\right)^{-1}$$

and therefore $F(z)$ equals the μ-measure of that part of M which lies under the hyperbola $xy = z/(2-z)$, $0 < z \leq 1$.

COROLLARY 5. *For almost all x one has*

$$\lim_{n\to\infty} \frac{1}{n} \sum_{j\leq n} D_j(x) = 1 - \frac{1}{2 \log 2} = 0{,}27865\ldots.$$

PROOF. With the F from theorem 5 the limit equals

$$\int_0^1 z dF(z) = 1 - \frac{1}{2 \log 2}.$$

Since the mean of $D(x)$ is for almost all x smaller than the mean of $\theta(x)$, compare with corollary 1, the estimate (1.1) is on the average better than the estimate (3.6). Fig. 1 depicts the graphs of the various distribution functions found so far. Each graph is marked with the number of the corresponding theorem.

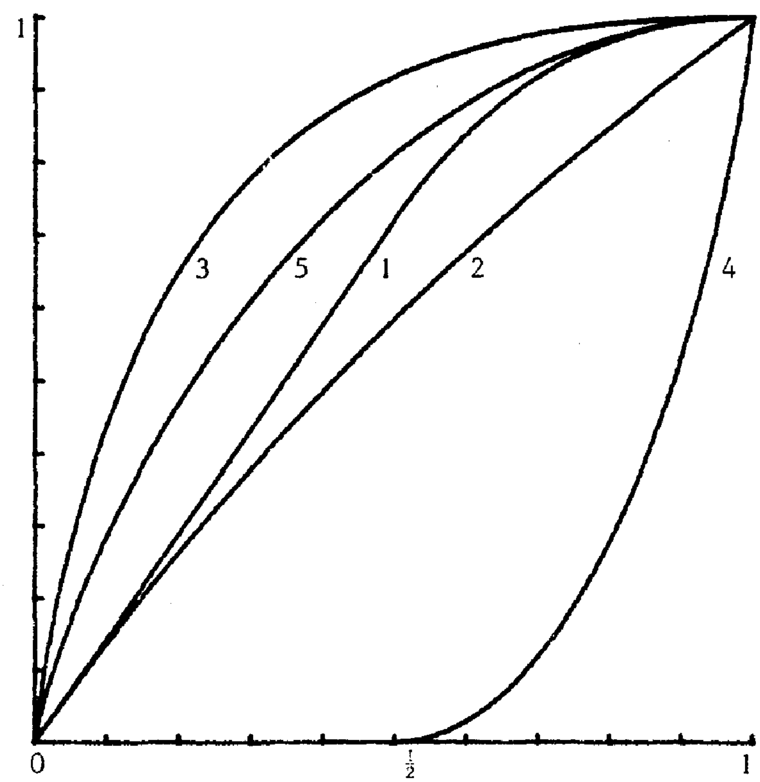

Fig. 1.

4. NAKADA'S α-EXPANSIONS

Recently, Nakada [8] introduced and studied a whole class of continued fraction expansions, based upon the operator $f_\alpha : [\alpha-1, \alpha] \to [\alpha-1, \alpha]$ with

$$(4.1) \qquad f_\alpha(x) := \left|\frac{1}{x}\right| - \left[\left|\frac{1}{x}\right| + 1 - \alpha\right], \quad x \neq 0.$$

Here and in the sequel we assume that $\frac{1}{2}\le\alpha\le 1$. For $\alpha=1$ one gets the regular continued fraction expansion with which we dealt until now. For $\alpha=\frac{1}{2}$ one obtains the nearest integer continued fraction expansion and for $\alpha=g$ the singular continued fraction expansion of Hurwitz, see [9], p. 166. In the same way as this is done for $\alpha=\frac{1}{2}$ or $\alpha=g$ one shows that the sequence of convergents $p_{\alpha,n}/q_{\alpha,n}$, $n=1,2,\ldots$ of the α-expansion forms a subsequence of the sequence p_n/q_n, $n=1,2,\ldots$ of the regular expansion. Hence

$$(4.2)\qquad \left|x-\frac{p_{\alpha,n}}{q_{\alpha,n}}\right|<\frac{1}{q_{\alpha,n}^2}.$$

