

Modular properties of matrix coefficients of corepresentations of a locally compact quantum group

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Plancherel (Mautner, Segal)

For G l.c. group, 2^{nd} countable, type I, unimodular.

$$\lambda \times \rho : G \times G \rightarrow L^2(G) : (x, y) \mapsto \lambda(x)\rho(y)$$

$$\lambda \times \rho \simeq \int_{\pi \in \hat{G}}^{\oplus} \pi \otimes \bar{\pi} d\mu(\pi)$$

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Intertwiner being given by

$$L^2(G) \rightarrow \int_{\pi \in \hat{G}}^{\oplus} \mathcal{H}_{\pi} \otimes \overline{\mathcal{H}_{\pi}} d\mu(\pi)$$

$$L^1(G) \cap L^2(G) \ni f \mapsto \int_{\pi \in \hat{G}}^{\oplus} \pi(f) d\mu(\pi),$$

where $\pi(f) = \int_G f(x)\pi(x) d_l x$ is Hilbert-Schmidt.

Plancherel (Duflo, Moore)

For G l.c. group, 2^{nd} countable, type I, **unimodular**.

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where $\pi(f) = \int_G f(x)\pi(x)dx$ and $(D_{\pi})_{\pi \in \hat{G}}$ a field of positive self-adjoint operators.

Locally compact quantum groups (Kustermans, Vaes)

A **von Neumann algebraic quantum group** consists of:

- a **von Neumann algebra** M ;
- a **comultiplication**, i.e. a unital normal $*$ -homomorphism $\Delta: M \rightarrow M \otimes M$ such that $(\Delta \otimes \iota)\Delta = (\iota \otimes \Delta)\Delta$;
- two normal semi-finite faithful **Haar weights** φ, ψ on M , i.e.

$$\begin{aligned} \varphi((\omega \otimes \iota)\Delta(x)) &= \varphi(x)\omega(1), & \forall \omega \in M_*^+, \forall x \in \mathfrak{m}_\varphi^+, \\ \psi((\iota \otimes \omega)\Delta(x)) &= \psi(x)\omega(1), & \forall \omega \in M_*^+, \forall x \in \mathfrak{m}_\psi^+. \end{aligned}$$

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- We denote $(\mathcal{H}, \pi, \Lambda)$ for the GNS-construction w.r.t. φ .
- $M \simeq \pi(M) \Rightarrow$ we assume that M acts on the GNS-space \mathcal{H} .

- To every quantum group (M, Δ) , one can associate a dual quantum group $(\hat{M}, \hat{\Delta})$ such that $(\hat{\hat{M}}, \hat{\hat{\Delta}}) = (M, \Delta)$.

Classical example

Let G be a locally compact group. Set $M = L^\infty(G)$ and

$$\begin{aligned} \Delta : L^\infty(G) &\rightarrow L^\infty(G) \otimes L^\infty(G) \simeq L^\infty(G \times G) \\ (x \mapsto f(x)) &\mapsto ((x, y) \mapsto f(xy)) \end{aligned}$$

Let φ and ψ be the integrals with respect to the left and right Haar measure.

Then, $\hat{M} = \mathcal{L}(G)$ the group von Neumann algebra and $\hat{\Delta}(\lambda(x)) = \lambda(x) \otimes \lambda(x)$, $x \in \hat{M}$.

Corepresentations

A unitary **corepresentation** of a l.c. quantum group (M, Δ) on a Hilbert space \mathcal{H}_U is a **unitary operator** $U \in M \otimes B(\mathcal{H}_U)$ that satisfies the relation $(\Delta \otimes \iota)U = U_{13}U_{23}$.

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Multiplicative unitary

There exists a unique **unitary element** $W \in B(\mathcal{H} \otimes \mathcal{H})$ such that

$$W^*(\Lambda(x) \otimes \Lambda(y)) = (\Lambda \otimes \Lambda)(\Delta(y)(x \otimes 1)), \quad \forall x, y \in \mathfrak{n}_\varphi.$$

- It follows that W is a corepresentation.
- For $(L^\infty(G), \Delta_G)$, $W \in L^\infty(G) \otimes B(L^2(G))$ corresponds to the left regular representation.

