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Optimal continued fractions

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ABSTRACT

A new continued fraction algorithm is given and analyzed. It yields approximations for an irrational real number by generating a subsequence of its regular continued fraction convergents that is optimal in several respects.

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0. INTRODUCTION

This paper is concerned with the approximation of real irrational numbers by rational numbers via continued fractions. Every semi-regular continued fraction expansion of an irrational number x determines a sequence $\{p_k/q_k\}_{k\geq 1}$ of increasingly good rational approximations, the convergents. A *fastest* expansion of x is an expansion for which the growth rate of the denominators q_k is maximal; it turns out that this means that these denominators grow asymptotically as fast as the denominators of the nearest integer continued fraction (NICF) convergents of that x (see section 3). *Closest* expansions are those for which $\sup\{\theta_k : \theta_k = q_k|q_k x - p_k|\}$ is minimal; every x admits an expansion for which $\theta_k < 1/2$ (for every $k \geq 1$), given by Minkowski's diagonal continued fraction (DCF). Since in general the NICF does not provide closest expansions and the DCF does not provide fastest expansions, one wonders whether there exist for every irrational number x expansions that give both closest and fastest approximation. In [Keller] it was shown that such semi-regular expansions do indeed always exist; in [Selenius] it was shown how such an expansion can be obtained once the regular continued fraction (RCF) expansion is known. In section 4 we present an algorithm to compute the optimal continued fraction (OCF) expansion of x, without using its regular expansion, and we show that it is guaranteed to be both a fastest and a closest expansion. This algorithm arises in a natural way from the geometrical interpretation of the RCF-algorithm. Some other interesting properties of OCF-expansions are derived in section 4; for instance, the smallest constant c such that $\min \{\theta_k, \theta_{k+1}\} < c$ for every x is minimal (namely $1/\sqrt{5}$) for the OCF.

The definition of the optimal continued fractions given in this article enables one to develop a metrical theory for the OCF; it turns out that the distribution of the θ_k associated to the OCF is also in several respects optimal. The metrical properties of the OCF will be given in a subsequent article (see [Bosma, Kraaikamp]).

1. SEMI-REGULAR CONTINUED FRACTIONS

(1.1) DEFINITION. A *semi-regular continued fraction* (SRCF) is a finite or infinite fraction

$$b_0 + \cfrac{\varepsilon_1}{b_1 + \cfrac{\varepsilon_2}{b_2 + \cfrac{\varepsilon_3}{b_3 + \ddots}}}$$

with $\varepsilon_n = \pm 1$, $b_0 \in \mathbb{Z}$ and $b_n \in \mathbb{Z}_{\geq 1}$ for $n \geq 1$, subject to the condition

(1.2) $\quad \varepsilon_{n+1} + b_n \geq 1$ for $n \geq 1$,

and with the restriction that in the infinite case

(1.3) $\quad \varepsilon_{n+1} + b_n \geq 2$ infinitely often;

moreover we demand (see Remark (1.12))

(1.4) $\quad \varepsilon_n + b_n \geq 1$ for $n \geq 1$.

In the sequel we will mainly be interested in infinite expansions of this kind. We will recall some basic facts about SRCF's that can all be found in [Perron] Ch. V.

Every infinite SRCF with $b_0 = 0$ determines a unique irrational real number x, $-1 < x < 1$, and conversely it can be shown that for every infinite sequence $\varepsilon_1, \varepsilon_2, \varepsilon_3, \ldots$ ($\varepsilon_n = \pm 1$) there is a unique expansion of the form (1.1) with $b_0 = 0$ for any irrational x, provided that $0 < \varepsilon_1 x < 1$. We will denote this expansion by $x = [0; \varepsilon_1 b_1, \varepsilon_2 b_2, \ldots]$.

(1.5) REMARKS. Various of the results on continued fractions for irrationals in this paper hold also for the finite expansions of rational numbers; it requires disproportionate circumstantiallity however, to formulate the results if one allows the expansion to break off, especially since ambiguities arise here. Therefore we will usually assume $x \in \Omega$, denoting the irrational numbers by Ω, though for sake of simplicity some of the examples will concern rational numbers. Often we will implicitly assume $b_0 = 0$.

Given $\varepsilon_1, \varepsilon_2, \varepsilon_3, \ldots$ the sequence of partial quotients $b_1, b_2, b_3, \ldots$ for $x \in \Omega \cap \cap [-1, 1]$ can be computed as follows. Let for $k \geq 1$ the operator $U^k : (-1, 1) \to \to (-1, 1)$ recursively be given by

$$(1.6) \qquad U^k x = |U^{k-1}x|^{-1} - [|U^{k-1}x|^{-1}] + \tfrac{1}{2}(\varepsilon_{k+1} - 1)$$

then one finds b_k from

$$(1.7) \qquad b_k = [|U^{k-1}x|^{-1}] - \tfrac{1}{2}(\varepsilon_{k+1} - 1).$$

The *convergents* (or approximants) of some SRCF-expansion of an irrational x are the primitive fractions p_n/q_n defined for $n \geq 1$ as the finite truncations:

$$\cfrac{\varepsilon_1}{b_1 + \cfrac{\varepsilon_2}{b_2 + \cfrac{\varepsilon_3}{b_3 + \ddots + \cfrac{\varepsilon_n}{b_n}}}}$$

which can be found recursively from

$$(1.8) \qquad \begin{cases} p_{-1} = 1, p_0 = 0 & p_n = b_n p_{n-1} + \varepsilon_n p_{n-2}, \\ q_{-1} = 0, q_0 = 1 & q_n = b_n q_{n-1} + \varepsilon_n q_{n-2}. \end{cases}$$

The *regular continued fraction expansion* (denoted RCF) is the SRCF for which $1 = \varepsilon_1 = \varepsilon_2 = \ldots$. We will denote data for RCF-expansions in the capitals P_n, Q_n and B_n.

From the regular expansions we will also need the *secondary convergents* $\frac{P_{k,i}}{Q_{k,i}}$, $(1 \leq i \leq B_k - 1)$ given by

$$(1.9) \qquad \begin{cases} P_{k,i} = iP_{k-1} + P_{k-2}, \\ Q_{k,i} = iQ_{k-1} + Q_{k-2}. \end{cases}$$

Notice that $P_k = P_{k, B_k - 1} + P_{k-1}$ etc., and that

$$(1.10) \qquad Q_{k-1} < Q_{k,1} < \ldots < Q_{k, B_k - 1} < Q_k \text{ for } k \geq 1.$$

The following lemma shows why these secondary convergents are of importance.

(1.11) LEMMA. *Every semi-regular convergent to any x is either a primary or a secondary regular convergent to that x.*

PROOF. See e.g. [Tietze].

(1.12) REMARK. In the definition of semi-regular continued fractions, one could do without restriction (1.4), which prohibits $\varepsilon_n b_n = -1$, as is done in [Perron]. But a convenient consequence of our convention is that the denominators of the convergents increase monotonically:

$$\forall n \geq 2 : q_n > q_{n-1}$$

as can be seen from (1.8) (compare [Perron] p. 158). That (1.4) can be imposed without serious loss of generality can be verified using the equality

$$[0; \varepsilon_1 b_1, \varepsilon_2 b_2, \ldots, \varepsilon_{n-1} b_{n-1}, -1, b_{n+1}, \ldots]$$
$$= [0; \varepsilon_1 b_1, \varepsilon_2 b_2, \ldots, \varepsilon_{n-1}(b_{n-1} - 1), b_{n+1} + 1, \ldots].$$

Next we describe a transformation that turns regular expansions into semi-regular ones, via what some authors call a *singularization* process. It is based on the equality

$$A + \cfrac{1}{1 + \cfrac{1}{B + x}} = A + 1 - \frac{1}{B + 1 + x}$$

which implies that in any SRCF-expansion we may replace

$$[\ldots, \varepsilon_{k-1} b_{k-1}, 1, b_{k+1}, \ldots] \text{ by } [\ldots, \varepsilon_{k-1}(b_{k-1} + 1), -(b_{k+1} + 1), \ldots],$$

obtaining thus another SRCF-expansion for the same x. We say that we have *singularized* some b_k equal to 1 if we apply the above operation to it.

(1.13) EXAMPLE. Take $x = 11/29$; it has the regular expansion

$$x = [0; 2, 1, 1, 1, 3].$$

Singularizing the "middle 1" we get the semiregular $x = [0; 2, 2, -2, 3]$, but successively singularizing the other ones yields $x = [0; 2, 1, 2, -4] = [0; 3, -3, -4]$, or in reversed order of singularization $x = [0; 2, -2, 1, 3] = [0; 3, -3, -4]$.

(1.14) REMARKS. (i) We see immediately from the above that never two consecutive 1's can be singularized. We also notice that every singularization step reduces the "length" of the expansion. See also section 3.

(ii) Of course one can invert the above transformation to obtain regular continued fractions out of semiregular ones; in fact one could prove (1.11) directly in this way.

