

Relational semantics for a fragment of linear logic

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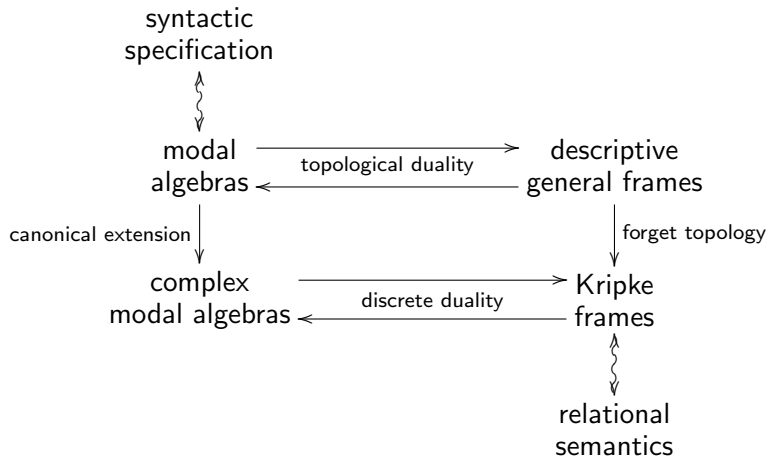
Radboud University Nijmegen

Brussels, February 2011

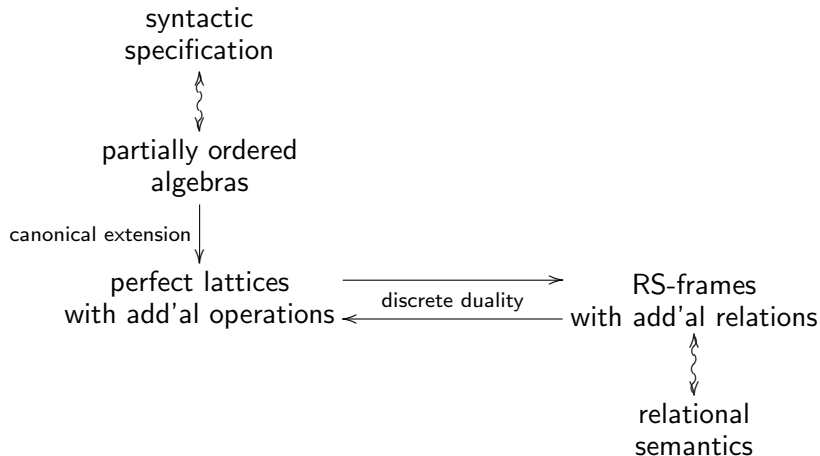
Aim: derive relational semantics for MALL

- 1 General method to obtain relational semantics
- 2 Application of the general theory to linear logic
- 3 'Canonical semantics' and phase spaces
- 4 Concluding remarks

Obtaining relational semantics



Obtaining relational semantics



Obtaining relational semantics

Basic logic: non-associative Lambek calculus (NLC)

Binary connectives \otimes , \rightarrow , \leftarrow

Rules expressing \rightarrow and \leftarrow are residuals of \otimes :

$$\frac{}{\phi \vdash \phi} \qquad \frac{\Gamma \vdash \phi \quad \Delta; \phi; \Sigma \vdash \psi}{\Delta; \Gamma; \Sigma \vdash \psi} \qquad \frac{\phi; \Gamma \vdash \psi}{\Gamma \vdash \phi \rightarrow \psi}$$
$$\frac{}{\phi; \phi \rightarrow \psi \vdash \psi} \qquad \frac{\Gamma; \phi \otimes \psi; \Delta \vdash \xi}{\Gamma; \phi; \psi; \Delta \vdash \xi} \qquad \frac{\Gamma; \phi \vdash \psi}{\Gamma \vdash \psi \leftarrow \phi}$$
$$\frac{}{\psi \leftarrow \phi; \phi \vdash \psi} \qquad \frac{\Gamma; \phi \otimes \psi; \Delta \vdash \xi}{\Gamma; \phi; \psi; \Delta \vdash \xi}$$

