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Metrical Theory for Optimal Continued Fractions

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The ergodic system underlying the optimal continued fraction algorithm is introduced and studied. In particular the distribution of the sequence $\theta_n(x)_{n\geqslant 1}$, which measures how well a number x is approximated by its convergents, is derived for almost all irrational numbers.

1. INTRODUCTION

In this paper we develop the metrical theory for the optimal continued fraction (OCF), introduced in [1]. Let us describe briefly its main features. For proofs of statements in this introduction the reader is referred to [1].

The OCF arose in the investigation of semi-regular continued fraction expansions (SRCF's) of a real number x, i.e., expansions of the form

$$x=[b_0;\varepsilon_1 b_1,\varepsilon_2 b_2,\dots]=b_0+\cfrac{\varepsilon_1}{b_1+\cfrac{\varepsilon_2}{b_2+\cdots}} \tag{1.1}$$

where $\varepsilon_k=\pm 1$, $b_k\in\mathbf{Z}_{\geqslant 1}$, $k\geqslant 1$, and with some constraints on b_k and ε_k. Usually we will assume that x is irrational, and thus that the expansion (1.1) is infinite.

A special case of an SRCF is the *regular continued fraction*, RCF, which is obtained by taking $\varepsilon_k=1$ for every k in (1.1).

The aim in introducing the OCF was to optimize two things simultaneously. In the first place one wishes the convergents $p_k/q_k=[b_0;\varepsilon_1 b_1,\varepsilon_2 b_2,\dots,\varepsilon_k b_k]$, yielded by the SRCF, to be good approximations of x for every $k\geqslant 1$, in the sense described below. For the regular continued fraction expansion $\mathrm{RCF}(x)=[B_0;B_1,B_2,\dots]$ it is known that the convergents, denoted now by P_n/Q_n $(n\geqslant 1)$, all satisfy

$$0<Q_n|Q_n x-P_n|<1. \tag{1.2}$$

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We define the approximation constants Θ_n by

$$\Theta_n := Q_n |Q_n x - P_n|, \qquad n \geq 1,$$

and, more generally, for any SRCF by

$$\theta_k := q_k |q_k x - p_k|, \qquad k \geq 1.$$

Minkowski knew already (see [9, Section 45]) that the subset of regular convergents for which $\Theta_n < \frac{1}{2}$ constitute together the sequence of convergents for a semi-regular continued fraction expansion, and that for values smaller than $\frac{1}{2}$ this is no longer true in general. Thus we could require $\theta_k < \frac{1}{2}$ for every k; this also means that we restrict to expansions whose convergents form a subsequence of the regular convergents, since every fraction P/Q satisfying $Q|Qx - P| < \frac{1}{2}$ is a regular convergent by Legendre's theorem.

On the other hand we would like to approximate x as fast as possible; that is, one would like the denominators q_k of the convergents p_k/q_k to grow as fast as possible. In other words, one wants the subsequence $(p_k/q_k)_{k \geq 1}$ of $(P_n/Q_n)_{n \geq 1}$ to be as sparse as possible; this density cannot be arbitrarily small, though, since we demand that the p_k/q_k together still constitute the sequence of convergents for a semi-regular continued fraction expansion, which turns out to imply that out of two consecutive regular convergents at most one can be skipped. Furthermore, we get subsequences of the regular convergents in this way if one skips only P_n/Q_n for which $B_{n+1} = 1$. *Fastest* expansions can now be defined as those in which always the maximal number of regular convergents is skipped, which means that whenever a stretch of m consecutive partial quotients B_{n+1} equal to 1 appears in the RCF-expansion, exactly $\lfloor (m+1)/2 \rfloor$ regular convergents are skipped. (Note that this implies that for fastest semi-regular expansions only a choice is left in deciding which regular convergents will be skipped whenever a sequence of 1's of *even* length is encountered.)

The OCF-algorithm (as in [1, 4.1], see Section 3) computes the OCF(x) without first computing RCF(x) and guarantees to give both fast and good approximations in the sense that for every x an SRCF is defined that

(i) is a fastest expansion and

(ii) has $\theta_k = q_k |q_k x - p_k| < \frac{1}{2}$ for every $k \geq 1$.

The main part of this article is devoted to an investigation of the distribution of the values of θ_k for the OCF-expansion of almost all irrational numbers x. In fact we will give in Section 4 for almost all x the distribution of pairs (θ_{k-1}, θ_k). In order to do so we describe in Section 4 the ergodic system underlying the OCF-operator. The construction of this system is in fact only a very special case of a general way of finding (or even defining) ergodic "subsystems" of the RCF-system, defined by semi-regular continued fraction operators; a more general approach based on this idea will be presented elsewhere (see [6]).

2. Regular Continued Fractions

Let $\lfloor x \rfloor$ denote the greatest integer not exceeding x. The regular continued fraction (RCF) operator T: $[0, 1) \to [0, 1)$, given by

$$T(x) = \begin{cases} \dfrac{1}{x} - \left\lfloor \dfrac{1}{x} \right\rfloor & x \neq 0, \\ 0 & x = 0, \end{cases}$$

associates by repeated application to every $x \in \mathbf{R} \backslash \mathbf{Q}$ an infinite expansion $\mathrm{RCF}(x) = [B_0; B_1, B_2, ...]$, with $B_n \in \mathbf{Z}_{\geqslant 1}$, if we define $B_0 := \lfloor x \rfloor$. This determines a sequence of rational approximations $P_n/Q_n = [B_0; B_1, B_2, ..., B_n]$, the convergents of x, whose numerators and denominators satisfy the recurrence relations

$$\begin{array}{llll} P_{-1} := 1, & P_0 := B_0, & P_{n+1} = B_{n+1} P_n + P_{n-1}, & (n \geqslant 0) \\ Q_{-1} := 0, & Q_0 := 1, & Q_{n+1} = B_{n+1} Q_n + Q_{n-1} & (n \geqslant 0). \end{array}$$