(iii) Noti⌄ that by using this singularization process only, one always gets semi-regular expansions for which every convergent is a primary regular convergent (but sometimes one of these is skipped); we will denote this property by $\mathrm{SRCF}(x) \subseteq \mathrm{RCF}(x)$.

So far we have encountered SRCF-expansions either with pregiven sequence $\varepsilon_1, \varepsilon_2 \dots$ or arising from singularization of RCF-expansions. In 1907 in [McKinney] an infinitude of SRCF-operators was defined that determine for every (irrational) x the infinite sequence $\varepsilon_1, \varepsilon_2 \dots$ as well as $b_1, b_2, \dots$; more recently these were studied in the following form (e.g. in [Nakada], [Nakada, Ito, Tanaka] and [Bosma, Jager, Wiedijk]).

(1.15) DEFINITION

Let $\frac{1}{2} \leq \alpha \leq 1$.

The operator $T_\alpha : [\alpha - 1, \alpha) \to [\alpha - 1, \alpha)$ is defined by

$$T_\alpha(x) = |x|^{-1} - [|x|^{-1} + 1 - \alpha]$$

and for $k \geq 1$ we put for $x \in [\alpha - 1, \alpha) \cap \Omega$:

$$b_{\alpha,k}(x) = [|T_\alpha^{k-1}(x)|^{-1} + 1 - \alpha]$$

and

$$\varepsilon_{\alpha,k} = \operatorname{sgn}\, T_\alpha^{k-1}(x) \text{ where we put } T_\alpha^0(x) = x.$$

As usual the convergents can be found from

$$p_{\alpha,-1} = 1, \quad p_{\alpha,0} = 0, \quad p_{\alpha,k} = b_{\alpha,k} p_{\alpha,k-1} + \varepsilon_{\alpha,k} p_{\alpha,k-2}$$
$$q_{\alpha,-1} = 0, \quad q_{\alpha,0} = 1, \quad q_{\alpha,k} = b_{\alpha,k} q_{\alpha,k-1} + \varepsilon_{\alpha,k} q_{\alpha,k-2}.$$

(1.16) REMARKS. (i) Notice that with $\alpha = 1$ the RCF-algorithm is found from (1.15) and that $\alpha = 1/2$ gives the *nearest integer continued fraction* (NICF), cf. [Perron] § 39, [Hurwitz-1], [Minnigerode]. The nearest integer convergents will be denoted from now on by R_k/S_k.

(ii) It is not very hard to show that (1.15) always defines a semi-regular expansion; moreover for these expansions $\mathrm{SRCF}(x) \subseteq \mathrm{RCF}(x)$ for every x, which can for instance be seen using the following criterion (cf. [Perron] § 40).

(1.17) PROPOSITION. Let $x \in \Omega$ have SRCF-expansion $[0; \varepsilon_1 b_1, \varepsilon_2 b_2, \dots]$. Then:

$$\mathrm{SRCF}(x) \subseteq \mathrm{RCF}(x) \Leftrightarrow \forall k \geq 1 : \varepsilon_k + \varepsilon_{k+1} > 2 - 2b_k.$$

2. APPROXIMATION

Every SRCF-expansion of an irrational number x gives an infinite sequence of increasingly good approximations to x, determined by the convergents p_k/q_k:

(2.1) $$\forall x \in \Omega, k \geq 0 \exists m \geq 1 : \left| x - \frac{p_{k+m}}{q_{k+m}} \right| < \left| x - \frac{p_k}{q_k} \right|.$$

In particular the approximation of x by its regular convergents has been studied thoroughly; in this case – and a fortiori for every expansion $\mathrm{SRCF}(x) \subseteq \mathrm{RCF}(x)$ – (2.1) holds with $m=1$. We briefly list the main properties (for proofs see e.g. [Perron], [Venkov]).

(2.2) THEOREM

$$\forall x \in \Omega, k \geq 0: \left| x - \frac{P_k}{Q_k} \right| < \frac{1}{Q_k Q_{k+1}}.$$

(2.3) COROLLARY

$$\forall x \in \Omega, k \geq 0: \left| x - \frac{P_k}{Q_k} \right| < \frac{1}{Q_k^2}.$$

(2.4) DEFINITION. For every $x \in \Omega \cap (-1,1)$ and every semi-regular continued fraction we define the sequence $\{\theta_k\}_{k\geq 1}$ by

$$\theta_k = \theta_k(x) = q_k |q_k x - p_k|,$$

with the p_k, q_k from (1.8). In particular for the regular continued fraction we write

$$\Theta_k = \Theta_k(x) = Q_k |Q_k x - P_k|.$$

(2.5) COROLLARY

$$\forall x \in \Omega, k \geq 0: \Theta_k < \frac{1}{B_{k+1}}.$$

PROOF. $\Theta_k = Q_k^2 \left| x - \frac{P_k}{Q_k} \right| < \frac{Q_k}{Q_{k+1}}$ by (2.2) and $Q_{k+1} = B_{k+1} Q_k + Q_{k-1} \geq B_{k+1} Q_k$.
In particular $\Theta_k < 1$ $(k \geq 0)$, and since for semi-regular expansions with $\mathrm{SRCF}(x) \subseteq \mathrm{RCF}(x)$ clearly $\{\theta_k\}_{k\geq 1} \subseteq \{\Theta_k\}_{k \geq 1}$, we also find $\theta_k < 1$ for these expansions.

Next we mention that every regular convergent is always a *best approximation* to x, that is there do not exist better approximations with smaller denominators:

(2.6) $$\forall r, s \in \mathbb{Z}, 0 < s \leq Q_k: \left| x - \frac{r}{s} \right| \leq \left| x - \frac{P_k}{Q_k} \right| \Rightarrow \frac{r}{s} = \frac{P_k}{Q_k}.$$

The converse does not hold: there exist secondary convergents which are also best approximations (cf. [Perron] p. 60).

The primary regular convergents do provide all *relative minima* to the linear form $Qx - P$, as is expressed by the following lemma, sharpening both (2.6) and (2.1) for the RCF (see [Venkov] p. 47).

(2.7) LEMMA

(i) $\forall x \in \Omega, k \geq 1: |Q_k x - P_k| < |Q_{k-1} x - P_{k-1}|.$

(ii) *$\forall x \in \Omega, k \geq 1$ and $r, s \in \mathbb{Z}$ with $0 < s \leq Q_k$ we have:*

$$|sx - r| \leq |Q_k x - P_k| \Rightarrow \frac{r}{s} = \frac{P_k}{Q_k}.$$

(iii) $\forall x \in \Omega, \dfrac{P}{Q} \in \mathbb{Q}$:

If for every $r, s \in \mathbb{Z}$ with $0 < s \leq Q : |sx - r| \leq |Qx - P| \Rightarrow \dfrac{r}{s} = \dfrac{P}{Q}$

then $\dfrac{P}{Q} = \dfrac{P_k}{Q_k}$ for some $k \geq 1$.

Now we mention some properties of the sequences $\{\Theta_k\}_{k \geq 1}$.

(2.8) THEOREM. [Vahlen]

$$\forall x \in \Omega, k \geq 1 : \min\{\Theta_k(x), \Theta_{k+1}(x)\} < \tfrac{1}{2}.$$

This theorem reveals only part of what is known about the two-dimensional distribution of the sequence $(\Theta_k(x), \Theta_{k+1}(x))$ $k = 1, 2, \ldots$; it was known that the sharper $\Theta_k + \Theta_{k+1} < 1$ holds (see [Brauer, Macon]), and more recently for almost all x the two-dimensional limiting distribution was given in [Jager-2].

Furthermore there are several results concerning three or even more consecutive Θ's; for future reference we list three of them in the following theorem. The first is a generalization of Borel's theorem (see [Borel]) and the third is known as Fujiwara's theorem; for a generalization of this see [Kopetzky, Schnitzer].

(2.9) THEOREM. $\forall x \in \Omega, k \geq 1$:

(i) $\min\{\Theta_k(x), \Theta_{k+1}(x), \Theta_{k+2}(x)\} < \dfrac{1}{\sqrt{(B_{k+2}^2 + 4)}}$.

(ii) $\max\{\Theta_k(x), \Theta_{k+1}(x), \Theta_{k+2}(x)\} > \dfrac{1}{\sqrt{(B_{k+2}^2 + 4)}}$.

(iii) *If $B_{k+2} > 1$ then: either $\Theta_{k+1} < \frac{2}{5}$*
or both $\Theta_k < \frac{2}{5}$ and $\Theta_{k+2} < \frac{2}{5}$.

PROOF

(i) See [Bagemihl, McLaughlin].
(ii) See [Tong].
(iii) See [Fujiwara], [Koksma] p. 34–35.

In terms of Θ we have the following "converse" to (2.3).

(2.10) THEOREM. [Legendre]
Let $x \in \Omega$ and $P, Q \in \mathbb{Z}$ with $Q > 0$ such that $\Theta = Q|Qx - P| < \frac{1}{2}$. Then

$\dfrac{P}{Q} = \dfrac{P_k}{Q_k}$ for some $k \geq 1$.

