Non-commutative integration on Quantum Groups

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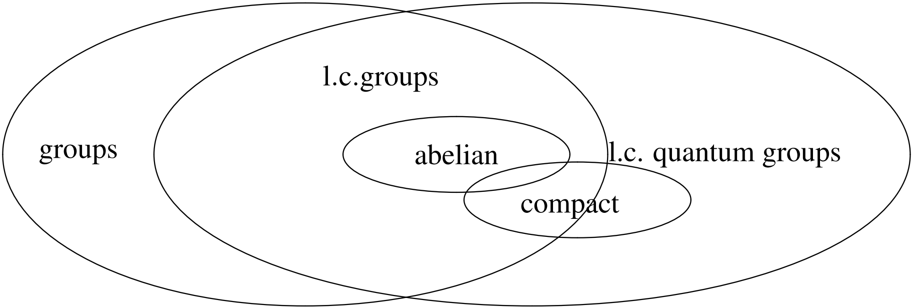
Quantum groups

The operator algebraic story

A philosophy of quantum groups.



Main motivation: extension of tools from harmonic analysis to 'interesting examples'.

Group →

Functions on the group
- Measurable functions.
- Continuous functions.
- Matrix coefficients.

↓

Generators and deformed relations
- Typical examples like $SU_q(2)$, $SU_q(1,1)$, $E_q(2)$, $\ldots$

Example $SU_q(2)$**:**

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- Let A be the $*$-algebra generated by α and γ. Relations:

$$\alpha^*\alpha + \gamma^*\gamma = 1, \qquad \alpha\alpha^* + \quad \gamma\gamma^* = 1,$$
$$\gamma\gamma^* = \gamma^*\gamma, \qquad \gamma\alpha = \alpha\gamma, \qquad \gamma^*\alpha = \alpha\gamma^*.$$

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- Consider the Hilbert space:

$$\mathcal{H} = L^2(\mathbb{N}) \otimes L^2(\mathbb{T}), \text{ standard basis } e_k \otimes \zeta^m,$$

with operators:

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- S, T generate a C*-algebra $C(SU_q(2))$ and von Neumann algebra $L^\infty(SU_q(2)) = B(l^2(\mathbb{N})) \otimes L^\infty(\mathbb{T})$.
- Δ lifts to both $C(SU_q(2))$ and $L^\infty(SU_q(2))$.

- There is a (normal) state on $L^\infty(SU_q(2))$ called the **Haar state** defined by

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- One checks that:

$$(\varphi \otimes \iota)\Delta(x) = \varphi(x) \cdot 1, \qquad (\iota \otimes \varphi)\Delta(x) = \varphi(x) \cdot 1.$$

Example, classical groups: let G be a compact group.

- Let A be the algebra of either:
  - Matrix coefficients of unitary irrep's of G.
  - $C(G)$.
  - $L^\infty(G)$.
- Define a **comultiplication** $\Delta_G : A \to A \otimes A$, by

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- Define **Haar states** $\varphi : A \to \mathbb{C}$ and $\psi : A \to \mathbb{C}$ as the integrals with respect to the left and right Haar measure (note: $\varphi = \psi$).

Locally compact quantum groups (Kustermans, Vaes)

A **von Neumann algebraic quantum group** consists of:

- a von Neumann algebra M;
- a comultiplication, i.e. a unital normal $*$-homomorphism $\Delta: M \to M \otimes M$ such that $(\Delta \otimes \iota)\Delta = (\iota \otimes \Delta)\Delta$;
- two normal semi-finite faithful Haar weights φ, ψ on M, i.e.

$$\begin{aligned} \varphi((\omega \otimes \iota)\Delta(x)) &= \varphi(x)\omega(1), \qquad \forall\, \omega \in M_*^+, \forall\, x \in \mathfrak{m}_\varphi^+, \\ \psi((\iota \otimes \omega)\Delta(x)) &= \psi(x)\omega(1), \qquad \forall\, \omega \in M_*^+, \forall\, x \in \mathfrak{m}_\psi^+. \end{aligned}$$

Pontrjagin duality.