In the sequel we will always assume x to be an irrational number to guarantee that all expansions are infinite. We denote

$$T_n = T_n(x) := T^n(T_0) \qquad (n \geqslant 1),$$

where $T_0 = T_0(x) := x - B_0$, and

$$V_n = V_n(x) := \frac{Q_{n-1}}{Q_n} \qquad (n \geqslant 0).$$

It is well-known that

$$T_n = [0; B_{n+1}, B_{n+2}, ...]$$

and

$$V_n = [0; B_n, B_{n-1}, ..., B_1].$$

The two-dimensional operator $\mathscr{T}$ is defined on $\Omega := ([0,1)\backslash\mathbf{Q}) \times [0,1]$ by

$$\mathscr{T}(x, y) := \left(Tx, \frac{1}{\left(\left\lfloor \frac{1}{x} \right\rfloor + y \right)} \right).$$

Note that

$$\mathscr{T}^n(x, y) = (T^n(x), [0; B_n, B_{n-1}, ..., B_2, B_1 + y]), \qquad \text{for} \quad n \geqslant 1,$$

where $\mathrm{RCF}(x) = [0; B_1, B_2, B_3, ...]$; in particular

$$\mathscr{T}^n(x, 0) = (T_n(x), V_n(x)), \qquad \text{for} \quad n \geqslant 0.$$

3. Optimal Continued Fractions

We recall the definition of the OCF as given in [1]. Let $x \in \mathbf{R}\backslash\mathbf{Q}$; the $\mathrm{OCF}(x)$ can be obtained in the following way. First let $a_0 \in \mathbf{Z}$ be such that $x \in (a_0 - \frac{1}{2}, a_0 + \frac{1}{2})$; put $t_0 = x - a_0$, $\varepsilon_1 = \mathrm{sgn}(t_0)$, and

$$\begin{aligned} r_{-1} &= 1, \qquad r_0 = a_0, \\ s_{-1} &= 0, \qquad s_0 = 1, \end{aligned} \tag{3.1}$$

and

$$v_0 = 0.$$

Suppose that $t_i, r_i, s_i, a_i, \varepsilon_{i+1}, v_i$ have been found for $i \leqslant k$, for some $k \geqslant 0$. Then $t_{k+1}, r_{k+1}, s_{k+1}, a_{k+1}, \varepsilon_{k+2}, v_{k+1}$ can be obtained as follows:

$$\begin{aligned} a_{k+1} &= \left\lfloor |t_k|^{-1} + \frac{\lfloor |t_k|^{-1} \rfloor + \varepsilon_{k+1} v_k}{2(\lfloor |t_k|^{-1} \rfloor + \varepsilon_{k+1} v_k) + 1} \right\rfloor, \\ t_{k+1} &= |t_k|^{-1} - a_{k+1}, \\ \varepsilon_{k+2} &= \mathrm{sgn}(t_{k+1}), \\ r_{k+1} &= a_{k+1} r_k + \varepsilon_{k+1} r_{k-1}, \\ s_{k+1} &= a_{k+1} s_k + \varepsilon_{k+1} s_{k-1}, \\ v_{k+1} &= \frac{s_k}{s_{k+1}}. \end{aligned} \tag{3.2}$$

Now $\mathrm{OCF}(x) = [a_0; \varepsilon_1 a_1, \varepsilon_2 a_2, ...]$.

It is not hard to see that

$$t_k = [0; \varepsilon_{k+1} a_{k+1}, \varepsilon_{k+2} a_{k+2}, \ldots]$$
$$v_k = [0; a_k, \varepsilon_k a_{k-1}, \ldots, \varepsilon_2 a_1].$$

The convergents $\{r_k/s_k\}_{k=-1}^{\infty}$ satisfy the recursion relations (3.1) and (3.2) and have the following properties: the sequence $\{r_k/s_k\}_{k=-1}^{\infty}$ forms a subsequence of the sequence of regular convergents $\{P_n/Q_n\}_{n=-1}^{\infty}$; if we define $n(k)$ for $k \geqslant 1$ in such a way that $r_k/s_k = P_{n(k)}/Q_{n(k)}$ then

$$n(k+1) = \begin{cases} n(k)+1 & \Leftrightarrow \quad \varepsilon_{k+2} = +1 \\ n(k)+2 & \Leftrightarrow \quad \varepsilon_{k+2} = -1 \end{cases}$$

when we set

$$n(0) = \begin{cases} 0 & \Leftrightarrow \quad x > 0 \\ 1 & \Leftrightarrow \quad x < 0. \end{cases}$$

The following lemma shows that the pair (T_n, V_n), defined in the previous section, tells us whether the regular convergent P_n/Q_n is an OCF-convergent or not. Define $\Gamma \subset \Omega$ by

$$\Gamma := \left\{ (T, V) \in \Omega : V < \min\left(T, \frac{2T-1}{1-T}\right) \right\}$$

(see Fig. 1) and put

$$\Xi := \Gamma^c = \left\{ (T, V) \in \Omega : V \geqslant \min\left(T, \frac{2T-1}{1-T}\right) \right\}.$$

(3.3) Lemma. *Let x be an irrational number and let $n \geqslant 1$. Then the following three assertions are equivalent*:

(3.4) P_n/Q_n *is not an OCF-convergent,*

(3.5) $B_{n+1} = 1$, $\Theta_{n-1} < \Theta_n$, *and* $\Theta_n > \Theta_{n+1}$,

(3.6) $(T_n, V_n) \in \Gamma$.

Proof. The equivalence of (3.4) and (3.5) is part of Corollary (4.20) of [1]. Here we prove that (3.5) and (3.6) are equivalent.