In this section we shall study the distribution of the values of the functions $\theta_{\alpha,n}(x)$, defined by

$$(4.3)\qquad \left|x-\frac{p_{\alpha,n}}{q_{\alpha,n}}\right|=\frac{\theta_{\alpha,n}(x)}{q_{\alpha,n}^2},\quad x\notin\mathbb{Q},\ \tfrac{1}{2}\le\alpha\le 1.$$

Let $k_\alpha:\mathbb{N}\to\mathbb{N}$ be the arithmetical function for which

$$\frac{p_{\alpha,n}}{q_{\alpha,n}}=\frac{p_{k_\alpha(n)}}{q_{k_\alpha(n)}},\quad n=1,2,\ldots.$$

Just as it was proved in [6] that

$$\lim_{n\to\infty}\frac{k_{\frac{1}{2}}(n)}{n}=\frac{\log 2}{\log G},\ \text{a.e.,}$$

one proves that

$$(4.4)\qquad \lim_{n\to\infty}\frac{k_\alpha(n)}{n}=\begin{cases}\dfrac{\log 2}{\log G}, & \tfrac{1}{2}\le\alpha\le g,\ \text{a.e.,}\\[2ex] \dfrac{\log 2}{\log(1+\alpha)}, & g\le\alpha\le 1\ \text{a.e.,}\end{cases}$$

using theorem 1 and its corollary 1 from [8].

Another proof of (4.4) is contained in proposition 2 of [8]. The larger the limit in (4.4), the faster the subsequence of α-convergents runs through the sequence p_n/q_n, $n=1,2,\ldots$. For numerical purposes one should therefore choose an α with $\frac{1}{2}\le\alpha\le g$. Formula (4.4) also shows that in this respect there is no preference for any particular number α from this interval. We will now describe how there might however be a preference in other respects.

It follows from theorem 1 that

$$\sup_{n,x}\theta_n(x)=1,$$

where the supremum is taken over all $n\in\mathbb{N}$ and all x, $0<x<1$, $x\notin\mathbb{Q}$. An old result of Hurwitz's [5], see also [9], p. 173 states that

$$(4.5)\qquad \sup_{n,x}\theta_{\frac{1}{2},n}(x)=\sup_{n,x}\theta_{g,n}(x)=g.$$

This means that for $\alpha=\frac{1}{2}$ and $\alpha=g$, the inequality (4.2) can be sharpened with a factor g in the right hand side. Otherwise stated, picking one's way through the sequence p_n/q_n by expanding in the nearest integer continued fraction or in the singular continued fraction, terms p_n/q_n with $g\le\theta_n(x)<1$ are always omitted. These considerations naturally lead to the question whether there are α's for which this g can be replaced by a still smaller number. To be more precise, if we define the function $c:[\frac{1}{2},1]\to[0,1]$ by

$$(4.6) \qquad c(\alpha):=\sup_{n,x}\theta_{\alpha,n}(x)$$

with the $\theta_{\alpha,n}(x)$ from (4.2), we want to determine

$$\inf_{\alpha} c(\alpha)$$

and also the α_0 for which this infimum is attained. The next theorem shows that this α_0 lies in the interval $[\frac{1}{2},g]$ as is to be expected from (4.4).

THEOREM 6. *The function c, defined by* (4.6), *is given by*

$$c(\alpha)=\max\left\{G\,\frac{1-\alpha}{1+g\alpha},\ \alpha\right\},\quad \tfrac{1}{2}\le\alpha\le 1.$$

Hence

$$\min_{\alpha} c(\alpha)=c(\alpha_0)=\alpha_0$$

with

$$\alpha_0=\tfrac{1}{2}(-2-\sqrt{5}+(6\sqrt{5}+15)^{\frac{1}{2}})=0{,}54731\ldots.$$

The proof of this theorem will be incorporated in that of the next theorem which gives for all α in $[\frac{1}{2},1]$ and almost all x the limiting distribution of the $\theta_{\alpha,n}(x)$.

