On generalizing canonical extension

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Nijmegen, November 2011

Motivation

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logic $\mathcal{L}$ $\quad \leftrightsquigarrow \quad$ **category of algebras $\mathcal{V}_{\mathcal{L}}$**

For all formulas ϕ, ψ in $\mathcal{L}$,

$$\phi \vdash \psi \text{ in } \mathcal{L} \quad \Leftrightarrow \quad \phi \vDash \psi \text{ in } \mathcal{V}_{\mathcal{L}}$$

Examples:

CPL $\leftrightsquigarrow$ Boolean algebras

IPL $\leftrightsquigarrow$ Heyting algebras

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Duality theory:

category of algebras $\leftrightsquigarrow$ category of topological spaces

Outline

1. Introduction to duality theory and canonical extension
2. Semantics for coherent first order logic ($\wedge$, $\vee$, $\bot$, $\top$, $\exists$):
   - Coherent categories
   - Coherent hyperdoctrines
3. Canonical extension in the categorical setting
4. Relation to other constructions (Makkai's topos of types)
5. Future work

Stone duality

Boolean algebras: structures $(B, \wedge, \vee, \neg, 0, 1)$

Boolean spaces: compact, totally disconnected, Hausdorff spaces.

Boolean algebras $\leftrightarrows$ **Boolean spaces**

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$Cl(X)$	$\leftarrow\!\shortmid$	X

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B	$\mapsto$	$(PrIdl(B), \tau_B)$

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$Cl(X) \xleftarrow{f^{-1}} Cl(Y)$	$\leftarrow\!\shortmid$	$X \xrightarrow{f} Y$
$B \xrightarrow{h} C$	$\mapsto$	$PrIld(B) \xleftarrow{h^{-1}} PrIdl(C)$

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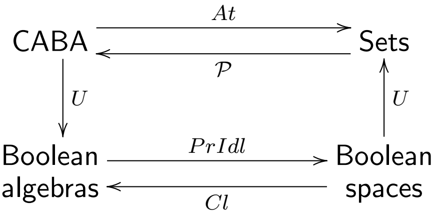
Stone Representation Theorem: every Boolean algebra is embeddable in a powerset algebra.

Proof: for a Boolean algebra B,

$$B \cong Cl(PrIdl(B)) \hookrightarrow \mathcal{P}(PrIdl(B))$$

Stone duality and canonical extension

Canonical extension: algebraic description of topological duality



CABA	=	complete and atomic Boolean algebras
Boolean spaces	=	compact, totally disconnected Hausdorff spaces

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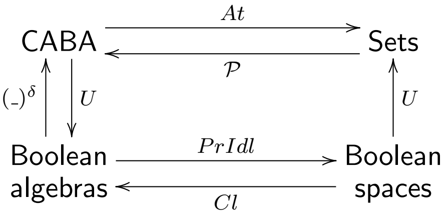

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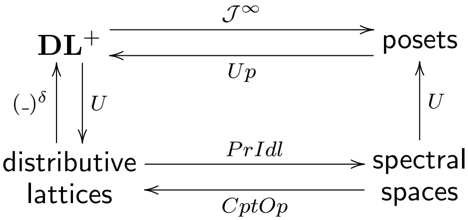
Canonical extension of distributive lattices

Canonical extension: algebraic description of topological duality



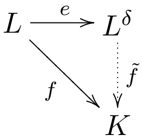
$\mathbf{DL}^+$	$=$	completely distributive algebraic lattices
spectral spaces	$=$	sober spaces with a basis of compact opens

Canonical extension of distributive lattices

$\mathbf{DL}^+$ = completely distributive algebraic lattices.

Canonical extension is left adjoint to $\mathbf{DL}^+ \hookrightarrow \mathbf{DL}$.

Universal characterization of canonical extension:



where $L \in \mathbf{DL}$ and $K, L^\delta \in \mathbf{DL}^+$.

Interpolation in propositional logic

Let $\mathbb{T}$ be a theory in intuitionistic propositional logic.