Irrational numbers do not admit infinitely many arbitrary close rational approximations; here we only mention Hurwitz' theorem (see [Perron] p. 49, [Hurwitz-2]).

(2.11) THEOREM. *For every $\theta < 1/\sqrt{5}$ there exist irrational numbers x such that only finitely many rational solutions p/q to the equation*

$$\left| x - \frac{p}{q} \right| < \frac{\theta}{q^2}$$

exist.

For the RCF the upperbound $\Theta_k < 1$ of (2.4) is best possible in the sense that

(2.12) $$\sup_{k,x} \Theta_k(x) = 1, \text{ so } \forall \Theta < 1 \ \exists x, k : Q_k |Q_k x - P_k| > \Theta.$$

However, for other SRCF-expansions the corresponding supremum of the $\theta_k(x)$ may be significantly smaller. In [Hurwitz-1] for instance it was shown that for the NICF it attains the value $g = (\sqrt{5} - 1)/2$. Since for every x, $\mathrm{NICF}(x) \subseteq \mathrm{RCF}(x)$ we might say that, taking the θ's as yardstick, the NICF picks systematically the better approximations out of the sequence of regular convergents.

We can find more generally $\sup \theta_{\alpha,n}(x)$ for the α-expansions of (1.15) using the following theorem. It gives the distribution of $\theta_{\alpha,n}(x)$ for almost all x and is proved as Theorem 7 in [Bosma, Jager, Wiedijk]. See also [Bosma, Kraaikamp].

As before $g = \frac{1}{2}(\sqrt{5} - 1)$ and throughout the rest of the paper we denote $G = g + 1 = \frac{1}{2}(\sqrt{5} + 1)$.

(2.13) THEOREM. *Let $\frac{1}{2} \leq \alpha \leq 1$.*

Define for $i = 1, \ldots, 5$ the functions $z_i : [\frac{1}{2}, 1] \to \mathbb{R}$ by

$$z_1(\alpha) = \frac{\alpha}{1 + g\alpha},\ z_2(\alpha) = \alpha,\ z_3(\alpha) = 1 - \alpha,\ z_4(\alpha) = G\,\frac{1-\alpha}{1+g\alpha},\ z_5(\alpha) = \frac{\alpha}{1+\alpha}$$

and $\Psi_{\alpha,i}(z) : [0,1] \to \mathbb{R}$ by

$$\Psi_{\alpha,i}(z) = \begin{cases} 0 & \text{for } 0 \leq z \leq z_i(\alpha) \\ \dfrac{z}{z_i(\alpha)} - \log \dfrac{z}{z_i(\alpha)} - 1 & \text{for } z_i(\alpha) \leq z \leq 1. \end{cases}$$

For $0 \leq z \leq 1$ let $A_\alpha(k, x, z)$ denote the number of integers j with $1 \leq j \leq k$ and $\theta_{\alpha,j}(x) \leq z$. Then for almost all x we have

$$\lim_{k \to \infty} \frac{1}{k} A_\alpha(k, x, z) = F_\alpha(z)$$

where $F_\alpha : [0,1] \to [0,1]$ *is given by*

$$(2.14)\qquad F_\alpha(z) = \begin{cases} \dfrac{1}{\log G}\{z - \Psi_{\alpha,1}(z) + \Psi_{\alpha,2}(z) - \Psi_{\alpha,3}(z) + \Psi_{\alpha,4}(z)\} & \text{if } \alpha \in [\tfrac{1}{2}, g] \\[2ex] \dfrac{1}{\log(1+\alpha)}\{z - \Psi_{\alpha,1}(z) + \Psi_{\alpha,2}(z)\} & \text{if } \alpha \in [g, 1]. \end{cases}$$

(2.15) COROLLARY. *Let* $\frac{1}{2} \le \alpha \le 1$. *Then*

$$\sup_{n,x} \theta_{\alpha,n}(x) = \sup_{n,x} \{q_{\alpha,n}|q_{\alpha,n}x - p_{\alpha,n}|\} = \max\left\{G\frac{1-\alpha}{1+g\alpha}, \alpha\right\}.$$

(2.16) COROLLARY.

$$\min_{\alpha} \sup_{n,x} \theta_{\alpha,n}(x) = \alpha_0 = \tfrac{1}{4}(-2 - \sqrt{5} + \sqrt{6\sqrt{5}+15}) = 0.5473\ldots$$

and this minimum is attained for $\alpha = \alpha_0$.

For proofs see [Bosma, Jager, Wiedijk].
However, still smaller values for the supremum can be achieved.

(2.17) EXAMPLE. Minkowski's *diagonal continued fraction* (DCF) can be defined by taking as convergents to any x all (irreducible) rationals p/q satisfying:

$$(2.18)\qquad \left|x - \frac{p}{q}\right| < \frac{1}{2}\frac{1}{q^2}.$$

numbered in order of increasing denominator.

From (2.10) we see that we take precisely those regular convergents for which $\Theta_k < \frac{1}{2}$; it can be shown that this does indeed yield a SRCF ([Perron], [Minkowski]). Here the supremum clearly cannot exceed 1/2 (and in fact it equals this). Notice that by Vahlen's theorem (2.8) the DCF picks at least one out of any two consecutive regular convergents to x.

(2.19) REMARK. It is an immediate consequence of results by Jager that for almost all x for every SRCF-expansion $\sup \theta_k(x) \ge 1/2$ (the supremum being taken over $k \ge 1$). Namely, it is shown in [Jager-2] that

$$\sup \min \{\theta_k(x), \theta_{k+1}(x)\} = 1/2$$

for almost all x, implying our assertion by Legendre's theorem (2.10) and the fact that at least one of any two regular convergents is contained in a semi-regular expansion of x for which $\text{SRCF}(x) \subseteq \text{RCF}(x)$, see (3.2).

3. FASTEST EXPANSIONS

In (1.11) we saw that every SRCF of x determines a subsequence of the sequence of primary and secondary regular convergents to the same x. In order to obtain quickly very good approximations, one would like this subsequence to be as "sparse" as possible; in this section we study the fastest way of running through the regular convergents. In doing this we pay special attention to those SRCF's for which SRCF(x) $\subseteq$ RCF(x), since only these can have $\theta_k(x) < 1/2$ for every k by (2.10).

As before, $x \in \Omega$, RCF(x) $= [0; B_1, B_2, \dots]$, the sequences of RCF and NICF-convergents are respectively denoted by $\{P_n/Q_n\}_{n\geq 1}$ and $\{R_k/S_k\}_{k\geq 1}$, while $\{p_k/q_k\}_{k\geq 1}$ denotes the sequence of convergents of some SRCF.

The next lemma shows that we cannot skip an arbitrary number of regular convergents in a semi-regular expansion.

(3.1) LEMMA. *Let $n \geq 1$.*

(i) *If $B_n > 1$ then for some $k \geq 1$:* $\dfrac{p_k}{q_k} \in \left\{\dfrac{P_{n-1}}{Q_{n-1}}, \dfrac{P_{n,1}}{Q_{n,1}}, \dots, \dfrac{P_{n,B_n-1}}{Q_{n,B_n-1}}\right\}$.

(ii) *If $B_n = 1$ then for some $k \geq 1$:* $\dfrac{p_k}{q_k} \in \left\{\dfrac{P_{n-1}}{Q_{n-1}}, \dfrac{P_{n-2}}{Q_{n-2}}\right\}$.

PROOF. This is a direct consequence of [Tietze] Satz 2.

(3.2) COROLLARY. *If SRCF(x) $\subseteq$ RCF(x) then $\{p_k/q_k\}_{k\geq 1}$ contains at least one out of every two consecutive regular convergents, it contains every P_n/Q_n with $B_{n+1} > 1$ ($n \geq 0$).*

It turns out that in the long run the NICF-expansion skips the maximal number of (primary) regular convergents; we first give another result from [Tietze].

(3.3) THEOREM. *For every $x \in \Omega$ and every semi-regular expansion:*

$$\forall k \geq 1: |q_k x - p_k| \geq |S_k x - R_k|.$$

Combining this with lemma (2.7)(i) we immediately find the following if SRCF(x) $\subseteq$ RCF(x).

(3.4) COROLLARY

Let p_k/q_k be the convergents of x for some SRCF-expansion satisfying SRCF(x) $\subseteq$ RCF(x). Then

$$\forall k \geq 1: \left| x - \frac{p_k}{q_k} \right| \geq \left| x - \frac{R_k}{S_k} \right|.$$

Dropping the condition SRCF(x) $\subseteq$ RCF(x) we can still derive a similar, slightly weaker result, using the following lemma.

(3.5) LEMMA

For the secondary (regular) convergents $\dfrac{P_{k,i}}{Q_{k,i}}$, ($i = 1, \dots, B_k - 1$) to any x we have for every k:

$$|Q_{k-2}x-P_{k-2}|>|Q_{k,1}x-P_{k,1}|>\ldots>|Q_{k,B_k-1}x-P_{k,B_k-1}|>$$
$$>|Q_{k-1}x-P_{k-1}|.$$

PROOF. By definition $|Q_{k,i}x-P_{k,i}|=|i(Q_{k-1}x-P_{k-1})+(Q_{k-2}x-P_{k-2})|$ and now the result is an immediate consequence of lemma (2.7)(i) and the well-known fact that two consecutive regular convergents are on opposite sides of x on the real line.