It is not hard to see that

$$\Theta_{n-1} < \Theta_n \quad \Leftrightarrow \quad T_n > V_n; \tag{3.7}$$

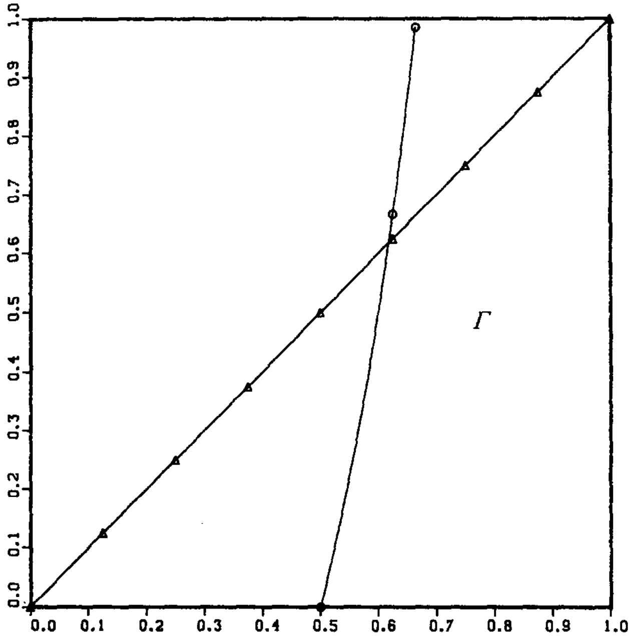

FIGURE 1

see, e.g., Lemma (4.13) of [1]. Furthermore, if $B_{n+1}=1$, then $Q_{n+1}=Q_n+Q_{n-1}$, and we have

$$\begin{aligned}\Theta_{n+1}&=Q_{n+1}\,|Q_{n+1}x-P_{n+1}|\\&=(Q_n+Q_{n-1})\,|(Q_n+Q_{n-1})x-(P_n+P_{n-1})|\\&=(Q_n+Q_{n-1})\,|(Q_{n-1}x-P_{n-1})+(Q_nx-P_n)|\\&=(Q_n+Q_{n-1})(|Q_{n-1}x-P_{n-1}|-|Q_nx-P_n|),\end{aligned}$$

since $|Q_nx-P_n|<|Q_{n-1}x-P_{n-1}|$ while Q_nx-P_n and $Q_{n-1}x-P_{n-1}$ have different signs. Thus

$$\Theta_{n+1}=\Theta_{n-1}\left(1+\frac{Q_n}{Q_{n-1}}\right)-\Theta_n\left(1+\frac{Q_{n-1}}{Q_n}\right).$$

Since

$$\Theta_{n-1}=\frac{V_n}{1+T_nV_n}\qquad\text{and}\qquad\Theta_n=\frac{T_n}{1+T_nV_n}$$

we have

$$T_n = \frac{\Theta_n}{\Theta_{n-1}} \frac{Q_{n-1}}{Q_n},$$

and therefore

$$B_{n+1} = 1 \text{ and } \Theta_{n+1} < \Theta_n \quad \Leftrightarrow \quad V_n < \frac{2T_n - 1}{1 - T_n}. \tag{3.8}$$

Combining (3.7) and (3.8) with the definition of Γ ends the proof of Lemma (3.3).

Next define $\mathscr{U}$ by

$$\mathscr{U}(T, V) = \begin{cases} \mathscr{T}(T, V) & \Leftrightarrow \quad \mathscr{T}(T, V) \in \Xi \\ \mathscr{T}^2(T, V) & \Leftrightarrow \quad \mathscr{T}(T, V) \notin \Xi \end{cases} \quad \text{for} \quad (T, V) \in \Xi.$$

We perform an easy calculation to show that this defines an operator $\mathscr{U}: \Xi \to \Xi$; put

$$\Delta := \left\{ (T, V) \in \Omega : V > \max\left(T, \frac{1+T}{2+T}\right) \right\}.$$

Note that Δ is the reflection of Γ in the diagonal of the unit square. In the sequel we abbreviate $G = (\sqrt{5} + 1)/2$ and $g = (\sqrt{5} - 1)/2$.

(3.9) LEMMA. $\mathscr{T}[\Gamma] = \Delta$.

Proof. One verifies that $\mathscr{T}$ transforms the part of the hyperbola defining Γ into the line segment connecting the points $(1, 1)$ and (g, g) and also that this line segment is transformed into the part of the hyperbola defining Δ; the lemma now easily follows.

(3.10) COROLLARY. $\mathscr{U}[\Xi] \subset \Xi$.

Proof. If $\mathscr{T}(x, y) \notin \Xi$ then $\mathscr{U}(x, y) = \mathscr{T}^2(x, y) \in \mathscr{T}[\Gamma] = \Delta \subset \Xi$. The result follows.

(3.11) *Remark.* If we put

$$\mathscr{U}^0(x, y) = \begin{cases} (x, y) & \Leftrightarrow \quad (x, y) \in \Xi \\ \mathscr{T}(x, y) & \Leftrightarrow \quad (x, y) \notin \Xi \end{cases} \quad \text{for} \quad (x, y) \in \Omega$$

then in fact we have $\mathscr{U}^k(x, 0) = \mathscr{T}^{n(k)}(x, 0)$ for $k \geqslant 0$ and $n(k)$ as before. In this view, Lemma (3.9) is just a reformulation of the fact that in the

OCF-expansion never two consecutive regular convergents are skipped: $\mathscr{T}^n(x, 0) \in \Gamma$ if and only if P_n/Q_n is not an OCF-convergent. But in this case necessarily $\mathscr{T}^{n+1}(x, 0) \in \Xi$, hence P_{n+1}/Q_{n+1} is an OCF-convergent.

4. Ergodic Systems

The most important fact in the metrical theory of continued fractions is that the continued fraction transformation is *ergodic*; it turns out that this is true not only for the RCF but also for the OCF. First we briefly indicate why this is so important and next we present both ergodic systems and their relation; for the generalities on ergodicity we refer to [10].