Plancherel theorem for quantum groups (Desmedt 2003, PhD Thesis)

Let (M, Δ) be a l.c.q.g, s.t. \hat{M} is type I, satisfying conditions (...). W decomposes as a direct integral of irreducible, unitary corepresentations:

$$W \simeq \int_{\text{IC}(M)}^{\oplus} \dim(U) \cdot U d\mu(U).$$

The intertwiner is given by the formula:

$$\xi(\alpha) \mapsto \int_{\text{IC}(M)}^{\oplus} (\alpha \otimes \iota)(U) \cdot D_U^{-1} d\mu(U), \quad \alpha \in \mathcal{I} \subseteq M_*.$$

- There is a similar theorem decomposing the right regular corepresentation. It comes with positive self-adjoint operators E_U , $U \in \text{IC}(M)$.

Modular automorphism group

There exists a unique **1-parameter group of automorphisms**

$\sigma_t^\varphi : M \rightarrow M$ for which φ satisfies the **KMS-property**.

σ_t^φ is called the **modular automorphism group**.

- σ_t^φ measures how far φ is from being tracial.

$$\varphi(xy) = \varphi(y\sigma_{-i}^\varphi(x)), \quad y \in \mathfrak{m}_\varphi, x \text{ analytic for } \sigma_t^\varphi.$$

Modular properties of matrix coefficients

Modular formula (C-Koelink, arXiv:1003.2278)

Suppose that (M, Δ) is unimodular, \hat{M} type I (. . .). Let $(\xi_U)_U, (\eta_U)_U$ be square integrable vector fields. The modular automorphism group of the Haar weight φ can be expressed as:

$$\sigma_t^\varphi \left(\int_{\text{IC}(M)} (\iota \otimes \omega_{\xi_U, \eta_U})(U) d\mu(U) \right) = \int_{\text{IC}(M)} (\iota \otimes \omega_{D_U^{2it}\xi_U, E_U^{2it}\eta_U})(U) d\mu(U).$$

- μ is the Plancherel measure;
- D_U, E_U the operators from the Plancherel theorems;
- $\text{IC}(M)$ eq. classes of irreducible, unitary corep's of M .

Some remarks:

- In the compact case, the integrals can be omitted.
- The formula is closely related to formulas contained in a preprint by Van Daele ¹.
- Denote $(\mathcal{H}, \pi, \Lambda)$ and $(\mathcal{H}, \iota, \Gamma)$ for the GNS-representations of φ and ψ . The proof is obtained by giving the polar decomposition of the closure of the operator

$$S_{\psi, \varphi, 0} : \Lambda(x) \mapsto \Gamma(x^*) \quad x \in \mathfrak{n}_\varphi \cap \mathfrak{n}_\psi^*,$$

in terms of matrix coefficients. So $S_{\psi, \varphi} = J_{\psi, \varphi} \nabla_\varphi^{\frac{1}{2}}$.

- In the non-unimodular case, one still has $\sigma_t^\psi(x) = \nabla_\varphi^{it} x \nabla_\varphi^{-it}$.

¹arXiv:math/0609502

Plancherel transform of $\widetilde{SU_q(1, 1)}$

Work by Koelink, Kustermans and Groenevelt^a:

$$W \simeq \bigoplus_{p \in \mathbb{q}^{\mathbb{Z}}} \left(\int_{[-1,1]}^{\oplus} W_{p,x} dx \oplus \bigoplus_{x \in I_p} W_{p,x} \right),$$

obtained by spectral decomposition of Casimir operator.

Q: What are $D_{p,x}$, $E_{p,x}$?

^aThe dual quantum group for the quantum group analog of the normalizer of $SU(1, 1)$ in $SL(2, \mathbb{C})$, to appear in IMRN (2010).