(3.6) PROPOSITION

Let p_k/q_k be the convergents of x for some SRCF-expansion. Then:

$$\forall k\geq 1:\ q_k<Q_{n+1},\ \textit{with } n \textit{ such that } \frac{P_n}{Q_n}=\frac{R_k}{S_k};$$

in particular

$$\left|x-\frac{p_k}{q_k}\right|>\left|x-\frac{P_{n+1}}{Q_{n+1}}\right|.$$

PROOF. Suppose that for some SRCF-expansion of x we find for some k:

$$\text{(3.7)}\qquad \left|x-\frac{p_k}{q_k}\right|\leq\left|x-\frac{P_{n+1}}{Q_{n+1}}\right|,\ \text{with } n \text{ such that } \frac{P_n}{Q_n}=\frac{R_k}{S_k}.$$

Then by (2.6) we have $q_k\geq Q_{n+1}$. Since every SRCF-convergent is at least a secondary RCF-convergent to x, we then have $p_k=P_{m,i}$ and $q_k=Q_{m,i}$ for some $m\geq n+2$ (by (1.10)), and we see from (3.5) that

$$|q_kx-p_k|=|Q_{m,i}x-P_{m,i}|<|Q_{m-2}x-P_{m-2}|\leq|Q_nx-P_n|=|S_kx-R_k|,$$

the second inequality by (2.7)(i) since $m-2\geq n$.

But this violates (3.3).

(3.8) REMARK. The sharper inequality (3.4) does not hold in general for SRCF-expansions as the following example shows.

(3.9) EXAMPLE. Let $x=[0;2,2,2,\ldots]=\sqrt{2}-1$. This is both the regular and the NICF-expansion to x, but we also have $x=[0;3,-2,-3,2,2,2,\ldots]$.

Here

$$\frac{P_1}{Q_1}=\frac{R_1}{S_1}=\tfrac{1}{2} \text{ and } \frac{p_1}{q_1}=\tfrac{1}{3} \text{ while } \frac{P_k}{Q_k}=\frac{R_k}{S_k}=\frac{p_k}{q_k} \text{ for } k\geq 2.$$

But $|x-\frac{1}{2}|>|x-\frac{1}{3}|$.

(3.10) DEFINITION. An SRCF-expansion of an irrational number x is called a *fastest* expansion of x if and only if $q_k\geq S_k$ infinitely often.

(3.11) REMARKS. (i) In particular the NICF-expansion itself is for every x a fastest one.

(ii) This notion of fastest continued fractions generalizes the obvious one for *shortest* expansions of a rational x (see e.g. [Blumer] or [Tietze]; also compare example (1.13)).

(iii) In connection with the singularization mentioned in section 1 we remark the following. If we restrict ourselves to SRCF-expansions for which all convergents are regular convergents, we can obtain fastest continued fractions only if we singularize the maximal number of partial quotients equal to 1; that is, in any sequence of m consecutive ones we have to singularize exactly $\left[\frac{m+1}{2}\right]$ (again compare example (1.13)); here and in the sequel $[x]$ will denote the floor function: $[x]$ is the greatest integer not exceeding x.

(3.12) NOTATION. To every sequence $\{p_k/q_k\}_{k\geq 1}$ of SRCF-convergents of an irrational x we associate a non-decreasing arithmetical function $n:\mathbb{N}\to\mathbb{N}$ defined by

$$Q_{n(k)}\leq q_k<Q_{n(k)+1}.$$

We fix the notation $n_{1/2}$ for the arithmetical function thus associated to the NICF-expansion $\{R_k/S_k\}_{k\geq 1}$ of x.

(3.13) COROLLARY. *A semi-regular expansion of x is a fastest expansion if and only if for infinitely many $k\geq 1$: $n(k)=n_{1/2}(k)$.*

In particular a semi-regular expansion of x for which $SRCF(x)\subsetneq RCF(x)$ is a fastest expansion if and only if $q_k=S_k$ infinitely often, and therefore if and only if

$$\left|x-\frac{p_k}{q_k}\right|=\left|x-\frac{R_k}{S_k}\right| \textit{ infinitely often.}$$

PROOF. By (3.6) $q_k<Q_{n_{1/2}(k)+1}$ for every $k\geq 1$ and the result is immediate from the definitions.

(3.14) LEMMA. *A sequence $\{p_k/q_k\}_{k\geq 1}$ of SRCF-convergents to some x forms a fastest expansion if and only if for every SRCF-expansion of x with convergents $\{p'_k/q'_k\}_{k\geq 1}$ we have for the associated arithmetical functions n respectively n' infinitely often: $n'(k)\leq n(k)$.*

PROOF. Suppose that $q_k\geq S_k$. Then $n'(k)>n(k)$ would imply by definition:

$$S_k\leq Q_{n(k)}\leq q_k<Q_{n(k)+1}\leq Q_{n'(k)}\leq q'_k$$

and since q'_k is at least a secondary regular convergent we see from (3.5) that

$$|q'_kx-p'_k|<|Q_{n(k)}x-P_{n(k)}|\leq|S_kx-R_k|,$$

violating (3.3). Therefore $n'(k)\leq n(k)$ whenever $q_k\geq S_k$ and this occurs infinitely often for fastest expansions.

The converse is immediate from the definitions: take $n'(k)=n_{1/2}(k)$. This proves (3.14).

The following theorem is a result from [Bosma, Jager, Wiedijk].

(3.15) THEOREM. *Let $n_\alpha(k)$ denote the arithmetical function from* (3.12) *for the α-expansions of* (1.15), *with* $1/2 \le \alpha \le 1$. *Then for almost all x:*

$$\lim_{k\to\infty} \frac{n_\alpha(k)}{k} = \begin{cases} \dfrac{\log 2}{\log G} & \textit{if } \frac{1}{2} \le \alpha \le g \\ \dfrac{\log 2}{\log(1+\alpha)} & \textit{if } g \le \alpha \le 1. \end{cases}$$

(3.16) REMARKS. (i) In particular we have for almost all x:

$$\lim_{k\to\infty} \frac{n_{1/2}(k)}{k} = \frac{\log 2}{\log G} = 1.44042\ldots,$$

see also [Adams], [Jager-1]. For every irrational x we have by (3.6) for every semi-regular expansion that $n(k) \le n_{1/2}(k)$, $k \ge 1$, while by (3.13) for fastest expansions equality holds infinitely often. Therefore

$$\limsup_{k\to\infty} \frac{n(k)}{k} = \frac{\log 2}{\log G},$$

for fastest expansions for almost all x.

Notice that thus (3.15) shows immediately that for $g < \alpha \le 1$ the α-expansions are for almost no x fastest expansions. On the other hand, using for instance (3.17), one can prove that for $1/2 \le \alpha \le g$ the α-expansions are indeed for every x fastest expansions.

(ii) In fact one can prove that for the arithmetical function $n(k)$ associated to a fastest expansion

$$n_{1/2}(k) - 1 \le n(k) < n_{1/2}(k) + 1 \quad k \ge 1.$$

Here the upper bound is immediate from (3.6), while the lower bound can be found by combining (3.1), (3.11)(iii) and (3.13). This implies that for almost all x we have for every fastest expansion

$$\lim_{k\to\infty} \frac{n(k)}{k} = \frac{\log 2}{\log G}.$$

(3.17) PROPOSITION. *With U as in* (1.6), *we have when* $SRCF(x) \subseteq RCF(x)$:

$$SRCF(x) \textit{ is a fastest expansion} \Leftrightarrow \forall k : |U^k(x)| \le g.$$

PROOF. See [Tietze].

(3.18) EXAMPLE. Minkowski's diagonal expansion does not in general give fastest expansions; for by Corollary 1 of [Bosma, Jager, Wiedijk] we have (with obvious notation) the value

$$\lim_{k\to\infty} \frac{n_D(k)}{k} = 2 \log 2 = 1.3862\ldots < \frac{\log 2}{\log G} \text{ for almost all } x.$$

Now the question arises whether there exist semi-regular continued fraction algorithms which give both fastest expansions for any x and have optimal approximation properties in the sense that sup $\theta_k \le 1/2$ for every x. In [Keller] this question was raised, and answered in the affirmative: it was shown there that in general there are infinitely many ways to get semi-regular expansions satisfying both conditions, out of the regular expansion. Selenius gave a singularization algorithm to achieve this (cf. [Selenius]). In these results it is necessary to compute the RCF-expansion first. In the next section we show how in a natural way an algorithm arises which satisfies both conditions and which computes a semi-regular expansion of a given x without making use of its regular expansion. Moreover, close inspection of this algorithm enables us to prove various metrical results for this expansion, such as the distribution of the sequence $\{\theta_k\}_{k\ge 1}$ for almost all x.