THEOREM 7. *Let the five functions* $z_i:[\frac{1}{2},1]\to\mathbb{R}$, $i=1,\ldots,5$ *be defined by*

$$z_1(\alpha):=\frac{\alpha}{1+g\alpha},$$

$$z_2(\alpha):=\alpha,$$

$$z_3(\alpha):=1-\alpha,$$

$$z_4(\alpha):=G\,\frac{1-\alpha}{1+g\alpha},$$

$$z_5(\alpha):=\frac{\alpha}{1+\alpha},$$

and for every α *with* $\frac{1}{2} \le \alpha \le 1$ *the five functions* $\psi_{\alpha,i}: [0,1] \to \mathbb{R}$, $i = 1, \ldots, 5$ *by*

$$\psi_{\alpha,i}(z) = \begin{cases} 0 & , \quad 0 \le z \le z_i(\alpha) \\ \dfrac{z}{z_i(\alpha)} - \log \dfrac{z}{z_i(\alpha)} - 1, & z_i(\alpha) \le z \le 1 \end{cases}.$$

Then the arithmetical function $\theta_\alpha(x)$, *defined by*

$$n \mapsto \theta_{\alpha,n}(x)$$

has for every $\alpha \in [\frac{1}{2}, g]$ *for almost all* x *the limiting distribution* F_α *where*

$$F_\alpha(z) = \frac{1}{\log G} (z - \psi_{\alpha,1}(z) + \psi_{\alpha,2}(z) - \psi_{\alpha,3}(z) + \psi_{\alpha,4}(z)), \quad 0 \le z \le 1,$$

and for every $\alpha \in [g, 1]$ *for almost all* x *the limiting distribution* F_α *where*

$$F_\alpha(z) = \frac{1}{\log (1+\alpha)} (z - \psi_{\alpha,5}(z) + \psi_{\alpha,2}(z)), \quad 0 \le z \le 1.$$

PROOF. First we deal with the case $\frac{1}{2} \le \alpha \le g$. Following Nakada [8] we define $M_\alpha \subset \mathbb{R}^2$ by

$$M_\alpha := \left(\left[\alpha - 1, \frac{1-2\alpha}{\alpha}\right] \times [0, 1-g)\right) \cup$$
$$\left(\left(\frac{1-2\alpha}{\alpha}, \frac{2\alpha-1}{1-\alpha}\right) \times [0, \tfrac{1}{2})\right) \cup$$
$$\left(\left[\frac{2\alpha-1}{1-\alpha}, \alpha\right] \times [0, g)\right)$$

and

$$\mathcal{T}_\alpha : M_\alpha \to M_\alpha$$

by

$$\mathcal{T}_\alpha(x, y) = \left(f_\alpha(x), \frac{1}{a + \varepsilon(x) y}\right)$$

where f_α is given by (4.1), $a = a_\alpha(x) := [|(1/x)| + 1 - \alpha]$ and $\varepsilon(x) := \operatorname{sgn} x$. Finally M_α is provided with the measure μ with density function

$$\frac{1}{\log G} \frac{1}{(1+xy)^2}.$$

Then

$$(M_\alpha, \mathcal{B}, \mu, \mathcal{T}_\alpha) \text{ is an ergodic system,}$$

see [8].

We split M_α into two parts, M_α^- and M_α^+, according to sgn $x=-1$ or sgn $x=+1$. In the same way as one proves (1.5) one shows that now

$$\theta_{\alpha,n}(x)=\left|\frac{1}{f_\alpha^n(x)}+\frac{q_{\alpha,n-1}}{q_{\alpha,n}}\right|^{-1}.$$

Hence $\theta_{\alpha,n}(x)\le z$ if and only if

$$\frac{1}{f_\alpha^n(x)}+\frac{q_{\alpha,n-1}}{q_{\alpha,n}}\ge\frac{1}{z} \text{ and } f_\alpha^n(x)>0$$

or

$$\frac{1}{f_\alpha^n(x)}+\frac{q_{\alpha,n-1}}{q_{\alpha,n}}\le-\frac{1}{z} \text{ and } f_\alpha^n(x)<0$$

Reasoning along the same lines as in the proof of theorem 1 we see that for almost all x the limiting distribution F_α exists and that

$$F_\alpha(z)=\mu\left(\Omega^+\left(\frac{1}{z}\right)\right)+\mu\left(\Omega^-\left(-\frac{1}{z}\right)\right),\quad 0\le z\le 1,$$

where $\Omega_\alpha^+((1/z))$ is that part of M_α^+ which lies *above* the hyperbola $(1/x)+y=$ $=(1/z)$ and $\Omega_\alpha^-(-(1/z))$ that part of M_α^- which lies *under* the hyperbola $(1/x)+y=-(1/z)$.