Let $(\Lambda, \mathscr{A}, \lambda)$ be a *complete probability space*; that is, Λ is a set, $\mathscr{A}$ a σ-algebra of *measurable* subsets of Λ, and the *measure* λ is a countably additive (non-negative) set function on $\mathscr{A}$ such that $\lambda(\Lambda) = 1$ and $\mathscr{A}$ contains all subsets of measure zero. A transformation $\mathscr{S}: \Lambda \to \Lambda$ is called an *endomorphism* of $(\Lambda, \mathscr{A}, \lambda)$ if it is onto and $\mathscr{S}^{-1}A$ is measurable for every $A \in \mathscr{A}$. An endomorphism is *measure-preserving* if $\lambda(\mathscr{S}^{-1}A) = \lambda(A)$ for every $A \in \mathscr{A}$. A set $A \in \mathscr{A}$ is called *invariant* under a transformation $\mathscr{S}$ if $\lambda(A \triangle \mathscr{S}^{-1}A) = 0$, where $\triangle$ denotes the symmetric difference, so $A \triangle \mathscr{S}^{-1}A = (A \backslash \mathscr{S}^{-1}A) \cup (\mathscr{S}^{-1}A \backslash A)$. We call $(\Lambda, \mathscr{A}, \lambda, \mathscr{S})$ an *ergodic system* if $(\Lambda, \mathscr{A}, \lambda)$ is a probability space and $\mathscr{S}$ is a measure-preserving endomorphism of $(\Lambda, \mathscr{A}, \lambda)$ satisfying the *ergodic property*

$$\text{for any } A \in \mathscr{A}: \qquad A \text{ is invariant} \quad \Leftrightarrow \quad \lambda(A) \in \{0, 1\}.$$

A consequence of the ergodic theorem (cf. [10]) that is of particular importance in our applications states that in an ergodic system for almost all $x \in \Lambda$ the orbit of x hits any $A \in \mathscr{A}$ a number of times that is proportional to $\lambda(A)$:

$$\lim_{n \to \infty} \frac{1}{n} \sum_{k=0}^{n-1} \chi_A(\mathscr{S}^k x) = \lambda(A) \qquad \text{for almost all } x,$$

where χ_A denotes the characteristic function on A.

Let $\mathscr{B}$ be the collection of Borel subsets of Ω and μ the measure induced on $\mathscr{B}$ by the density function $(\log 2)^{-1}(1 + xy)^{-2}$, i.e.,

$$\mu(A) := \frac{1}{\log 2} \iint_A \frac{dx\, dy}{(1 + xy)^2} \qquad \text{for every} \quad A \in \mathscr{B}.$$

Then the following fundamental fact was proved in [4].

(4.1) THEOREM. *With the above notation, $(\Omega, \mathscr{B}, \mu, \mathscr{T})$ forms an ergodic system.*

Next we construct the ergodic system for the OCF.

(4.2) LEMMA. $\mu(\Xi) = \dfrac{\log G}{\log 2}$.

Proof. $\mu(\Xi) = 1 - \mu(\Gamma)$ and

$$\begin{aligned}\mu(\Gamma) &= \frac{1}{\log 2}\iint_\Gamma \frac{dT\,dV}{(1+TV)^2}\\ &= \frac{1}{\log 2}\int_{1/2}^{g} \frac{2T-1}{2T^2-2T+1}\,dT + \frac{1}{\log 2}\int_g^1 \frac{T}{T^2+1}\,dT\\ &= 1 + \frac{\log g}{\log 2}.\end{aligned}$$

Hence we find

$$\mu(\Xi) = -\frac{\log g}{\log 2} = \frac{\log G}{\log 2}.$$

(4.3) *Remark.* In fact this lemma shows that for almost all x the OCF-convergents form a subsequence of the RCF-convergents with density $\log G/\log 2$; in other words, Lemma (4.2) is a probabilistic reformulation of the fact that the OCF always forms *fastest* expansions, cf. (3.16) and (4.28) of [1].

Let $\mathscr{B}_\Xi$ be the collection of Borel subsets of the set Ξ that was defined in the previous section. Also, let μ_Ξ be the probability measure induced on $\mathscr{B}_\Xi$ by the measure μ above; that is, μ_Ξ is the renormalized restriction of the measure on Ω to Ξ. From (4.2) we see that this means that μ_Ξ is induced by the density function $(\log G)^{-1}(1+xy)^{-2}$ on Ξ. The transformation $\mathscr{U}$ was defined in the previous section.

(4.4) THEOREM. *$(\Xi, \mathscr{B}_\Xi, \mu_\Xi, \mathscr{U})$ forms an ergodic system.*

Proof. By construction $\mu_\Xi(\Xi) = 1$. Since $\mathscr{T}$ is invertible, so is $\mathscr{U}$. First we will show now that $\mathscr{U}$ is measure preserving with respect to μ_Ξ. Let $A \in \mathscr{B}_\Xi$ and put

$$A_1 := A \cap \Delta, \qquad A_2 := A \backslash A_1; \tag{4.5}$$

hence $A = A_1 \cup A_2$ and $A_1 \cap A_2 = \varnothing$. Since $A_1 \subset \Delta$ we have $\mathscr{T}^{-1}A_1 \subset \Gamma$ and hence $\mathscr{U}^{-1}A_1 = \mathscr{T}^{-2}A_1$. Now $\mathscr{T}^{-1}A_2 \cap \mathscr{T}^{-2}A_1 = \varnothing$ and $\mathscr{U}^{-1}A =$

$\mathscr{T}^{-1}A_2 \cup \mathscr{T}^{-2}A_1$, so $\mathscr{U}$ is measure preserving since $\mathscr{T}$ is ergodic and therefore measure preserving.

This line of argument also contains the core of a proof for the ergodicity of $\mathscr{U}$. Since we know that $\mathscr{U}$ is measure preserving, it suffices to prove for ergodicity that

$$A \in \mathscr{B}_{\Xi} \text{ and } \mathscr{U}^{-1}A = A \quad \Rightarrow \quad \mu_{\Xi}(A) = 0 \text{ or } 1. \tag{4.6}$$

Suppose that some $A \in \mathscr{B}_{\Xi}$ satisfies $\mathscr{U}^{-1}A = A$. Let A_1 and A_2 be defined as in (4.5), and define moreover

$$A_3 := \mathscr{T}^{-1}A_1 \qquad \text{and} \qquad B := A_1 \cup A_2 \cup A_3.$$

Now

$$\mathscr{T}^{-1}B = \mathscr{T}^{-1}A_1 \cup \mathscr{T}^{-1}A_2 \cup \mathscr{T}^{-2}A_1 = \mathscr{T}^{-1}A_1 \cup \mathscr{U}^{-1}(A_1 \cup A_2)$$
$$= \mathscr{T}^{-1}A_1 \cup \mathscr{U}^{-1}(A) = A_3 \cup A_1 \cup A_2 = B,$$

so B is invariant under $\mathscr{T}$. Since $\mathscr{T}$ is ergodic with respect to the measure μ on Ω, the measure of B must be o or 1; that immediately implies $\mu_{\Xi}(A) = 0$ or 1 and we have proved (4.6). This completes the proof of (4.4).