4. GEOMETRICAL INTERPRETATION AND A NEW ALGORITHM

As in [Klein] we will give a geometrical interpretation of the RCF-algorithm and we will show how a similar interpretation can be given for the α-expansions. Then we present a new SRCF algorithm inspired by this and derive its main properties.

The idea behind the geometrical interpretation is to represent (irreducible) rational numbers r/s by integral vectors in the first quadrant of $\mathbb{R}^2$, namely r/s ($0 \le r/s \le 1$) is represented by (s, r), and to represent irrational numbers x with $0 < x < 1$ by lines L with slope x. The approximation of x by its convergents comes down to systematically finding integral vectors close to the line L. More precisely, starting with $V_{-1} = (0,1)$, $V_0 = (1,0)$ we find $V_1, V_2, \ldots$ by taking $V_1 = B_1 V_0 + V_{-1}$ with $B_1 \ge 1$ in $\mathbb{Z}$ maximal with respect to the property that V_1 is on the same side of L as V_{-1}. Next we find $V_2 = B_2 V_1 + V_0$ with $B_2 \ge 1$ maximal such that V_2 and V_0 are on the same side of L, etc..

For general α-expansions the following modification is made; instead of choosing B_k to be the largest integer λ such that V_k lies on the same side of L as V_{k-2}, one chooses either $a_{\alpha,k} = \lambda$ or $\lambda + 1$, depending on α as follows.

The line L intersects $m_k : \mu V_{k-1} + V_{k-2}$ (μ in $\mathbb{R}$) somewhere, say in point S, between the lattice points $\lambda V_{k-1} + V_{k-2}$ and $(\lambda+1) V_{k-1} + V_{k-2}$; now choose

$a_{\alpha,k} = \lambda \quad \Leftrightarrow$ S lies between $\lambda V_{k-1} + V_{k-2}$ and $(\lambda + \alpha) V_{k-1} + V_{k-2}$

$a_{\alpha,k} = \lambda + 1 \quad \Leftrightarrow$ S lies between $(\lambda + \alpha) V_{k-1} + V_{k-2}$ and $(\lambda + 1) V_{k-1} + V_{k-2}$.

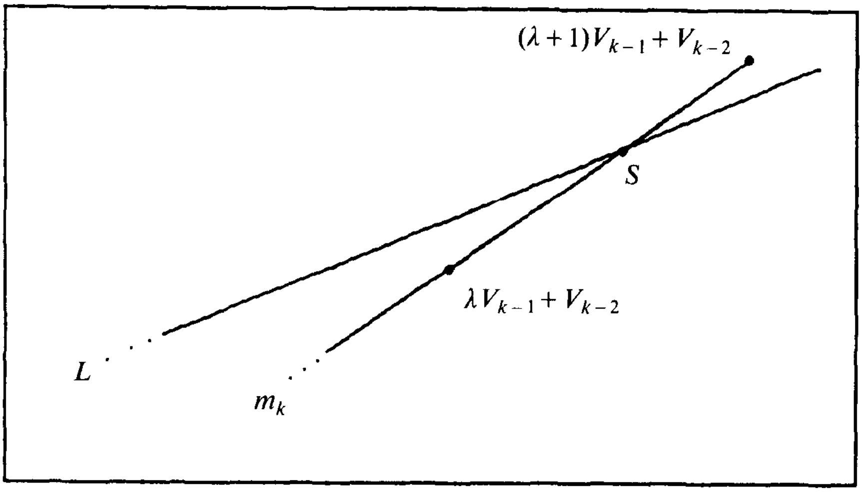

Figure 1

In the former case we continue as usual, in the latter $a_{\alpha,k+1}$ is found as before from $V_{k+1}=a_{\alpha,k+1}V_k-V_{k-1}$, that is, we take $\varepsilon_{k+1}=-1$. Notice that V_k and $-V_{k-1}$ are again on opposite sides of L.

Viewed this way α gives a criterion whether to take $a_{\alpha,k}=\lambda$ or $\lambda+1$.

This suggests the possibility of *varying* α during the expansion process, in order to make in every step the optimal choice in the sense that $\theta=s|sx-r|$ is minimized.

This leads to the following algorithm, as we will see below.

(4.1) DEFINITION. The *optimal continued fraction* expansion of an irrational number x satisfying $-1/2<x<1/2$, denoted by OCF(x), is defined recursively as follows.

Put

$$r_{-1}=1,\ r_0=0,$$

$$s_{-1}=0,\ s_0=1,$$

$$t_0=x,$$

$$\varepsilon_1=\text{sgn }(t_0)=\text{sgn }(x),$$

and let for $k\geq 1$:

$$b_k=[|t_{k-1}|^{-1}],$$

$$v_k=b_ks_{k-1}+\varepsilon_ks_{k-2}\text{ and }u_k=b_kr_{k-1}+\varepsilon_kr_{k-2},$$

$$\alpha_k=\frac{v_k+s_{k-1}}{2v_k+s_{k-1}}.$$

Let the partial fractions be given by:

$$a_k=[|t_{k-1}|^{-1}+1-\alpha_k]$$

and the convergents r_k/s_k by:

$$r_k = a_k r_{k-1} + \varepsilon_k r_{k-2}$$

$$s_k = a_k s_{k-1} + \varepsilon_k s_{k-2}.$$

Next put

$$t_k = |t_{k-1}|^{-1} - a_k$$

and

$$\varepsilon_{k+1} = \operatorname{sgn}(t_k).$$

For arbitrary (irrational) x we define $\mathrm{OCF}(x) = [a_0; \varepsilon_1 a_1, \varepsilon_2 a_2, \ldots]$ where a_0 in $\mathbb{Z}$ is such that $-1/2 < x - a_0 < 1/2$ and where $[0; \varepsilon_1 a_1, \varepsilon_2 a_2, \ldots] = \mathrm{OCF}(x - a_0)$.

(4.2) REMARKS. Notice that the auxiliary integers b_k are computed in the same way as the integers λ in the introduction of this section. That is, the lattice points W_k and W_k', resp. given by

$$(b_k s_{k-1} + \varepsilon_k s_{k-2}, b_k r_{k-1} + \varepsilon_k r_{k-2})$$

and

$$((b_k + 1)s_{k-1} + \varepsilon_k s_{k-2}, (b_k + 1)r_{k-1} + \varepsilon_k r_{k-2}),$$

are on opposite sides of the line determined by x. The choice between W_k and W_k' then depends on α_k. In the algorithm defined by (4.1) the u_k are not used at all, but we refer to them in the proofs.

First we show in lemma (4.3) that this algorithm has the announced property, namely that in the k^{th} step out of the two possibilities

$$\frac{r_k}{s_k} = \frac{u_k}{v_k} = \frac{b_k r_{k-1} + \varepsilon_k r_{k-2}}{b_k s_{k-1} + \varepsilon_k s_{k-2}}$$

and

$$\frac{r_k}{s_k} = \frac{u_k + r_{k-1}}{v_k + s_{k-1}} = \frac{(b_k + 1)r_{k-1} + \varepsilon_k r_{k-2}}{(b_k + 1)s_{k-1} + \varepsilon_k s_{k-2}}$$

that one is chosen which minimizes $s_k|s_k x - r_k|$.

(4.3) LEMMA

$$\frac{r_k}{s_k} = \frac{u_k}{v_k} \Leftrightarrow v_k|v_k x - u_k| < (v_k + s_{k-1})|(v_k + s_{k-1})x - (u_k + r_{k-1})|.$$

PROOF. As we can easily see from (4.2), we can choose $0 < \tau < 1$ such that

$$x = \frac{(b_k + \tau)r_{k-1} + \varepsilon_k r_{k-2}}{(b_k + \tau)s_{k-1} + \varepsilon_k s_{k-2}}.$$

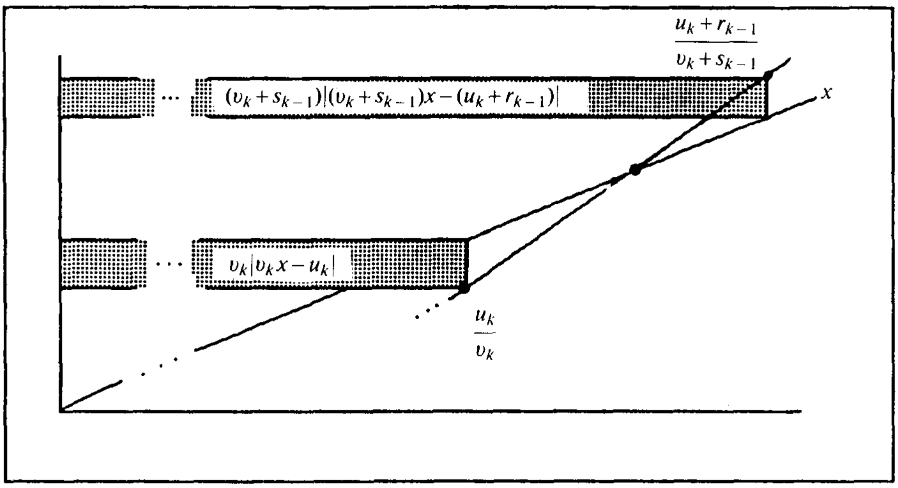

Figure 2

Then from the definition (and the introduction to this section) we can see that

$$\frac{r_k}{s_k} = \frac{u_k}{v_k} \Leftrightarrow \tau < \alpha_k.$$

But

$$\tau < \alpha_k = \frac{v_k + s_{k-1}}{2v_k + s_{k-1}} \Leftrightarrow$$

$$\frac{1}{\tau} > 1 + \frac{v_k}{v_k + s_{k-1}} \Leftrightarrow$$

$$\frac{\tau}{1-\tau} < \frac{v_k + s_{k-1}}{v_k}.$$

On the other hand we have

$$\frac{\tau}{1-\tau} = \frac{|v_k x - u_k|}{|(v_k + s_{k-1})x - (u_k + r_{k-1})|},$$

as can be seen by comparing the similar triangles from figure 2. Combining these gives the desired result.