Plancherel transform of $\widetilde{SU_q(1,1)}$ (Desmedt)

$$W \simeq \bigoplus_{\rho \in q^{\mathbb{Z}}} \left(\int_{[-1,1]}^{\oplus} W_{\rho,x} dx \oplus \bigoplus_{x \in I_{\rho}} W_{\rho,x} \right),$$

- Desmedt computes $D_{\rho,x}$, $E_{\rho,x}$, $x \in I_{\rho}$, $\rho \in q^{\mathbb{Z}}$.
- Using special function theory, orthogonality relations.

Plancherel transform of $\widetilde{SU_q(1, 1)}$ (Koelink, M.C.)

$$W \simeq \bigoplus_{p \in q^{\mathbb{Z}}} \left(\int_{[-1, 1]}^{\oplus} W_{p, x} dx \oplus \bigoplus_{x \in I_p} W_{p, x} \right),$$

- We find

$$\begin{aligned} D_{p, x} e_m^{\epsilon \eta} &= pq^m c(p, x) e_m^{\epsilon \eta}, \\ E_{p, x} e_m^{\epsilon \eta} &= q^{-m} c(p, x) e_m^{\epsilon \eta}, \end{aligned}$$

$x \in [-1, 1]$ and $\{e_m^{\epsilon \eta} \mid \epsilon, \eta \in \{-, +\}, m \in \mathbb{Z}\}$ is a basis for $\mathcal{H}_{p, x}$. Function $c(p, x)$ depends on the Plancherel measure.

- Using the modular formula.
- Method is also applicable to the discrete series (and generally shorter).

Fourier transforms

Goal is ...

To define for $p \in [1, 2]$ a Fourier transform

$$\mathcal{F}_p : L^p(M) \rightarrow L^q(\hat{M}),$$

where $\frac{1}{p} + \frac{1}{q} = 1$.

Background/Motivation:

- Motivated by algebraic L^2 -Fourier transform by Van Daele (arXiv:math/0609502).
- Application of interpolation theory, i.e. non-commutative versions of the Riesz-Thorin theorem.

Riesz-Thorin

Let T_0, T_1 be bounded maps

$$T_0 : L^{p_0}(X, \mu) \rightarrow L^{q_0}(Y, \nu);$$

$$T_1 : L^{p_1}(X, \mu) \rightarrow L^{q_1}(Y, \nu),$$

such that the restrictions of T_0 and T_1 to $L^{p_0}(X, \mu) \cap L^{p_1}(X, \mu)$ agree.

Then for $\theta \in [0, 1]$ there exists a unique bounded map

$$T : L^p(X, \mu) \rightarrow L^q(Y, \nu),$$

where $\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}$ and $\frac{1}{q} = \frac{1-\theta}{q_0} + \frac{\theta}{q_1}$. Moreover,

$$\|T\| \leq \|T_0\|^{1-\theta} \|T_1\|^\theta.$$

Idea of what we will do:

- Define a L^2 -Fourier transform, motivated by classical transform.
- Define the intersection of M and M_* in such a way that one can ...
- ... determine a L^1 -Fourier transform that is compatible with the L^2 -Fourier transform.
- Use interpolation theory (Riesz-Thorin-like theory) to define a L^p -Fourier transform.

What are non-commutative L^p -spaces?

- There are various definitions of non-commutative L^p -spaces:
 - Haagerup;
 - Connes/Hilsum;
 - Terp;
 - Izumi;
 - L^p -spaces associated with semi-finite von Neumann algebras.
- All definitions are equivalent (however, the latter is less general).

L^p -spaces with respect to a trace

Let M be a von Neumann algebra with trace τ .

- For a positive, self-adjoint operator $a \in M$, we put

$$\tau(a) = \sup_{n \in \mathbb{N}} \tau \left(\int_0^n \lambda dE_\lambda \right),$$

where $a = \int_0^\infty \lambda dE_\lambda$.

- For $p \in [1, \infty)$, put

$$L^p(M, \tau) = \{a \in M \mid a \text{ cl., d.d.}, \tau(|a|^p) < \infty\}$$

and set $\|a\|_p = \tau(|a|^p)^{1/p}$.