Obtaining relational semantics

Algebraic semantics for NLC are given by **residuated algebras**, i.e. structures $(P, \otimes, \rightarrow, \leftarrow)$, where P is a poset and

$$x \otimes y \leq z \quad \Leftrightarrow \quad y \leq x \rightarrow z \quad \Leftrightarrow \quad x \leq z \leftarrow y.$$

Canonical extension yields:

$P \hookrightarrow P^\delta$ embedding in a perfect lattice

$\otimes^\delta: P^\delta \times P^\delta \rightarrow P^\delta$ completely join preserving map.

where a **perfect lattice** is a complete lattice \mathbf{L} , which is join generated by $\mathcal{J}^\infty(\mathbf{L})$ and meet generated by $\mathcal{M}^\infty(\mathbf{L})$.

Discrete duality for perfect lattices

A **two-sorted frame** is a triple (X, Y, \preceq) , where $\preceq \subseteq X \times Y$.

We have a discrete duality:

Perfect lattices

RS-frames

$$\mathbf{L} \quad \xrightarrow{\mathcal{F}} \quad (\mathcal{J}^\infty(\mathbf{L}), \mathcal{M}^\infty(\mathbf{L}), \preceq)$$

where $x \preceq y \Leftrightarrow x \leq_{\mathbf{L}} y$

$$\{A \subseteq X \mid (A^u)^l = A\} \quad \xleftarrow{\mathcal{G}} \quad F = (X, Y, \preceq)$$

Duality for residuated perfect lattices

Capturing the fusion, $\otimes: \mathbf{L} \rightarrow \mathbf{L}$, dually:

Perfect res. lattices

$$(\mathbf{L}, \otimes)$$

$$\xrightarrow{\mathcal{F}}$$

Relational RS-frames

$$(\mathcal{F}(\mathbf{L}), R_{\otimes} \subseteq X^3)$$

$$\mathcal{F}(\mathbf{L}) = (\mathcal{J}^{\infty}(\mathbf{L}), \mathcal{M}^{\infty}(\mathbf{L}), \preceq)$$

$$R_{\otimes}(x_1, x_2, x_3) \Leftrightarrow x_3 \leq x_1 \otimes x_2$$

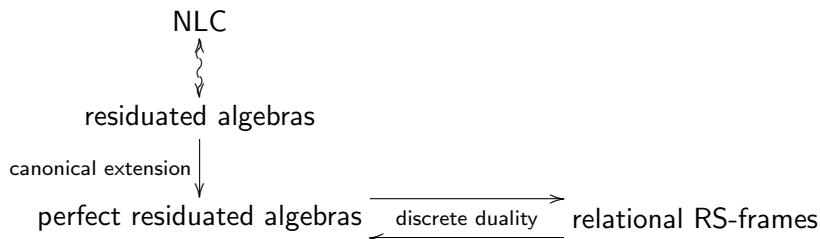
$$(\mathcal{G}(F), \otimes_R)$$

$$\xleftarrow{\mathcal{G}}$$

$$(F, R)$$

$$x_1 \otimes_R x_2 = R[x_1, x_2, -]$$

Obtaining relational semantics



Define a **satisfaction relation** \Vdash on frames s.t.

$$\phi \Vdash \psi \text{ holds in } F \quad \Leftrightarrow \quad \phi \leq \psi \text{ holds in } \mathcal{G}(F)$$

Thm: Relational RS-frames provide complete semantics for NLC.

Linear logic

Introduced by Jean-Yves Girard (1987)

Formulas represent resources which may be used exactly once.

Deduction corresponds to transformation of resources.

Consequence: no contraction and weakening.

Example:

$\text{€}1, \text{€}1 \vdash \text{drink}$ does not imply $\text{€}1 \vdash \text{drink}$.

Exponential ! expresses unlimited availability of a resource.

Algebraic semantics for MALL

Algebraic semantics for MALL are given by **classical linear algebras** (CL-algebras), i.e. structures $(\mathbf{L}, \otimes, \rightarrow, \leftarrow, 1, 0)$, where

- 1 $(\mathbf{L}, \otimes, \rightarrow, \leftarrow)$ is a residuated algebra;
- 2 the fusion \otimes is associative and commutative and has a unit 1;
- 3 \mathbf{L} is a bounded lattice;
- 4 for all $a \in \mathbf{L}$, $(a \rightarrow 0) \rightarrow 0 = a$.