We now give some consequences of definition (4.1).

(4.4) LEMMA. *For every irrational number* $x \in [-1/2, 1/2]$ *and for every* $k \geq 1$:

$$\alpha_k - 1 < t_k < \alpha_k.$$

PROOF. By definition we have

$$t_k = |t_{k-1}|^{-1} - a_k = |t_{k-1}|^{-1} - [|t_{k-1}|^{-1} + 1 - \alpha_k]$$

which implies (4.4) immediately.

(4.5) LEMMA. *For every x and for every $k \geq 1$:*

$$(4.6) \qquad a_k = \begin{cases} b_k & \text{if and only if } \varepsilon_{k+1} = 1 \\ b_k + 1 & \text{if and only if } \varepsilon_{k+1} = -1, \end{cases}$$

$$(4.7) \qquad s_k > s_{k-1},$$

$$(4.8) \qquad \tfrac{1}{2} < \alpha_k < 1.$$

PROOF. By induction on k. For $k=1$ the restriction $-1/2 < x < 1/2$ gives $b_1 \geq 2$. From the definitions we see $v_1 = b_1$ so $\alpha_1 = (b_1+1)/(2b_1+1)$ whence $1/2 \leq \alpha_1 \leq 3/5$. From this we see that $a_1 \geq b_1$ and thus that $s_1 = a_1 \geq 2 > s_0 = 1$ and also that $a_1 = b_1$ or $a_1 = b_1 + 1$, the latter being equivalent to $\alpha_1 < |t_0|^{-1}$, that is to $\varepsilon_2 = -1$.

If (4.6)-(4.8) hold for $k-1$ then using (4.4) we see

$$-1/2 \leq \alpha_{k-1} - 1 < t_{k-1} < \alpha_{k-1} < 1$$

whence $b_k \geq 1$, with equality only possible when $\varepsilon_k = 1$. Now $b_k \geq 1$ gives $v_k > 0$, which in its turn implies $1/2 < \alpha_k < 1$. But then it is clear that $a_k = b_k$ or $a_k = b_k + 1$, the latter occurring if and only if $\varepsilon_{k+1} = -1$. Finally $s_k > s_{k-1}$.

This proves (4.5).

The t_k may be regarded as the k^{th} iterate of the operator t, which acts on an x with $\mathrm{OCF}(x) = [0; \varepsilon_1 a_1, \varepsilon_2 a_2, \ldots]$ by a shift: $tx = [0; \varepsilon_2 a_2, \varepsilon_3 a_3, \ldots]$. That is,

$$(4.9) \qquad t_k(x) = t^k([0; \varepsilon_1 a_1, \varepsilon_2 a_2, \ldots]) = [0; \varepsilon_{k+1} a_{k+1}, \varepsilon_{k+2} a_{k+2}, \ldots],$$

just as the operator T (which is the T_1 from (1.15)) shifts the regular expansion of x:

$$(4.10) \qquad T_m(x) = T^m([0; B_1, B_2, \ldots]) = [0; B_{m+1}, B_{m+2}, \ldots].$$

In the following lemma we give the connection between both. Its significance will become clear once we have shown that every OCF-convergent is a regular convergent.

(4.11) LEMMA. *Let x be an irrational number and let $k \geq 1$ be an integer.*

(i) *If there exists an $n \geq 1$ such that*

$$P_{n-1} = r_{k-1} \quad P_n = r_k$$

$$Q_{n-1} = s_{k-1} \quad Q_n = s_k,$$

then $\varepsilon_{k+1} = 1$ and $t_k = T_n$.

(ii) *If there exists an $n \geq 1$ such that*

$$P_{n-2} = r_{k-1} \quad P_n = r_k$$

$$Q_{n-2} = s_{k-1} \quad Q_n = s_k,$$

then $\varepsilon_{k+1} = -1$ and $t_k = -T_{n-1} T_n$.

PROOF. From (4.9) it is not hard to show inductively that for every k:

$$x=\frac{r_{k-1}(a_k+t_k)+\varepsilon_k r_{k-2}}{s_{k-1}(a_k+t_k)+\varepsilon_k s_{k-2}}=\frac{r_k+t_k r_{k-1}}{s_k+t_k s_{k-1}},$$

in particular that

$$x=\frac{P_{k-1}(B_k+T_k)+P_{k-2}}{Q_{k-1}(B_k+T_k)+Q_{k-2}}=\frac{P_k+T_kP_{k-1}}{Q_k+T_kQ_{k-1}}. \tag{4.12}$$

Substituting the hypotheses of (i) yields immediately $t_k=T_n$, and since $T_n>0$, we find also $\varepsilon_{k+1}=1$. In the second case, combination of the above equalities leads to

$$t_k(P_{n-2}Q_n-P_nQ_{n-2})+t_kT_n(P_{n-2}Q_{n-1}-P_{n-1}Q_{n-2})=$$
$$=-T_n(P_nQ_{n-1}-P_{n-1}Q_n).$$

Now use that $Q_n=B_nQ_{n-1}+Q_{n-2}$ and $P_n=B_nP_{n-1}+P_{n-2}$, and the well-known (see e.g. [Perron] p. 16)

$$P_nQ_{n-1}-P_{n-1}Q_n=(-1)^{n-1}$$

to obtain

$$t_k=\frac{-T_n}{B_n+T_n}=-T_{n-1}T_n;$$

comparison of the signs completes the proof of (4.11).

Before we formulate the main theorem of this section we state and prove one more fact about regular continued fractions, which turns out to be very useful to us.

(4.13) LEMMA. *Let $m\geq 1$. Then:*

$$\Theta_{m-1}<\Theta_m \Leftrightarrow T_m>Q_{m-1}/Q_m=[0;B_m,B_{m-1},\ldots,B_1].$$

PROOF. Suppose for simplicity (the other case being similar) that $P_m-Q_mx>0$, then $P_{m-1}-Q_{m-1}x<0$. From e.g. (4.12) we then see

$$T_m=\frac{P_m-Q_mx}{Q_{m-1}x-P_{m-1}}=\frac{\Theta_m}{\Theta_{m-1}}\,\frac{Q_{m-1}}{Q_m}$$

since $\Theta_m=Q_m(P_m-Q_mx)$ etc.. The assertion easily follows.

(4.14) PROPOSITION. *For every irrational number we have, with the notations of* (4.1), *for every integer $k\geq 1$:*

(4.15) *there exists an $n=n(k)\geq 1$ such that*

$$r_k=P_{n(k)}$$

and

$$s_k = Q_{n(k)},$$

and the thus defined arithmetical function $n: \mathbb{N} \to \mathbb{N}$ *satisfies:*

$$(4.16) \quad n(k) = \begin{cases} n(k-1)+1 \Leftrightarrow \varepsilon_{k+1} = 1 & \text{where we set } n(0)=0 \text{ when } \varepsilon_1 = 1, \\ n(k-1)+2 \Leftrightarrow \varepsilon_{k+1} = -1 & \text{where we set } n(0)=1 \text{ when } \varepsilon_1 = -1. \end{cases}$$

Moreover we have

$$(4.17) \quad \varepsilon_{k+1} = -1 \Rightarrow B_{n(k)} = 1$$

and

$$(4.18) \quad s_k|s_k x - r_k| < 1/2.$$

PROOF. By induction on k.

First consider the case $k=1$.