Question: does $\mathbb{T}$ have the **interpolation property**, i.e.,

for all formulas $\phi(p, q)$ and $\psi(p, r)$ with $\phi(p, q) \vdash_{\mathbb{T}} \psi(p, r)$, there exists a formula $\theta(p)$ s.t.

$$\phi(p, q) \vdash_{\mathbb{T}} \theta(p) \quad \text{and} \quad \theta(p) \vdash_{\mathbb{T}} \psi(p, r).$$

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>
> $$\phi(p,q) \vdash_{\mathbb{T}} \theta(p) \quad \text{and} \quad \theta(p) \vdash_{\mathbb{T}} \psi(p,r).$$

Question: are monomorphisms stable under pushout in $\mathcal{V}_{\mathbb{T}}$?

Interpolation in first order logic

Let $\mathbb{T}$ be a theory in intuitionic first order logic.

Question: does $\mathbb{T}$ have the **interpolation property**, i.e.,

for all sentences ϕ, ψ with $\phi \vdash_{\mathbb{T}} \psi$, there exists a sentence θ s.t.

1. $\phi \vdash_{\mathbb{T}} \theta$ and $\theta \vdash_{\mathbb{T}} \psi$;
2. every relation and function symbol which occurs in θ occurs in both ϕ and ψ.

Open problem for some first order intuitionistic theories, e.g.,

$\mathbb{T} = \text{IFOL} + (\phi \to \psi) \vee (\psi \to \phi)$.

Algebraic semantics for coherent logic

We start from

Signature:	$\Sigma = (f_0, \ldots, f_{k-1}, R_0, \ldots, R_{l-1})$
Set of var's:	$X = \{x_0, x_1, \ldots\}$
Equality:	$=$
Connectives:	$\wedge, \vee, \top, \bot, \exists$
Derivability notion:	$\vdash$ (given by axioms and rules)

Question:

What properties does the logic over Σ have?

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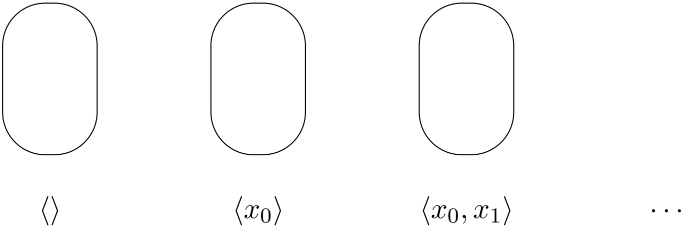
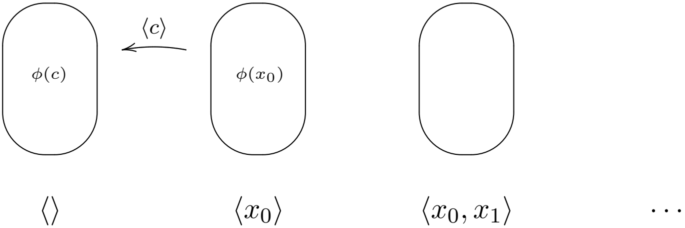
First observation:

For each sequence of variables $\vec{x} = \langle x_0, \ldots, x_{n-1} \rangle$,

$(Fm(\vec{x})/_{\vdash \cap \dashv}, \vdash)$ is a distributive lattice.


$\langle\rangle$
$\langle x_0 \rangle$
$\langle x_0, x_1 \rangle$
$\cdots$

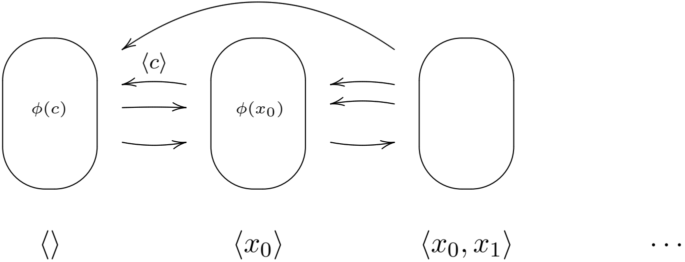
Algebraic semantics for coherent logic



Substitutions:

$$\begin{array}{ccc} x_0 & \mapsto & c \\ \phi(x_0) & \mapsto & \phi(c) \end{array}$$