In

$$\text{(4.7)}\qquad \mu\left(\Omega_\alpha^-\left(-\frac{1}{z}\right)\right)=\frac{1}{\log G}\iint_{\Omega_\alpha^-(-(1/z))}\frac{dxdy}{(1+xy)^2}$$

we perform the substitution $U(x,y)=(-x/(x+1),1-y)$. Under this substitution the hyperbola $(1/x)+y=-(1/z)$ changes into the hyperbola $(1/x)+y=(1/z)$ and a simple calculation then shows that the right hand side of (4.7) equals the μ-measure of that part of $U(\Omega_\alpha^-(-(1/z)))$ which lies above the hyperbola $(1/x)+y=(1/z)$. Putting

$$M_\alpha^*:=M_\alpha^+\cup UM_\alpha^-$$

we thus see that $F_\alpha(z)$ equals the μ-measure of that part of M_α^* which lies above the hyperbola $(1/x)+y=(1/z)$.

Fig. 2 shows a picture of M_α^-, M_α^+ and UM_α^- with $\alpha=0{,}55$. The boundary of M_α^* is a polygon with the six vertices $P_1,\dots,P_6$, where

$$P_1=(\alpha,g),$$

$$P_2=(\alpha,0),$$

$$P_3=\left(\frac{1-\alpha}{\alpha},\ 1\right),$$

$$P_4=\left(\frac{1-\alpha}{\alpha},\ g\right),$$

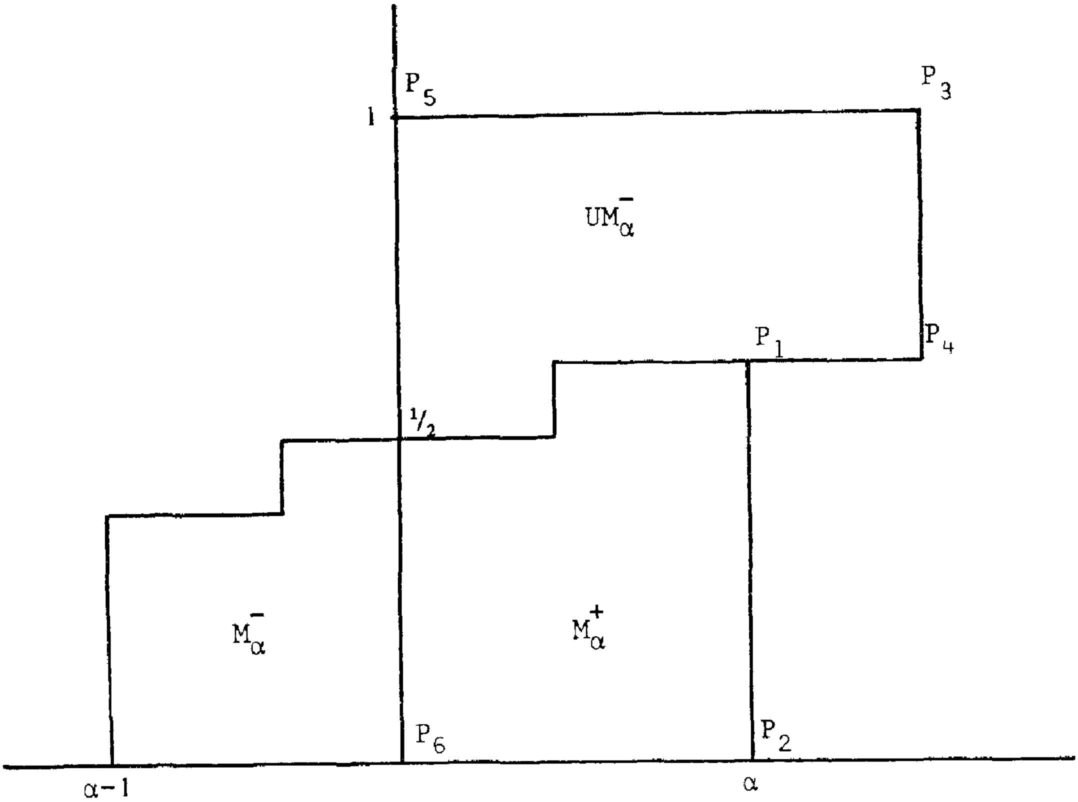

Fig. 2.

$$P_5 = (0, 1),$$

$$P_6 = (0, 0).$$

The points P_5 and P_6 will play no further rôle. Note that for $\alpha = g$ the points P_1 and P_4 coincide and do in fact not occur among the vertices.