Relational semantics for MALL

Obtaining relational semantics for MALL:

- 1 Correspondence:** Give necessary and sufficient conditions on frames F to ensure $\mathcal{G}(F)$ is a CL-algebra.
- 2 Canonicity:** For a CL-algebra \mathbf{L} , \mathbf{L}^δ is again a CL-algebra

We define

$$\mathcal{K} = \{F \mid \mathcal{G}(F) \text{ is a CL-algebra}\}$$

$$\mathcal{K}^+ = \{\mathcal{G}(F) \mid F \in \mathcal{K}\}.$$

Then: $\mathbf{CL}^\delta \subseteq \mathcal{K}^+ \subseteq \mathbf{CL}$.

Correspondence

Relational semantics for MALL are given by frames

$F = (X, Y, \preceq, R, U, Z)$ satisfying ...

Example: How to ensure U is the unit of \otimes in $\mathcal{G}(F)$?

Note that:

$$\begin{aligned}x \otimes U \leq y &\Leftrightarrow \bigvee \{x \otimes x' \mid x' \leq U\} \leq y \\&\Leftrightarrow \forall x' \in U. x \otimes x' \leq y \\&\Leftrightarrow \forall x' \in U. R[x, x', _]^l \subseteq \{y\}^l \\&\Leftrightarrow \forall x' \in U \forall x''. R[x, x', x''] \rightarrow x'' \preceq y\end{aligned}$$

Hence, U is the unit of the fusion in $\mathcal{G}(F)$ iff F satisfies:

$$\forall x \forall y. x \preceq y \Leftrightarrow \forall x' \in U \forall x''. R[x, x', x''] \rightarrow x'' \preceq y.$$

Simplifying the canonical semantics

How to get a simpler presentation of the frames (X, Y, \preceq, R, U, Z) ?

Properties of CL-algebras

Let \mathbf{L} be a perfect CL-algebra.

1 $\mathcal{J}^\infty(\mathbf{L}) \cong \mathcal{M}^\infty(\mathbf{L})^\partial$.

The isomorphism is given by $x \mapsto x \rightarrow 0$.

2 for $x_1, x_2 \in \mathcal{J}^\infty(\mathbf{L})$, $x_1 \leq x_2 \rightarrow 0 \Leftrightarrow x_1 \otimes x_2 \leq 0$.

3 $1 = 0 \rightarrow 0$.

Conclusion: we only have to remember (X, R, Z) .

Phase spaces

A **phase space** is a tuple $(M, \cdot, 1, \perp)$ where $(M, \cdot, 1)$ is a commutative monoid and $\perp \subseteq M$.

Define an operation on $\wp(M)$ by

$$A^\perp = \{m \mid \forall n \in A. m \cdot n \in \perp\}.$$

A **fact** is a subset $F \subseteq M$ such that $(F^\perp)^\perp = F$.

The facts of M form a CL-algebra $\mathcal{Fct}(M)$ s.t.

$$\phi \Vdash \psi \text{ holds in } M \quad \Leftrightarrow \quad \phi \Vdash \psi \text{ holds in } \mathcal{Fct}(M).$$

Canonical semantics and phase semantics

Let $P = (X, R, Z)$ be a CL-frame. Define a phase space by

$$M_P = \wp(X)$$

$$A \cdot_P B = \bigcup \{R[a, b, _]\mid a \in A, b \in B\}$$

$$\perp_P = \{A \in \wp(X) \mid A \subseteq Z_\downarrow\}.$$

Theorem: $\mathcal{G}(P) \cong \mathcal{Fct}(M_P)$

Concluding remarks

Advantages of working with CL-frames

- 1 CL-frames are in a duality with perfect CL-algebras, allowing a modular treatment of additional axioms and operations.
- 2 Size difference between CL-frames and phase spaces.

Work in progress:

- 1 Analysing the properties of the exponential !
- 2 Describing relational semantics for !