Every quantum group (M, Δ) has a (Pontrjagin) dual quantum group $(\hat{M}, \hat{\Delta})$ such that

$$(\hat{\hat{M}}, \hat{\hat{\Delta}}) = (M, \Delta)$$

- For $(L^\infty(G), \Delta_G)$, the dual vNA is the group vNA $\mathcal{L}(G)$.
- For $SU_q(2)$ the dual vNA is a direct sum of matrix algebras.

Corepresentation theory

- A (unitary) **corepresentation** for (M, Δ) is a unitary $U \in M \otimes B(\mathcal{H}_U)$ satisfying:

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- Example for $SU_q(2)$:

$$\begin{pmatrix} \alpha & -q\gamma^* \\ \gamma & \alpha \end{pmatrix}$$

- There is a very special corepresentation $W \in M \otimes B(\mathcal{H})$ which classically corresponds to the **left regular representation.**
- $\hat{M} = \{(\omega \otimes \iota)(W) \mid \omega \in M_*\}$.

Fourier theory on l.c. quantum groups

Group and its dual group.

- Let G be a l.c. **abelian** group.
- Put

$$\hat{G} = \{\pi \mid \pi \text{ irreducible, unitary representation of } G\} / \sim .$$

- G abelian, so every representation in $\hat{G}$ is 1-dimensional.
- $\hat{G}$ is called the **dual group**. Group multiplication:

$$(\pi_1 \cdot \pi_2)(x) = \pi_1(x)\pi_2(x).$$

Theorem (Pontrjagin duality)

$\hat{\hat{G}} \simeq G$.

Let G be a l.c. abelian group with dual $\hat{G}$.

- L^1-**Fourier transform**: $\mathcal{F}_1 : L^1(G) \to L^\infty(\hat{G}) : f \mapsto \hat{f}$, where

$$\hat{f}(\pi) = \int_G f(x)\pi(x)d_l x.$$

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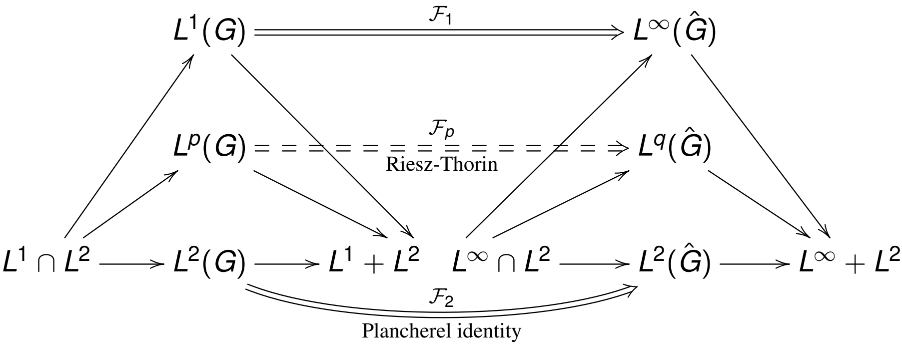


Question: does the L^p-Fourier transform have a generalization in this l.c. quantum group setting?[1]

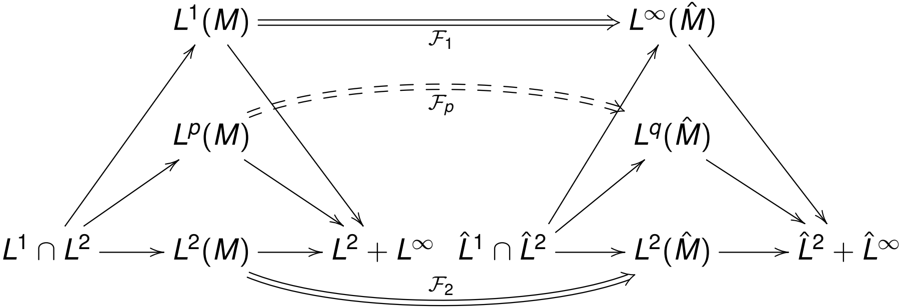
Quantum groups
- Duals.
- L^2- and L^1-FT.
- Algebraic FT (Van Daele).

$\longleftrightarrow$

NC L^p-spaces
- Riesz-Thorin properties.
- Non-semi-finite techniques.
- Haagerup, Connes, Hilsum, Kosaki, Terp, Izumi.