The four functions z_i, $i = 1, \dots, 4$, from theorem 7 are defined in such a way that the hyperbola $(1/x) + y = (1/z_i(\alpha))$ passes through the point P_i, $i = 1, \dots, 4$. Fig. 3 shows the graphs of the functions z_i, $i = 1, \dots, 5$.

Let P_0 be a point with coordinates x_0 and y_0, $0 \le x_0 \le 1$, $0 \le y_0 \le 1$ and define z_0 by $(1/x_0) + y_0 = (1/z_0)$. Denote by A or by $A(P_0, z)$ that part of $\mathbb{R}^2$ which lies to the right of the line $x = x_0$, under the line $y = y_0$ and above the hyperbola $(1/x) + y = (1/z)$. Note that for $0 < z < z_0$ this set is empty. We calculate $\mu(A)$. For $z \ge z_0$

$$\mu(A(P_0, z)) = \frac{1}{\log G} \int_{x_0}^{(z^{-1}-y_0)^{-1}} \left(\int_{z^{-1}-x^{-1}}^{y_0} \frac{dy}{(1+xy)^2} \right) dx =$$

$$\frac{1}{\log G} \int_{x_0}^{(z^{-1}-y_0)^{-1}} \left[\frac{y}{1+xy} \right]_{y=z^{-1}-x^{-1}}^{y=y_0} dx =$$

$$\frac{1}{\log G} \int_{x_0}^{(z^{-1}-y_0)^{-1}} \left(\frac{y_0}{1+xy_0} - \frac{1}{x} + \frac{z}{x^2} \right) dx =$$

$$\frac{1}{\log G} \left[\log \left(\frac{1}{x} + y_0 \right) - \frac{z}{x} \right]_{x=x_0}^{x=(z^{-1}-y_0)^{-1}} = \frac{1}{\log G} \left(\frac{z}{z_0} - \log \frac{z}{z_0} - 1 \right).$$

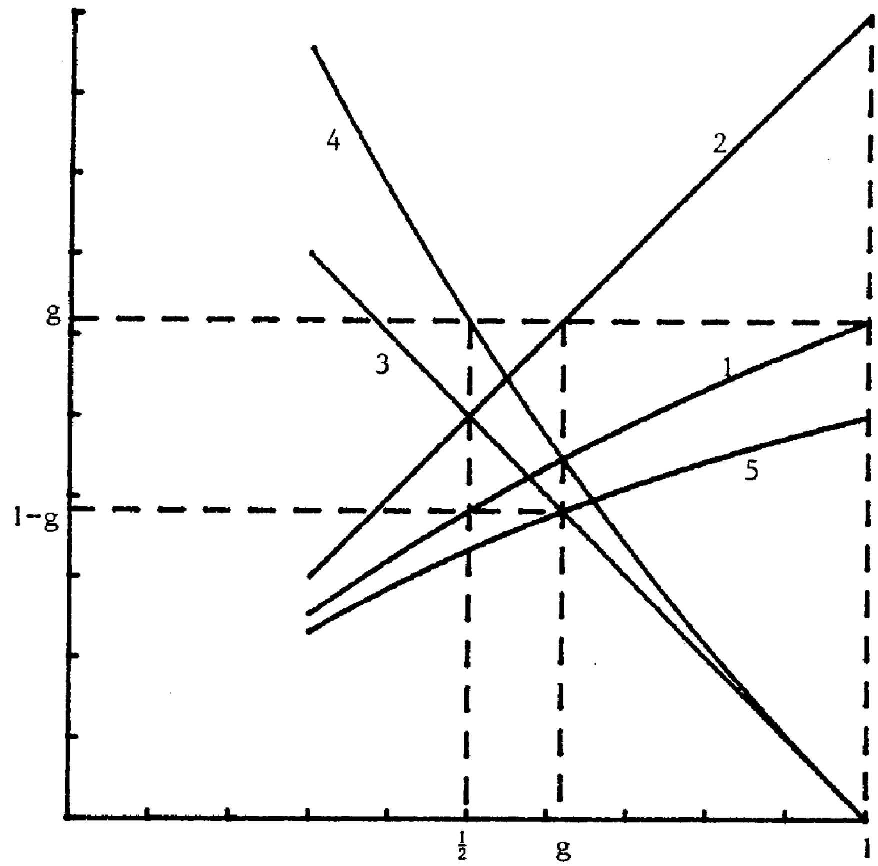

Fig. 3.