Type I factor

Let $M = B(\mathcal{H})$. Then $L^p(M, \text{Tr})$ are exactly the compact operators for which the singular values form a sequence in $l^p(\mathbb{N})$.

Type I von Neumann algebra

Let $M = \int_X^\oplus B(\mathcal{H}_x) d\mu(x)$. Then $L^p(M, \int \text{Tr}_x(\cdot) d\mu(x))$ are exactly the fields of operators $(a_x)_{x \in X}$ for which $x \mapsto \text{Tr}_x(a_x)$ is in $L^p(X)$.

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Warning: Connes/Hilsum definition is far more technical.

Connes/Hilsum definition of non-commutative L^p -spaces

- Let ϕ be a n.s.f. weight on the commutant M' .
- A closed, d.d. operator x is called **γ -homogeneous** ($\gamma \in \mathbb{R}$) if $ax \subseteq x\sigma_{i\gamma}^\phi(a)$ for all $a \in M'$ analytic w.r.t. ϕ .

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- Let φ be a weight on M and let $d = d\varphi/d\phi$ be the spatial derivative, then d is closed, d.d. and -1 -homogeneous.
- In fact, if a is a closed, d.d., positive, -1 -homogeneous operator, then $a = d\varphi_a/d\phi$ for some n.s. weight φ_a on M . For such a , put $\|a\|_1 = \varphi_a(1)$.

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- The **L^p -space** $L^p(M)$ is defined as the space of all **closed, d.d., $-1/p$ -homogeneous operators a such that $\| |a|^p \|_1 < \infty$.**
- For $a \in L^p(M)$, put $\|a\|_p = \| |a|^p \|_1^{\frac{1}{p}}$.

Facts:

- There is an isometric isomorphism $M_* \simeq L^1(M)$.
- $L^2(M)$ carries an inner product for which it is a Hilbert space. Moreover, $L^2(M) \simeq \mathcal{H}_\varphi$, the GNS-space.
- If ϕ is a trace, then γ -homogeneous is just affiliated. In that case $L^p(M) = L^p(M, \tau)$ where τ is a trace on M such that $d\tau/d\phi = 1$.
- If ϕ is not a trace $L^p(M) \cap L^{p'}(M) = \{0\}$, because of homogeneity relation:

$$ax \subseteq x\sigma_{-i/p}^\phi(a) \text{ for all } a \in M' \text{ analytic w.r.t. } \phi.$$

$L_0 := \{x \in \mathfrak{n}_\varphi \mid \exists {}_x\varphi \in M_* \text{ s.t. } \forall y \in \mathfrak{n}_\varphi : {}_x\varphi(y^*) = \varphi(y^*x)\}$.

We have $\mu_\infty : L_0 \subseteq M$ and $\mu_1 : L_0 \hookrightarrow M_* : x \mapsto {}_x\varphi$.

Interpolation

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$$L_1 := \{x \in \mathfrak{n}_\varphi^* \mid \exists \varphi_x \in M_* \text{ s.t. } \forall y \in \mathfrak{n}_\varphi : \varphi_x(y) = \varphi(xy)\}.$$

Similarly, $\mu'_\infty : L_1 \subseteq M$ and $\mu'_1 : L_1 \hookrightarrow M_* : x \mapsto \varphi_x$.

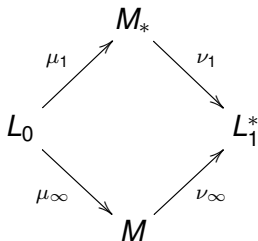
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Where $\nu_1 = \mu'_\infty|_{M_*}$, $\nu_\infty = \mu'_1$.