The previous theorem shows that there is an ergodic subsystem of the RCF underlying the OCF. Next we want to find a somewhat more intrinsic description of the ergodic system of the OCF: instead of considering a subsystem of $(T_n, V_n)_{n \geqslant 1}$, we study the corresponding sequence $(t_k, v_k)_{k \geqslant 1}$ in its own right. The connection between (t_k, v_k) and (T_n, V_n) is given in the following lemma; $n(k)$ is defined in Section 3.

(4.7) LEMMA. *We have*

$$\varepsilon_{k+1} = +1 \quad \Rightarrow \quad t_k = T_{n(k)} \quad \textit{and} \quad v_k = V_{n(k)};$$
$$\varepsilon_{k+1} = -1 \quad \Rightarrow \quad t_k = -T_{n(k)-1}T_{n(k)} \quad \textit{and} \quad v_k = 1 - V_{n(k)}.$$

Proof. The relation between t_k and T_n is easily found, and was also given in (4.11) of [1].

For v_k we have the following. Either $s_{k-1} = Q_{n(k)-1}$, $s_k = Q_{n(k)}$, and $v_k = V_{n(k)}$ in case $\varepsilon_{k+1} = 1$, or $\varepsilon_{k+1} = -1$ and then $s_{k-1} = Q_{n(k)-2} = Q_{n(k)} - Q_{n(k)-1}$, $s_k = Q_{n(k)}$, hence $v_k = 1 - V_{n(k)}$.

(4.8) DEFINITIONS. Let $\Upsilon \subset (-1, 1) \times (-1, 1) \subset \mathbf{R} \times \mathbf{R}$ be defined by

$$\Upsilon = \Big\{ (t, v) \in (-1, 1) \times (-1, 1) :$$

$$v \leqslant \min\left(\frac{2t+1}{t+1}, \frac{t+1}{t+2}\right) \text{ and } v \geqslant \max\left(0, \frac{2t-1}{1-t}\right)\Big\}.$$

See also Fig. 2.

Define the operator $\mathscr{W}$ on Υ as

$$\mathscr{W}(t, v) := \left(\left|\frac{1}{t}\right| - \beta(t, v), \frac{1}{\beta(t, v) + \operatorname{sgn}(t) v} \right),$$

where

$$\beta(t, v) := \left\lfloor \left|\frac{1}{t}\right| + \frac{\left\lfloor \left|\frac{1}{t}\right| \right\rfloor + \operatorname{sgn}(t) v}{2\left(\left\lfloor \left|\frac{1}{t}\right| \right\rfloor + \operatorname{sgn}(t) v \right) + 1} \right\rfloor.$$

Let μ_Υ be induced on Υ by $(\log G)^{-1}(1+xy)^{-2}$. Note that for $x \in (-\frac{1}{2}, \frac{1}{2}) \backslash \mathbf{Q}$ we have

$$\mathscr{W}^k(x, 0) = (t_k, v_k) \qquad \text{for} \quad k \geqslant 0.$$

(4.9) THEOREM. *$(\Upsilon, \mathscr{B}_\Upsilon, \mu_\Upsilon, \mathscr{W})$ forms an ergodic system.*

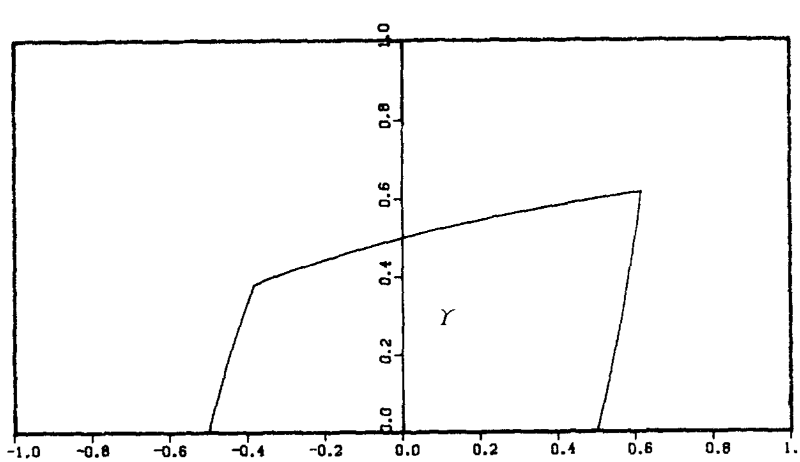

FIGURE 2

Proof. To prove (4.9) one checks that $\mathscr{W}$ is a measure-preserving composition of $\mathscr{U}$ and a transformation mapping (T_n, V_n) to $(t_k, v_k) = (-T_{n-1}T_n, 1 - V_n)$ if $(T_n, V_n) \in \Delta$, leaving the rest of Ξ invariant. We leave the details to the reader.

(4.10) COROLLARY. *For almost all x the sequence $(t_k, v_k)_{k \geqslant 1}$ is distributed over Υ according to the density function*

$$\frac{1}{\log G} \frac{1}{(1+tv)^2}.$$

Proof. This is an immediate consequence of (4.9); one may apply the same techniques as in the proof of Theorem 3 of [5] or that of Theorem 5 in [7].