Algebraic semantics for coherent logic



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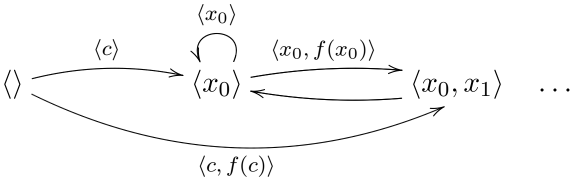
Algebraic semantics for coherent logic

Contexts and substitutions form a category $\mathbf{B}$:

Objects: contexts $\vec{x}$

Morphism $\vec{x} \to \vec{y}$: m-tuple $\langle t_0, \ldots, t_{m-1} \rangle$
s.t. $m = length(\vec{y})$ and $FV(t_i) \subseteq \vec{x}$



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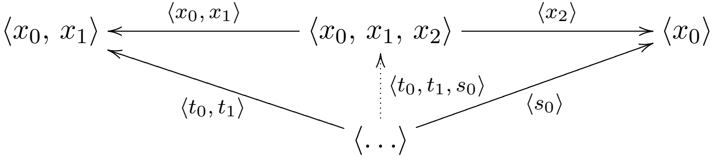
This category has **finite products**:

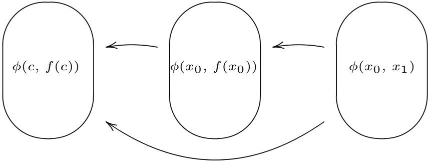


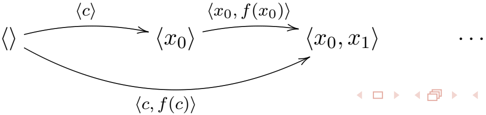
Algebraic semantics for coherent logic

Formulas and substitutions: functor $\mathbf{B}^{op} \to \mathbf{DL}$

$$\vec{x} \quad \mapsto \quad Fm(\vec{x})$$

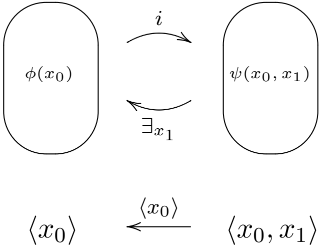
$$\vec{x} \xrightarrow{\langle t_0, \ldots, t_{m-1} \rangle} \vec{y} \quad \mapsto \quad \begin{array}{rcl} Fm(\vec{y}) & \to & Fm(\vec{x}) \\ \phi(\vec{y}) & \mapsto & \phi[\vec{t}/\vec{y}] \end{array}$$





Algebraic semantics for coherent logic

Existential quantification: related to the inclusion map



$$\dfrac{\exists_{x_1}(\psi(x_0, x_1)) \quad \vdash \quad \phi(x_0)}{\psi(x_0, x_1) \quad \vdash \quad \phi(x_0)}$$

Algebraic semantics for coherent logic

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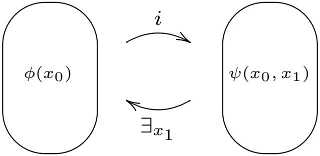
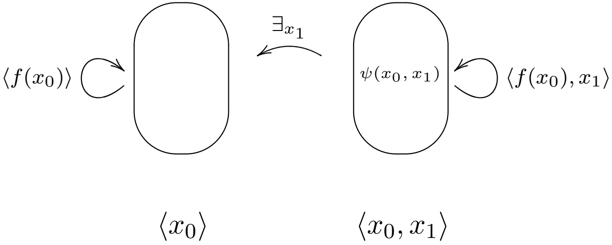


$$\langle x_0 \rangle \xleftarrow{\langle x_0 \rangle} \langle x_0, x_1 \rangle$$

$$\begin{array}{ccc} \exists_{x_1}(\psi(x_0, x_1)) & \vdash_{x_0} & \phi(x_0) \\ \hline\hline \psi(x_0, x_1) & \vdash_{x_0, x_1} & i(\phi(x_0)) \end{array}$$