If $x>0$, that is if $\varepsilon_1 = 1$, then $b_1 = [|t_0^{-1}|] = [x^{-1}] = B_1$. Therefore if $\varepsilon_2 = 1$ we see by (4.6) that $s_1 = B_1 Q_0 + Q_{-1} = Q_1$ so $n(1)=1$. If on the other hand $\varepsilon_2 = -1$ we get $s_1 = (B_1+1)Q_0 + Q_{-1} = Q_1 + Q_0$. If $B_2 = 1$ this equals Q_2 so we have $n(1)=2$ while $B_2 > 1$ and $\varepsilon_2 = -1$ are mutually exclusive: assumption of both leads to $Q_1 < s_1 = v_1 + s_0 = Q_1 + Q_0 < Q_2$, and likewise $P_1 < r_1 = u_1 + r_0 = P_1 + P_0 < P_2$. By lemma (4.3) this would imply

$$Q_1|Q_1 x - P_1| > (Q_1+Q_0)|(Q_1+Q_0)x - (P_1+P_0)|$$

which contradicts (3.5). Moreover, according to (4.3), the choice between $n(1)=1$ and $n(1)=2$ if $B_2=1$ is made in such a way that $s_1|s_1 x - r_1|$ is minimized and this minimal value is smaller than 1/2 by Vahlen's theorem (2.8). If $B_2 > 1$ we find (4.18) as a consequence of (2.5). If on the other hand $x<0$, $\varepsilon_1 = -1$ then we must have $\mathrm{RCF}(x) = [-1; B_1, B_2, \dots]$ with $B_1 = 1$, since $x > -1/2$. Therefore in this case

$$b_1 = [|x^{-1}|] = \left[\cfrac{-1}{-1 + \cfrac{1}{B_1 + \cfrac{1}{B_2 + T_2}}} \right] = [B_2 + T_2 + 1] = B_2 + 1.$$

For $\varepsilon_2 = 1$, combination with (4.6) gives $s_1 = B_2 + 1 = Q_2$ so $n(1)=2$. When $\varepsilon_2 = -1$ again (4.6) gives $s_1 = B_2 + 2 = Q_2 + Q_1$; just as above, either we get a contradiction from $B_3 > 1$ and $\varepsilon_2 = -1$ or we find that $s_1 = Q_3$ in which case $n(1)=3$. The same argument as above shows that (4.18) holds here.

This settles the case $k=1$.

Suppose next that (4.15)–(4.17) hold up to $k-1$ (inclusive).

Below we will prove that, with $m-1=n(k-1)$:

$$\text{(4.19)}\quad \text{either}\ \begin{cases} r_k=u_k=P_m \\ s_k=v_k=Q_m \end{cases} \quad \text{or}\ \begin{cases} r_k=u_k+r_{k-1}=P_m+P_{m-1} \\ s_k=v_k+s_{k-1}=Q_m+Q_{m-1}. \end{cases}$$

This will finish the proof as we will first demonstrate, before proving (4.19). To this end we distinguish the cases $B_{m+1}=1$ and $B_{m+1}>1$.

If $B_{m+1}=1$ then

$$P_m+P_{m-1}=P_{m+1}$$

$$Q_m+Q_{m-1}=Q_{m+1}$$

yielding (4.15) and (4.16) using (4.6), while by (4.3) the choice in (4.19) is made in such a way that $s_k|s_kx-r_k|$ is minimized. By Vahlen's theorem (2.8) we then get (4.18).

If $B_{m+1}>1$ then

$$P_m+P_{m-1}=P_{m,1}<P_{m+1}$$

$$Q_m+Q_{m-1}=Q_{m,1}<Q_{m+1}$$

so the second possibility in (4.19) gives a secondary convergent. By Legendre's theorem (2.10) we have $Q_{m,1}|Q_{m,1}x-P_{m,1}|>1/2$ while (2.5) implies here $Q_m|Q_mx-P_m|<1/2$. This gives (4.18) using (4.3), which now implies $n(k)=m$ and thus also (4.15) and (4.17). By (4.6) we get (4.16).

Remains to prove (4.19); again we distinguish two cases.

If $\varepsilon_k=1$ we have by (4.16)

$$r_{k-2}=P_{m-2},\quad r_{k-1}=P_{m-1}$$

$$s_{k-2}=Q_{m-2},\quad s_{k-1}=Q_{m-1},$$

and by lemma (4.11) we find $t_{k-1}=T_{m-1}$. Therefore $b_k=[t_{k-1}^{-1}]=[T_{m-1}^{-1}]=B_m$ and we are done bdy the definition of u_k and v_k:

$$u_k=b_kr_{k-1}+\varepsilon_kr_{k-2}=B_mP_{m-1}+P_{m-2}=P_m$$

$$v_k=b_ks_{k-1}+\varepsilon_ks_{k-2}=B_mQ_{m-1}+Q_{m-2}=Q_m.$$

If $\varepsilon_k=-1$ we have

$$r_{k-2}=P_{m-3},\quad r_{k-1}=P_{m-1}$$

$$s_{k-2}=Q_{m-3},\quad s_{k-1}=Q_{m-1},$$

and by lemma (4.11) we find $t_{k-1}=-T_{m-2}T_{m-1}$. Therefore $b_k=[|t_{k-1}|^{-1}]=$ $=[T_{m-2}^{-1}T_{m-1}^{-1}]=[B_{m-1}T_{m-1}^{-1}+1]=[T_{m-1}^{-1}+1]=B_m+1$, since by (4.17) we have $B_{m-1}=1$. Then

$$u_k=(B_m+1)P_{m-1}-P_{m-3}=B_mP_{m-1}+(P_{m-1}-P_{m-3})=$$
$$=B_mP_{m-1}+P_{m-2}=P_m$$

and likewise

$$v_k = Q_m.$$

This completes the proof of (4.14).

(4.20) COROLLARY. *Notations as in* (4.14). *Let* $m \geq 1$.

(i) *If* $n(k) \neq m$ *for every* $k \geq 1$ *then there exists* $k \geq 1$ *such that* $n(k) = m+1$.

(ii) *If* $B_{m+2} > 1$ *then there exists* $k \geq 1$ *such that* $n(k) = m+1$.

(iii) *If* $B_{m+2} = 1$ *then there exists* $k \geq 1$ *such that* $n(k) = m+1$ *if and only if*
either $\Theta_m > \Theta_{m+1}$ *and for every* $k \geq 1$: $n(k) \neq m$
or $\Theta_{m+1} < \Theta_{m+2}$ *and for some* $k \geq 1$: $n(k) = m$.

PROOF

(i) Immediate from (4.16).

(ii) Either for every $k \geq 1$ we have $n(k) \neq m$ and then we are done dy (i), or for some k we have $n(k) = m$ and by (4.16) and (4.17) the assumption that $n(k+1) = m+2$ implies that $B_{n(k+1)} = B_{m+2} = 1$, thus contradicting our hypothesis.

(iii) If $n(k) \neq m$ for every $k \geq 1$ then by (ii) we must have $B_{m+1} = 1$, and in this case we see as before that by lemma (4.3) necessarily $\Theta_m > \Theta_{m+1}$. Next suppose that for a certain k we do have $n(k) = m$. By the same argument, from $B_{m+2} = 1$ we get $n(k+1) = m+1$ if and only if $\Theta_{m+1} < \Theta_{m+2}$.

This proves (4.20).

We stated corollary (4.20) for easy reference; since it gives a criterion to decide whether a given regular convergent is an OCF-convergent, it will turn out to be very useful in studying the properties of the subsequence OCF(x) of RCF(x).

(4.21) PROPOSITION. *With notations as before we have*

$$\forall x \in \Omega,\ k \geq 0:\ -\tfrac{1}{2} < t_k < g = \tfrac{1}{2}(\sqrt{5} - 1).$$

PROOF. By induction on k; the case $k = 0$ is settled by definition.

Fix x and suppose that the assertion holds up to $k-1$ (inclusive); write $m - 1 = n(k-1)$. We distinguish three cases.

(i) $\varepsilon_{k+1} = -1$.

In this case $t_k < 0$ and the assertion for k follows from (4.4) and (4.8).

(ii) $\varepsilon_{k+1} = 1$, $\varepsilon_k = 1$.

By (4.15) and (4.16)

$$r_{k-2} = P_{m-2} \quad r_{k-1} = P_{m-1} \quad r_k = P_m$$

$$s_{k-2} = Q_{m-2} \quad s_{k-1} = Q_{m-1} \quad s_k = Q_m,$$

and by (4.11)(i): $t_k = T_m = [0;\ B_{m+1}, B_{m+2}, B_{m+3} \ldots]$. If $B_{m+1} > 1$, the latter is clearly smaller than $g = [0; 1, 1, \ldots]$, so suppose that $B_{m+1} = 1$. Now by (4.3) we induce from $n(k) = m$ that $\Theta_m < \Theta_{m+1}$ whence by lemma (4.13)

(4.22) $T_{m+1} > [0; B_{m+1}, B_m, \dots B_1] = [0; 1, B_m, \dots B_1]$.

If we assume $B_m = 1$ then by $n(k-1) = m-1$ we similarly see that $\Theta_{m-1} < \Theta_m$ and hence that $T_m > [0; B_m, B_{m-1}, \dots B_1]$; but this would give a contradiction with (4.22) since then

$$T_{m+1} = (B_m + T_m)^{-1} < (1 + [0; B_m, B_{m-1}, \dots B_1])^{-1} =$$
$$= [0; B_{m+1}, B_m, \dots B_1].$$

Therefore $B_m > 1$ and as we saw before we have $b_k = B_m$ so

$$\alpha_k = \frac{v_k + s_{k-1}}{2v_k + s_{k-1}} = \frac{b_k s_{k-1} + \varepsilon_k s_{k-2} + s_{k-1}}{2(b_k s_{k-1} + \varepsilon_k s_{k-2}) + s_{k-1}} = \frac{Q_{m+1}}{2Q_m + Q_{m-1}} =$$

$$= \frac{Q_{m+1}}{Q_{m+1} + Q_m} = [0; 1, B_{m+1}, B_m, \dots B_1] = [0; 1, 1, B_m, \dots B_1] < g$$

since $B_m \geq 2$; but $t_k < \alpha_k$ by (4.4) and case (ii) is finished.