(4.11) COROLLARY. *For every $x \in (-\frac{1}{2}, \frac{1}{2}) \backslash \mathbf{Q}$ we have, for every $k \geqslant 1$,*

$$-\frac{1}{2} \leqslant t_k \leqslant g \qquad \text{and} \qquad 0 \leqslant v_k \leqslant g.$$

5. DISTRIBUTION OF θ_k

In this section we study the distribution of the sequence $(\theta_{k-1}, \theta_k)_{k \geqslant 1}$ and hence that of $(\theta_k)_{k \geqslant 1}$, where $\theta_k = \theta_k(x) = s_k |s_k x - r_k|$, $k \geqslant 1$. We already know that for every x the sequence $(\theta_k)_{k \geqslant 1}$ forms a subsequence of $(\Theta_k)_{k \geqslant 1}$ with the property that $\theta_k \leqslant \frac{1}{2}$ for all $k \geqslant 1$. Here we will give the metric theory; our main theorem gives for almost all x the limiting distribition of the sequence $(\theta_{k-1}, \theta_k)_{k \geqslant 1}$. The method we use here was previously employed in [5] and in [7] to derive the corresponding results for the RCF and the NICF, respectively. First some more notation. By Π we will denote the following subspace of $\mathbf{R} \times \mathbf{R}$ (see Fig. 3):

$$\Pi := \{(w, z) \in \mathbf{R} \times \mathbf{R} : w > 0, z > 0, 4w^2 + z^2 < 1, w^2 + 4z^2 < 1\}.$$

(5.1) THEOREM. (i) *For every irrational number x, $(\theta_{k-1}, \theta_k) \in \Pi$ for every $k \geqslant 1$;*

(ii) *For almost all x the sequence $(\theta_{k-1}, \theta_k)_{k \geqslant 1}$ is distributed over Π according to the density function f, where*

$$f(w, z) = \frac{1}{\log G} \left(\frac{1}{\sqrt{1 - 4wz}} + \frac{1}{\sqrt{1 + 4wz}} \right), \tag{5.2}$$

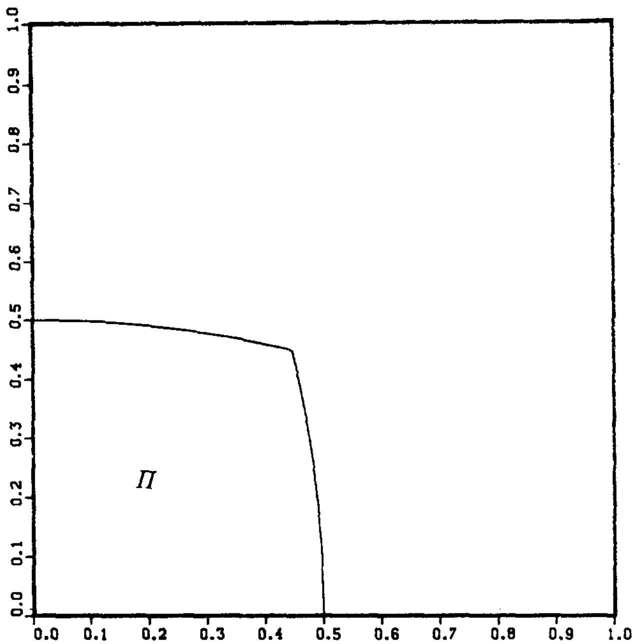

FIG. 3. The region Π.

so for almost all x we have, for every $a \geqslant 0$, $b \geqslant 0$,

$$\lim_{k \to \infty} \frac{1}{k} \# \{ j : j \leqslant k, \theta_j < a \text{ and } \theta_{j+1} < b \} = \int_0^a \int_0^b f(w, z)\, dw\, dz \qquad (5.3)$$

with $f(w, z)$ as in (5.2) for $(w, z) \in \Pi$ and zero otherwise.

Proof. It is a direct consequence of Theorem (4.7) that for every x,

$$(t_k, v_k) \in \Upsilon \qquad \text{for all} \quad k \geqslant 1. \qquad (5.4)$$

We now define a two-to-one mapping $\psi : \Upsilon \mapsto \Pi$ as

$$\psi(t, v) := \left(\frac{v}{1 + tv}, \frac{\varepsilon(t)\, t}{1 + tv} \right) \qquad \text{for} \quad (t, v) \in \Pi,$$

where $\varepsilon(t) = \operatorname{sgn}(t)$. Then clearly $\psi(0, v) = (v, 0)$ and $\psi(t, 0) = (0, \varepsilon(t)\, t)$. Furthermore it is not hard to see that ψ maps

$\left(t, \dfrac{2t-1}{1-t}\right)$, for $\dfrac{1}{2} \leqslant t \leqslant g$, to (w, z) satisfying $w^2 + 4z^2 = 1$, with $\dfrac{1}{\sqrt{5}} \leqslant w \leqslant \dfrac{1}{2}$

and that ψ maps

$$\left(t, \frac{1+t}{2+t}\right), \text{ for } 0 \leqslant t \leqslant g, \text{ to } (w, z) \text{ satisfying } 4w^2 + z^2 = 1, \text{ with } 0 \leqslant w \leqslant \frac{1}{\sqrt{5}}.$$

Thus $\Upsilon_1 := \{(t, v) \in \Upsilon : \varepsilon(t) = 1\}$ is mapped onto Π, and since ψ likewise maps

$$\left(t, \frac{1+t}{2+t}\right), \text{ for } -g^2 \leqslant t \leqslant 0, \text{ to } (w, z) \text{ satisfying } 4w^2 + z^2 = 1, \text{ with } 0 \leqslant w \leqslant \frac{1}{\sqrt{5}}$$

and

$$\left(t, \frac{2t+1}{1+t}\right), \text{ for } -\frac{1}{2} \leqslant t \leqslant -g^2, \text{ to } (w, z) \text{ satisfying } w^2 + 4z^2 = 1,$$

$$\text{with } \frac{1}{\sqrt{5}} \leqslant w \leqslant \frac{1}{2},$$

we also find that $\Upsilon_{-1} := \{(t, v) \in \Upsilon : \varepsilon(t) = -1\}$ is mapped onto Π.

Next we show that in fact for every x

$$\psi(t_k, v_k) = (\theta_{k-1}, \theta_k), \qquad k \geqslant 1. \tag{5.5}$$

Namely, from the well known relation

$$x = \frac{r_k + t_k r_{k-1}}{s_k + t_k s_{k-1}}, \tag{5.6}$$

which holds for every semi-regular expansion, we get

$$\theta_k = \frac{\varepsilon_{k-1} t_k}{1 + t_k v_k}. \tag{5.7}$$

But we find also

$$\theta_{k-1} = \frac{\varepsilon_k t_{k-1}}{1 + t_{k-1} v_{k-1}} = \varepsilon_k \left(\frac{1}{\dfrac{1}{t_{k-1}} + v_{k-1}} \right) = \varepsilon_k \frac{1}{\varepsilon_k (a_k + t_k) + \dfrac{s_{k-2}}{s_{k-1}}}$$

$$= \left(t_k + \frac{a_k s_{k-1} + \varepsilon_k s_{k-2}}{s_{k-1}} \right)^{-1} = \left(t_k + \frac{1}{v_k} \right)^{-1};$$

hence

$$\theta_{k-1} = \frac{v_k}{1 + t_k v_k}. \tag{5.8}$$

By (5.4)–(5.8) we have that $(\theta_{k-1}, \theta_k) \in \Pi$ for every x (and $k \geqslant 1$). This proves (5.1)(i).