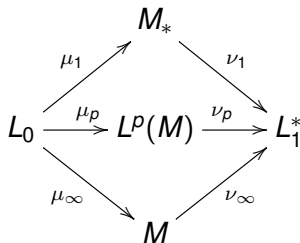
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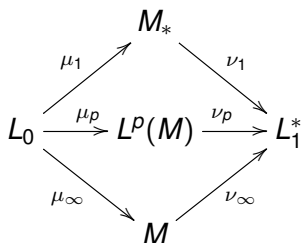
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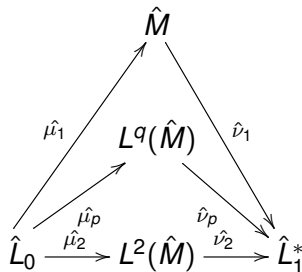
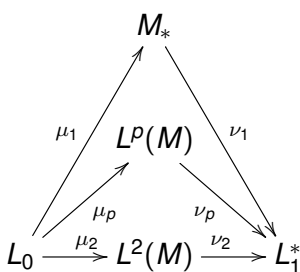


- We can prove that $\nu_\infty(\mu_\infty(L_0)) = \nu_\infty(M) \cap \nu_1(M_*)$, i.e. L_0 is the intersection of M in M_* if considered as subspaces of L_1^* .
- More general, the diagram defines intersections between various L^p -spaces.
- We can prove that $L^p(M)$ are complex interpolation spaces of (M, M_*) .

Now, let (M, Δ) be a l.c. quantum group and let $(\hat{M}, \hat{\Delta})$ be its dual. Let $\frac{1}{p} + \frac{1}{q} = 1$, $p \in [1, 2]$.

Theorem (C)

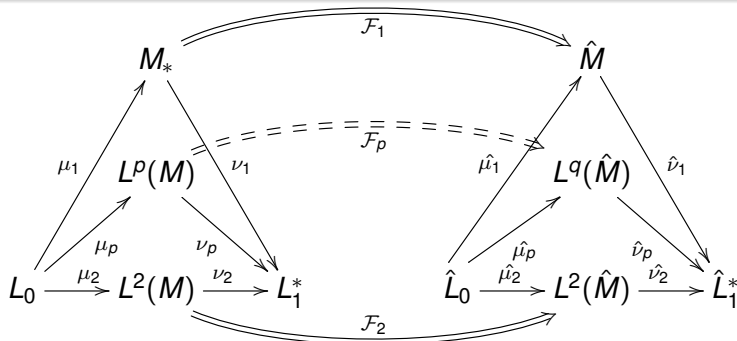
- $L^p(M)$ is a complex int. space between $L^2(M)$ and M_* .
- $L^q(\hat{M})$ is a complex int. space between $L^2(\hat{M})$ and \hat{M} .



Now, let (M, Δ) be a l.c. quantum group and let $(\hat{M}, \hat{\Delta})$ be its dual. Let $\frac{1}{p} + \frac{1}{q} = 1$, $p \in [1, 2]$.

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Now let $\mathcal{F}_2 : \mathcal{H} \rightarrow \mathcal{H}$ be the identity.

- Let G be abelian. Let $(L^\infty(G), \Delta_G)$ be the usual quantum group and let $(\mathcal{L}(G), \hat{\Delta}_G)$ be its dual ($\mathcal{L}(G)$ denotes the group von Neumann algebra). Then,

$$(\mathcal{L}(G), \hat{\Delta}_G) \simeq (L^\infty(\hat{G}), \Delta_{\hat{G}})$$

by means of the L^2 -Fourier transform.

Conclusion: the L^2 -Fourier transform is indeed trivial in the classical abelian case, since it is implicitly used in the construction of the dual quantum group.

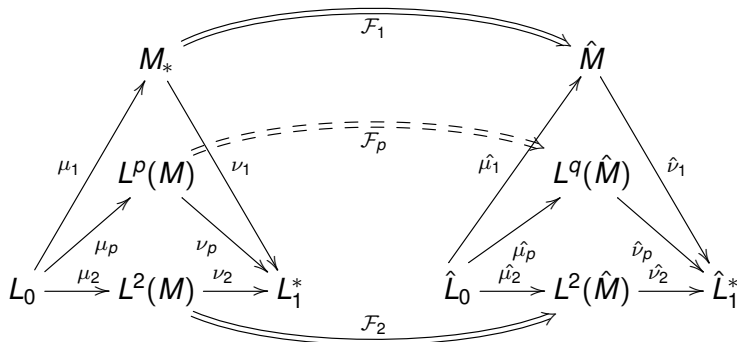
Inverse theorem

\mathcal{F}_2 is unitary and $\mathcal{F}_2^{-1} = \hat{\mathcal{F}}_2$.