In particular

$$\mu(A(P_i, z)) = \frac{1}{\log G}\,\psi_{\alpha,i}(z),\quad i=1,\ldots,4,\quad 0\le z\le 1.$$

Denote by V the quadrangle with vertices P_6, $(z, 0)$, $Q=(z, 1)$ and P_5. Then one deduces easily from a picture that the μ-measure of that part of M_α* which lies above the hyperbola $(1/x)+y=(1/z)$, i.e. $F_\alpha(z)$, equals

$$\mu(V)+\mu(A(Q,z))-\frac{1}{\log G}\,(\psi_{\alpha,1}(z)-\psi_{\alpha,2}(z)+\psi_{\alpha,3}(z)-\psi_{\alpha,4}(z)),\quad 0\le z\le 1.$$

Thus, since

$$\mu(V)=\frac{\log\,(1+z)}{\log G},$$

theorem 7 follows, for $\frac{1}{2}\le\alpha\le g$.

We make some remarks. For

$$\max\,(z_1(\alpha), z_2(\alpha), z_3(\alpha), z_4(\alpha)) = \max\,(z_2(\alpha), z_4(\alpha))\le z\le 1$$

the above yields

$$1=F(z)=\frac{1}{\log G}\left(z\left(1-\frac{1}{z_1(\alpha)}+\frac{1}{z_2(\alpha)}-\frac{1}{z_3(\alpha)}+\frac{1}{z_4(\alpha)}\right)-\log\frac{z_2(\alpha)z_4(\alpha)}{z_1(\alpha)z_3(\alpha)}\right).$$

Therefore

$$\frac{1}{z_1(\alpha)}-\frac{1}{z_2(\alpha)}+\frac{1}{z_3(\alpha)}-\frac{1}{z_4(\alpha)}=1 \tag{4.8}$$

and

$$\frac{z_1(\alpha)z_3(\alpha)}{z_2(\alpha)z_4(\alpha)}=G$$

which can, of course, also be verified directly.

The functions $\psi_{\alpha,i}$, $i=1,\dots,5$ are continuously differentiable and hence, so is F_α.

The case $g\leq\alpha\leq 1$ is proved similarly. Here one starts with

$$M_\alpha:=\left(\left[\alpha-1,\ \frac{1-\alpha}{\alpha}\right]\times[0,\tfrac{1}{2})\right)\cup \left(\left(\frac{1-\alpha}{\alpha},\ \alpha\right)\times[0,1)\right)$$

and with the measure μ_α with density function

$$\frac{1}{\log\,(1+\alpha)}\,\frac{1}{(1+xy)^2},$$

see [8]. The rest of the proof of this case is then obvious. We observe that the above reasoning also proves immediately theorem 6.

We consider some special values of α. First $\alpha=1$. We have $z_5(1)=\frac{1}{2}$ and therefore $\psi_{1,5}(z)=2z-\log\,2z-1$, $\frac{1}{2}\leq z\leq 1$. Hence F_1 is the function F from (1.3), in accordance with theorem 1. Next $\alpha=\frac{1}{2}$. We have

$$z_1(\tfrac{1}{2})=1-g,\ z_2(\tfrac{1}{2})=z_3(\tfrac{1}{2})=\tfrac{1}{2},\ z_4(\tfrac{1}{2})=g$$

and hence

$$F_{\frac{1}{2}}(z)=\begin{cases}\dfrac{z}{\log G}, & 0\leq z\leq 1-g\\[2ex] \dfrac{1}{\log G}\left(z-\dfrac{z}{1-g}+\log\dfrac{z}{1-g}+1\right), & 1-g\leq z\leq g.\\[2ex] 1, & g\leq z\leq 1\end{cases}$$

This is the function F from (1.6).

Finally we take $\alpha=g$. We have $z_5(g)=g/(1+g)=1-g$ and thus we see that $F_{\frac{1}{2}}=F_g$.

Fig. 4 shows the graphs of the functions F_1, $F_{0,8}$, $F_{\frac{1}{2}}$ and F_{α_0}, with the α_0 from theorem 6.