We now consider $\psi_1 = \psi \mid \Upsilon_1$ and $\psi_{-1} = \psi \mid \Upsilon_{-1}$; each of these is a C^1-diffeomorphism. The Jacobian J of ψ is easily seen to be $\varepsilon(t)(tv-1)(1+tv)^{-3}$. The distribution of $(\theta_{k-1}, \theta_k)_{k \geqslant 1}$ over Π can then be determined for almost all x. Namely, since $(t_k, v_k)_{k \geqslant 1}$ is distributed over Υ according to $(\log G)^{-1}(1+tv)^{-2}$ by (4.8), both ψ_1 and ψ_{-1} contribute

$$\frac{1}{|\det J| \log G} \frac{1}{(1+tv)^2} = \frac{1}{\log G} \frac{1+tv}{1-tv}$$

to the distribution of $(\theta_{k-1}, \theta_k)_{k \geqslant 1}$. Since by (5.7) and (5.8)

$$\varepsilon(t_k) = \varepsilon_{k+1} = 1 \qquad \Rightarrow \qquad \left(\frac{1 - t_k v_k}{1 + t_k v_k}\right)^2 = 1 - 4\theta_{k-1}\theta_k,$$

and

$$\varepsilon(t_k) = \varepsilon_{k+1} = -1 \qquad \Rightarrow \qquad \left(\frac{1 - t_k v_k}{1 + t_k v_k}\right)^2 = 1 + 4\theta_{k-1}\theta_k,$$

the density is given for almost all x by

$$f(w, z) = \frac{1}{\log G}\left(\frac{1}{\sqrt{1-4wz}} + \frac{1}{\sqrt{1+4wz}}\right).$$

This completes the proof of (5.1).

(5.9) COROLLARY. (i) *For every irrational number x we have*

$$0 < \theta_{k-1} + \theta_k < \frac{2}{\sqrt{5}} \qquad \textit{for every} \quad k \geqslant 1.$$

(ii) *For almost all x the sequence $(\theta_{k-1} + \theta_k)_{k \geqslant 1}$ is distributed over $[0, 2/\sqrt{5}]$ according to the density function g, where*

$$g(z)=\begin{cases} \dfrac{1}{\log G}\left(\log\sqrt{1+z}-\log\sqrt{1-z}+\arctan z\right) & \textit{for} \quad 0\leqslant z\leqslant\dfrac{1}{2} \\ \dfrac{1}{2\log G}\left(\log\left(\dfrac{5\sqrt{5-4z^2}-5z}{\sqrt{5-4z^2}+z}\right)+2\arcsin\left(\dfrac{-3z+2\sqrt{5-4z^2}}{5\sqrt{2+z^2}}\right)\right) & \\ & \textit{for} \quad \dfrac{1}{2}\leqslant z\leqslant\dfrac{2}{\sqrt{5}}. \end{cases}$$

Proof. (i) Immediate from (5.1)(i).

(ii) Integrate (5.2) for $w+z$ constant.

For a picture of g see Fig. 4.

(5.10) COROLLARY. *For every irrational number x we have*

$$\min(\theta_{k-1}, \theta_k) < \frac{1}{\sqrt{5}} \qquad \textit{for every} \quad k \geqslant 1.$$

Proof. Immediate from (5.9)(i).

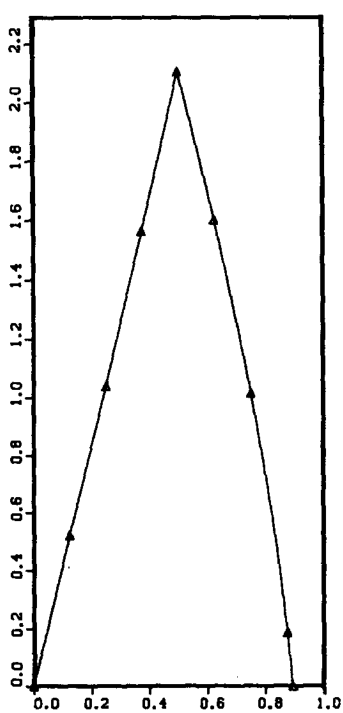

FIG. 4. The function g.

(5.11) *Remarks.* Notice that by Hurwitz' theorem the constant $1/\sqrt{5}$ in (5.10) is "best possible"; compare with (4.30) of [1]. The result also holds for rational x.

(5.12) COROLLARY. (i) *For every irrational number x we have*

$$0 \leqslant |\theta_{k-1} - \theta_k| \leqslant \tfrac{1}{2} \qquad \textit{for every} \quad k \geqslant 1.$$

(ii) *For almost all x the sequence $(|\theta_{k-1} - \theta_k|)_{k \geqslant 1}$ is distributed over $[0, \frac{1}{2}]$ according to the density function g^*, where*

$$g^*(z) = \frac{1}{\log G}\left(\log\left(\frac{5\sqrt{5-4z^2}-5z}{1+z}\right) - \arctan z + \arcsin\left(\frac{-3z+2\sqrt{5-4z^2}}{5\sqrt{1+z^2}}\right)\right).$$

Proof. Straightforward.

From Theorem (5.1) we can also deduce the distribution of the sequence $(\theta_k)_{k \geqslant 1}$.