The diagram determines \mathcal{F}_1 . We find

$$\mathcal{F}_1 : M_* \rightarrow \hat{M} : \omega \mapsto (\omega \otimes \iota)(W),$$

which is indeed a bounded map (!).



Theorem: L^p -Fourier transform (C)

Let (M, Δ) l.c. quantum group. Let $p \in [1, 2]$ and set q by $\frac{1}{p} + \frac{1}{q} = 1$.

There exists a unique bounded linear map

$$\mathcal{F}_p : L^p(M) \rightarrow L^q(\hat{M})$$

such that

$$\hat{\nu}_q \circ \mathcal{F}_p \circ \mu_p = \hat{\nu}_\infty \circ \mathcal{F}_1 \circ \mu_1 = \hat{\nu}_2 \circ \mathcal{F}_2 \circ \mu_2.$$

Moreover, $\|\mathcal{F}_p\| \leq 1$.

For $\omega_1, \omega_2 \in M_*$, define the **convolution product**

$$\omega_1 * \omega_2 = (\omega_1 \otimes \omega_2) \circ \Delta \in M_*.$$

Theorem: Convolution product (C)

Let (M, Δ) l.c. quantum group. Let $p \in [1, 2]$.

For $\omega \in M_*$, there is a bounded operator $\omega *^p : L^p(M) \rightarrow L^p(M)$, which is uniquely determined by $\omega *^p \mu_p(x) = \rho_p(\omega * x\varphi)$, where $\rho_p : L^1(M) \cap L^2(M) \hookrightarrow L^p(M)$ is the natural embedding.

Classically, this operator corresponds to the convolution product.

Moreover, for $\omega \in M_*$, $a \in L^p(M)$,

$$\mathcal{F}_1(\omega)\mathcal{F}_p(a) = \mathcal{F}_p(\omega *^p a).$$

Remarks on the dependence of the choice of $M \cap M_*$.

- Recall that we made a choice for the intersection of M and M_* .
- There are various choices for this intersection depending on a parameter $z \in \mathbb{C}$ (Izumi). In case M is the vNA of a quantum group only choices $z = -\frac{1}{2} + it$, $t \in \mathbb{R}$ gives a bounded (!) L^1 -Fourier transform.
- Here, we have presented the case $z = -\frac{1}{2}$. Also the cases $z = -\frac{1}{2} + it$, $t \in \mathbb{R} \setminus \{0\}$ are worked out.
- In an article by Daws and Runde² the natural value of this parameter z is questioned. The Fourier transform addresses this question.

²Reiter's properties P1 and P2 for locally compact quantum groups. Math. Anal. Appl. **2** (2010), 352-365.

Relation to Plancherel transformation

Suppose that (M, Δ) is unimodular, \hat{M} type I (...).

Let $\xi = (\xi_U)_U, \eta = (\eta_U)_U$ be square integrable vector fields such that $\eta \in \text{Dom}(D^k)$ for $k \in \mathbb{Z}$ and such that $(\theta_{\xi_U, D_U^{2/p} \eta_U})_U$ is essentially bounded and such that $(\theta_{\xi_U, D_U^{2/p-2} \eta_U})_U$ is trace class.

For $p \in [1, 2], \frac{1}{p} + \frac{1}{q} = 1,$

$$\nu_\infty^{-1} \circ \nu_p \circ \hat{\mathcal{F}}_p \left(\int_{\text{IC}(M)}^\oplus \theta_{\xi_U, \eta_U} d\mu(U) \right) = \int_{\text{IC}(M)} (\iota \otimes \omega_{\xi_U, D_U^{\frac{2}{p}-2} \eta_U})(U^*) d\mu(U).$$