[1] MC, *The L^p-Fourier transform on locally compact quantum groups*, to appear in J. Operator Theory.

We like a Fourier transform on NC L^p-spaces on QG's:



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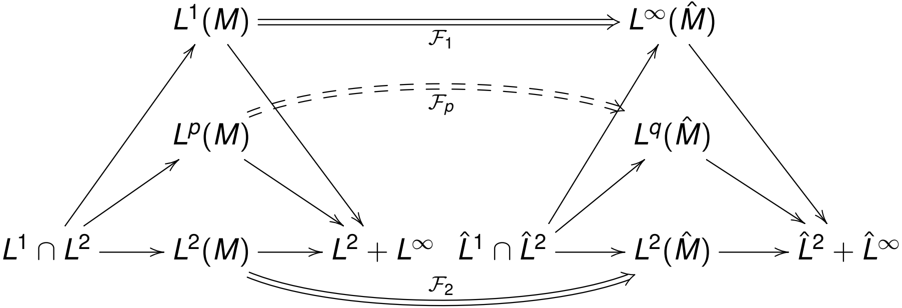

No-go 'theorem' (MC)

Using L^p-spaces w.r.t. a trace on M, there are generally no reasonable maps $\mathcal{F}_1, \mathcal{F}_2$ making the diagram commutative.

Let M be a von Neumann algebra acting on $\mathcal{H}$. Let ϕ be a weight on the commutant M'. Let σ^{ϕ} be its modular automorphism group:

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$L^p(M)$ consists of operators x that are:

- closed, densely defined;
- satisfy the homogenety relation:

$$ax \subseteq x\sigma^\phi_{-i/p}(a) \qquad \text{for certain } a \in M'.$$

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- There is a state $\omega \in M_*$ such that $|x|^p$ equals the spatial derivative $d\omega/d\phi$.

Example: set $d = d\varphi / d\phi$. If $x \in M$ is such that $\varphi(x^* x) < \infty$. Then,

$$[x d^{1/p}] \in L^p(M) \qquad \text{for } p \in [1,2].$$

Theorem (MC)

- Let (M, Δ) be a l.c. quantum group. Let $p \in [1, 2]$ and set q by $1/p + 1/q = 1$.
- Consider spaces $L^p(M)$ and $L^q(\hat{M})$ by choosing a weight ϕ on M' and $\hat{\phi}$ on $\hat{M}'$.
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- Set the corresponding spatial derivatives $d = d\varphi/d\phi$ and $\hat{d} = d\hat{\varphi}/d\hat{\phi}$.

Then, there is bounded **Fourier transform** defined as:

$$\mathcal{F}_p : L^p(\phi) \to L^q(\hat{\phi}) : [ad^{1/p}] \mapsto [\lambda(a\varphi)\hat{d}^{1/q}],$$

for certain $a \in M$. Here,

$$\lambda(a\varphi) = (\varphi \otimes \iota)(W(a \otimes 1)).$$

Some related results:

- We define a convolution product on L^p-spaces associated with M and show that it is turned into a product under the FT.
- We have an extended no-go theorem addressing a question by Daws and Runde [2].
- We determine intersections of L^p-spaces and prove density properties.
- Pairings between $L^p(M)$ and $L^p(\hat{M})$.

[2] M. Daws, V. Runde, *Reiter's properties (P1) and (P2) for locally compact quantum groups*, Math. Anal. Appl. **2** (2010), 352-365.

Related matters and outlook

- Modular properties of matrix coefficients [3].
- Quantum Gelfand pairs [4].

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[3] MC, Erik Koelink, *Modular properties of matrix coefficients of corepresentations of a l.c.q.g.*, J. Lie Theory **21** (2011), 905-928.

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Related matters and outlook

- Modular properties of matrix coefficients [3].
- Quantum Gelfand pairs [4].
- What other NC-spaces play a role in QG theory?
- Can other NC-spaces be defined for arbitrary QG's?
- Are there relations with Dirac operators on QG's?
- More particular, are there non-compact examples on $SU_q(1,1)$, $E_q(2)$ and how are they related to $SU_q(2)$?
- Fourier multipliers?
- ...

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