For $\frac{1}{2}\leq\alpha\leq g$ the right hand side of (4.2) has in view of (3.1) and (4.4) the same asymptotic behaviour. From (4.3) we might for these values of α then

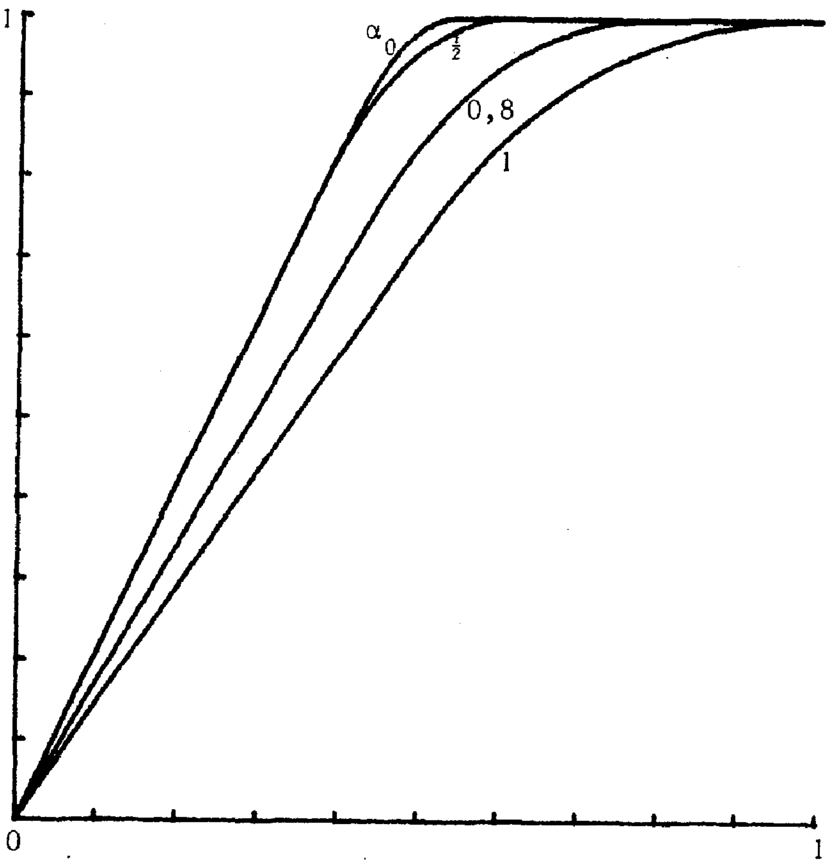

Fig. 4.

deduce a preference for that α for which the numerator $\theta_{\alpha,n}(x)$ is, in the mean, as small as possible. To this end we define the function $M:[\frac{1}{2},g]\to[0,1]$ by

$$(4.9)\qquad M(\alpha):=\lim_{n\to\infty}\frac{1}{n}\sum_{j\le n}\theta_{\alpha,j}(x),\ \text{a.e.}$$

We have

THEOREM 8. *For the function M defined in* (4.9) *one has*

$$M(\alpha)=\frac{1}{\log G}\,\frac{\alpha^2-\alpha+\frac{1}{2}}{\alpha+G},\quad \tfrac{1}{2}\le\alpha\le g.$$

Hence

$$\min_{\alpha} M(\alpha)=M(\alpha_1)=\frac{1}{\log G}\,((8G+6)^{\frac{1}{2}}-2G-1)=0{,}241959\ \ldots$$

with

$$\alpha_1=-G+(2G+\tfrac{3}{2})^{\frac{1}{2}}=0{,}55821\ \ldots.$$

PROOF. By definition $M(\alpha)$ is the first moment of F_α and therefore $M(\alpha)=$ $=\int_0^1 z\,dF_\alpha(z)$. Substituting the expression for $F_\alpha(z)$ found in theorem 7 and

using (4.8) one arrives easily at the form for $M(\alpha)$ given above. The rest of the proof then is a straightforward calculation.
The mean of the θ's in the case of the nearest integer continued fraction equals

$$M(\tfrac{1}{2}) = \frac{1}{\log G}\,(\tfrac{1}{2}\sqrt{5}-1) = 0{,}24528\ \ldots$$

which is already very close to the minimum value found in theorem 8. Comparing these values with corollary 2 we see that we have gained almost a factor $\frac{2}{3}$ on the regular continued fraction.

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