(5.13) THEOREM. *For every irrational number x we have*

$$\theta_k \in (0, \tfrac{1}{2}) \qquad \textit{for every} \quad k \geqslant 1,$$

and for almost all x the sequence $(\theta_k)_{k \geqslant 1}$ is distributed over $(0, \frac{1}{2})$ according to the density function h, where

$$h(z) = \begin{cases} \dfrac{1}{\log G} & \textit{for} \quad 0 \leqslant z \leqslant \dfrac{1}{\sqrt{5}} \\[2ex] \dfrac{1}{\log G} \dfrac{\sqrt{1-4z^2}}{z} & \textit{for} \quad \dfrac{1}{\sqrt{5}} \leqslant z \leqslant \dfrac{1}{2}, \end{cases}$$

so for almost all x

$$\lim_{k \to \infty} \frac{1}{k} \# \{j : j \leqslant k \textit{ and } \theta_j(x) \leqslant z\} = \int_0^z h(w)\, dw$$

$$= \begin{cases} \dfrac{z}{\log G} & \textit{for} \quad 0 \leqslant z \leqslant \dfrac{1}{\sqrt{5}}, \\[2ex] \dfrac{1}{\log G}\left(\sqrt{1-4z^2} + \log\left(G\,\dfrac{1-\sqrt{1-4z^2}}{2z}\right)\right) & \textit{for} \quad \dfrac{1}{\sqrt{5}} \leqslant z \leqslant \dfrac{1}{2}. \end{cases}$$

Proof. Determine $\iint f(v, w)\, dw\, dv$ over $(v, w) \in \Pi$ such that $v < z$ (with the notation of (5.1)).

(5.14) *Remarks.* Figure 5 depicts function $H(z) = \int h(w)\, dw$ for the sequence $(\theta_k)_{k \geq 1}$. In Fig. 6 we compare this function H with the functions H_1, $H_{1/2}$, and H_α. Here H_1 is the distribution function for the sequence Θ_n of approximation constants of the regular continued fraction expansion, $H_{1/2}$ the one for the nearest integer continued fraction, and H_α the one for the α-expansion with $\alpha = \alpha_0 = 0.55821...$, which gives of all α-expansions the one with smallest first moment of H_α. For all these results see [2].

Seeing this, one might wonder whether the OCF always gives an expansion with minimal mean of the θ's. That this is true in a very strong sense will be shown in a forthcoming paper [3]. Here we confine ourselves to computing this mean in (5.15).

(5.15) COROLLARY. *For almost all irrational x*

$$\lim_{N \to \infty} \sum_{j \leq N} \theta_j(x) = \frac{1}{4 \log G} \arctan \frac{1}{2} = 0.24087... . \tag{5.16}$$

Proof. Compute the first moment of H.

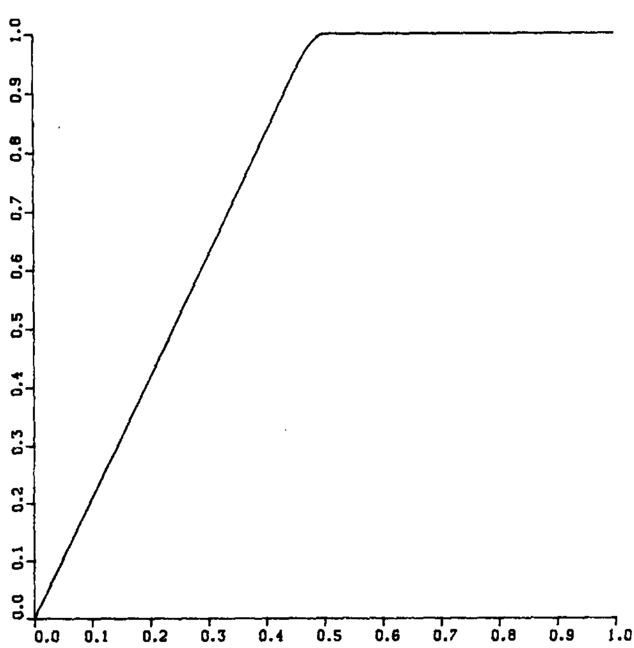

FIG. 5. The distribution of $(\theta_k)_{k \geq 1}$.

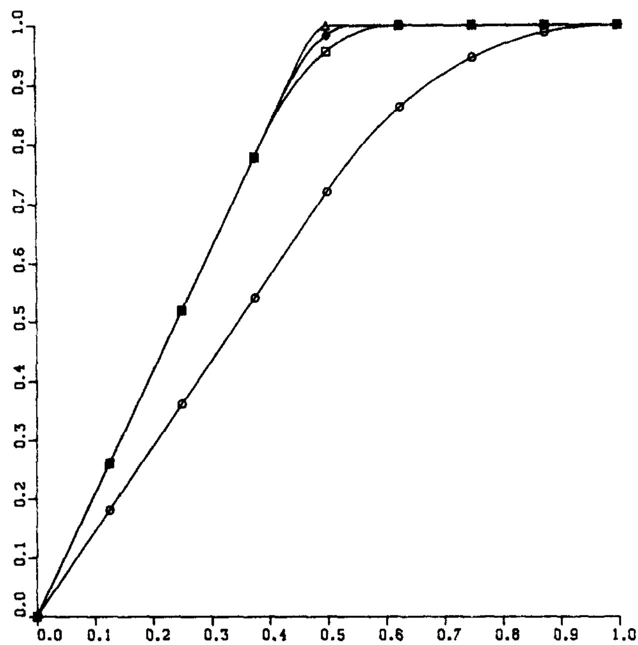

FIG. 6. The distribution functions H, H_α, $H_{1/2}$, H_1 (from left to right).

(5.17) *Remarks.* The value given in (5.16) should be compared to the corresponding values

$$\frac{1}{4\log 2} = 0.36067\ldots \qquad \text{for the RCF,}$$

$$0.25 \qquad \text{for Minkowski's diagonal continued fraction (DCF),}$$

$$\frac{\sqrt{5}-2}{2\log G} = 0.24528\ldots \qquad \text{for the NICF and Hurwitz' singular continued fraction,}$$

$$\frac{\sqrt{8G+6}-2G-1}{\log G} = 0.24195\ldots \qquad \text{for the } \alpha\text{-expansion with } \alpha = \alpha_0 = 0.55821\ldots \text{ as in (5.14).}$$

For all this see [2, 3, 6].

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