(iii) $\varepsilon_{k+1} = 1$, $\varepsilon_k = -1$.

By (4.15) and (4.16) now

$$r_{k-2} = P_{m-3} \quad r_{k-1} = P_{m-1} \quad r_k = P_m$$
$$s_{k-2} = Q_{m-3} \quad s_{k-1} = Q_{m-1} \quad s_k = Q_m.$$

By (4.17), $B_{m-1} = 1$ and by (4.11) $0 < t_k = T_m = [0; B_{m+1}, B_{m+2}, \dots]$. Again, either $B_{m+1} \geq 2$ and hence $t_k < 1/2 < g$ immediately, or $B_{m+1} = 1$ in which case we proceed as follows. From (4.3) and $n(k) = m$ we see $\Theta_m < \Theta_{m+1}$ and thus by lemma (4.13):

(4.23) $T_{m+1} > [0; B_{m+1}, B_m, \dots B_1]$.

Now $t_k = T_m = (B_{m+1} + T_{m+1})^{-1} < [0; 1, 1, B_m, B_{m-1}, \dots B_1]$ and we are done if we prove:

(4.24) $[0; B_m, B_{m-1}, \dots B_1] < g$.

We may assume that $B_m = 1$ since else (4.24) holds trivially. Using (4.3) and $\varepsilon_k = -1$ as before we get $\Theta_{m-2} > \Theta_{m-1}$ and so by lemma (4.13): $T_{m-1} < [0; B_{m-1}, B_{m-2}, \dots B_1]$. Since (with abuse of notation) $T_{m-1} = [0; B_m, B_{m+1}, T_{m+1}] = [0; 1, 1, T_{m+1}]$ this yields

(4.25) $[0; 1, 1, T_{m+1}] < [0; B_{m-1}, B_{m-2}, \dots, B_1]$.

Combination of (4.23) and (4.25) yields:

(4.26) $$\begin{cases} [0; 1, 1, B_{m+1}, B_m, \dots, B_1] = \\ = [0; 1, 1, 1, 1, B_{m-1}, B_{m-2}, \dots, B_1] < [0; B_{m-1}, B_{m-2}, \dots, B_1]. \end{cases}$$

Let us write for abbreviation

$$\mu = [0; B_{m-1}, B_{m-2}, \dots, B_1].$$

Then

$$[0; 1, 1, 1, 1, B_{m-1}, B_{m-2}, \ldots, B_1] = \frac{2\mu+3}{3\mu+5}$$

and (4.26) then implies

$$\mu^2 + \mu - 1 > 0$$

that is

$$\mu > g.$$

Therefore

$$[0; B_m, B_{m-1}, B_{m-2}, \ldots, B_1] = [0; 1, B_{m-1}, B_{m-2}, \ldots, B_1] =$$
$$= (1+\mu)^{-1} < (1+g)^{-1} = g$$

which proves (4.24).

(4.27) COROLLARY. *The partial quotients of an OCF-expansion satisfy:*

for every $k \geq 1$: $a_k \geq 2$.

PROOF. By definition $a_k = [|t_{k-1}|^{-1} + 1 - \alpha_k]$ and by (4.8) $1 - \alpha_k \geq 0$. By the previous proposition now either $-1/2 < t_{k-1} < 1/2$ and therefore $[|t_{k-1}|^{-1} + 1 - \alpha_k] \geq |t_{k-1}|^{-1} > 2$, or $1/2 < t_{k-1} < g$. The latter implies $g + 1 < t_{k-1}^{-1} < 2$ and then necessarily $a_k = 2$ by again the previous proposition, since $t_{k-1}^{-1} - a_k = t_k$.

We have now derived the main properties of the OCF and summarize them in the main theorem of this paper.

(4.28) MAIN THEOREM. *For every irrational x the OCF-expansion of x is:*
(i) *a semi-regular continued fraction,*
(ii) *a fastest expansion,*
which satisfies
(iii) $\mathrm{OCF}(x) \subseteq \mathrm{RCF}(x)$
and for every $k \geq 1$:
(iv) $\theta_k = s_k |s_k x - r_k| < 1/2$.

PROOF

(i) The requirements of definition (1.1) for semi-regularity are met obviously except for (1.3); that is, we have to prove that for every x in the OCF(x) infinitely often $a_k + \varepsilon_{k+1} \geq 2$. If this were not true, we would have for some $m \geq 1$: $a_m = a_{m+1} = \ldots = 2$, $\varepsilon_{m+1} = \varepsilon_{m+2} = \ldots = -1$ since $a_k \geq 2$ by (4.27). Then $t_m = [0; \varepsilon_{m+1} a_{m+1}, \varepsilon_{m+1} a_{m+1}, \ldots] = -1$. This contradicts (4.21).

(ii) This is an immediate consequence of (4.21), using (3.17).

(iii) This is (4.15).

(iv) This is (4.18).

We conclude this section by proving two more properties of the OCF-convergents. These are respectively the analogues for the OCF of the theorems of Vahlen (2.8) and Legendre (2.10).

(4.29) THEOREM. *Let* $\{r_k/s_k\}_{k\geq 1}$ *be the sequence of OCF-convergents to some* $x\in\Omega$, *and let* $\theta_k = s_k|s_kx-r_k|$. *Then*

$$\forall k\geq 1\colon \min\{\theta_k,\theta_{k+1}\}<\frac{1}{\sqrt{5}}.$$

PROOF. By (4.15) and (4.16) either $(\theta_k,\theta_{k+1})=(\Theta_m,\Theta_{m+1})$ or $(\theta_k,\theta_{k+1})=$ $=(\Theta_m,\Theta_{m+2})$ for some $m\geq 1$.

Suppose that $(\theta_k,\theta_{k+1})=(\Theta_m,\Theta_{m+1})$. If $B_{m+2}=1$ then by (4.20)(iii) we see that $\Theta_{m+1}<\Theta_{m+2}$ and we get the result by Borel's theorem (2.9)(i):

$$\min\{\theta_k,\theta_{k+1}\}=\min\{\Theta_m,\Theta_{m+1}\}=\min\{\Theta_m,\Theta_{m+1},\Theta_{m+2}\}<1/\sqrt{5}.$$

If $B_{m+2}>1$ we use Fujiwara's theorem (2.9)(iii): either $\theta_{k+1}=\Theta_{m+1}<2/5<$ $<1/\sqrt{5}$, or $\theta_k=\Theta_m<2/5<1/\sqrt{5}$.

Next we consider the case that $(\theta_k,\theta_{k+1})=(\Theta_m,\Theta_{m+2})$; then by (4.16) and (4.17) $B_{m+2}=1$ *and by* (4.20)*(iii)* $\Theta_{m+1}>\Theta_{m+2}$. Again

$$\min\{\theta_k,\theta_{k+1}\}=\min\{\Theta_m,\Theta_{m+2}\}=\min\{\Theta_m,\Theta_{m+1},\Theta_{m+2}\}<1/\sqrt{5}$$

by Borel's theorem.

This proves (4.29).

(4.30) REMARK. Notice that the constant $1/\sqrt{5}$ in (4.29) is best possible for any continued fraction expansion: a smaller constant would lead to a contradiction with Hurwitz' theorem (2.11).

(4.31) THEOREM. *Let* $x\in\Omega$ *and* $r,s\in\mathbb{Z}$ *with* $s>0$ *such that* $\theta=s|sx-r|<\dfrac{1}{\sqrt{5}}$. *Then* $\dfrac{r}{s}=\dfrac{r_k}{s_k}$ *for some* $k\geq 1$, *i.e.* $\dfrac{r}{s}$ *is an OCF-convergent of* x.

PROOF. Suppose that $\theta<1/\sqrt{5}$ but that r/s is not an OCF-convergent. Since $1/\sqrt{5}<1/2$ we have by Legendre's theorem (2.10) that r/s is a RCF-convergent: $r=P_m$, $s=Q_m$ for some $m\geq 1$. By (4.15)-(4.17) then for some $k\geq 1$:

$$r_{k-1}=P_{m-1},\quad r_k=P_{m+1}$$

$$s_{k-1}=Q_{m-1},\quad s_k=Q_{m+1}$$

and

$$B_{m+1}=1.$$

Now (4.20)(iii) implies that both $\Theta_{m-1}\leq\Theta_m$ and $\Theta_{m+1}\leq\Theta_m$ while $\Theta_m=\theta<$ $<1/\sqrt{5}$. So we find $\max\{\Theta_{m-1},\Theta_m,\Theta_{m+1}\}<1/\sqrt{5}$; this contradicts (2.9)(ii) and we have proved (4.31).

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