Algebraic semantics for coherent logic

Existential quantification: interaction with substitutions

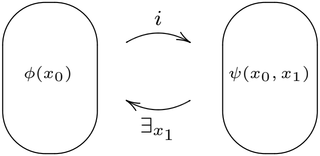


$$\exists_{x_1}(\psi(x_0, x_1))[f(x_0)/x_0] \;=\; \exists_{x_1}(\psi(f(x_0), x_1))$$

(Beck-Chevalley)

Algebraic semantics for coherent logic

Existential quantification: interaction with substitutions



$$\langle x_0 \rangle \xleftarrow{\langle x_0 \rangle} \langle x_0, x_1 \rangle$$

$$\exists_{x_1}[i(\phi(x_0)) \wedge \psi(x_0, x_1)] = \phi(x_0) \wedge \exists_{x_1}[\psi(x_0, x_1)]$$

(Frobenius)

Algebraic semantics for coherent logic

A **coherent hyperdoctrine** is a functor $P\colon \mathbf{B}^{\mathbf{op}} \to \mathbf{DL}$ s.t.

1. $\mathbf{B}$ is a category with finite limits;
2. for all $A \xrightarrow{\alpha} B \in \mathbf{B}$, $P(\alpha)$ has a left adjoint $\exists_\alpha$ with
   - **Frobenius reciprocity**, *i.e.*, for all $a \in P(A)$, $b \in P(B)$,
   $$\exists_\alpha(a \wedge P(\alpha)(b)) = \exists_\alpha(a) \wedge b$$
   - **Beck-Chevalley condition**, *i.e.*, for every pullback square
   $$\begin{array}{ccc} Q & \xrightarrow{\alpha'} & B \\ {\scriptstyle \beta'}\downarrow & & \downarrow{\scriptstyle \beta} \\ A & \xrightarrow[\alpha]{} & C \end{array}$$
   in $\mathbf{B}$, $P(\beta) \circ \exists_\alpha = \exists_{\alpha'} \circ P(\beta')$.

Algebraic semantics for coherent logic

Examples of coherent hyperdoctrines:

- **Syntactic hyperdoctrine**

  $\mathbf{B}$ = contexts and substitutions

$$\begin{array}{rccc} \mathcal{F}\colon & \mathbf{B}^{op} & \to & \mathbf{DL} \\ & \vec{x} & \mapsto & Fm(\vec{x})/_{\vdash \cap \dashv} \end{array}$$

- **Powerset hyperdoctrine**

  $\mathbf{B} = \mathbf{Set}$

$$\begin{array}{rccc} \mathcal{P}\colon & \mathbf{B}^{op} & \to & \mathbf{DL} \\ & A & \mapsto & \mathcal{P}(A) \\ & A \xrightarrow{f} B & \mapsto & \mathcal{P}(B) \xrightarrow{f^{-1}} \mathcal{P}(A). \end{array}$$

Coherent hyperdoctrines and categories

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A **coherent category** is a category $\mathbf{C}$ satisfying

1. $\mathbf{C}$ has finite limits;
2. $\mathbf{C}$ has stable finite unions;
3. $\mathbf{C}$ has stable images.

Coherent hyperdoctrines and coherent categories

Proposition: there is a 2-categorical adjunction

$$\mathcal{A}\colon \mathbf{CHyp} \leftrightarrows \mathbf{Coh} \colon \mathcal{S},$$

where $\mathcal{A} \dashv \mathcal{S}$ and $\mathcal{A}(\mathcal{S}(\mathbf{C})) \simeq \mathbf{C}$.

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For $\mathbf{C} \in \mathbf{Coh}$, $\mathcal{S}(\mathbf{C}) = \mathcal{S}_{\mathbf{C}} \colon \mathbf{C}^{op} \to \mathbf{DL}$, $A \mapsto Sub_{\mathbf{C}}(A)$

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For $P\colon \mathbf{B}^{op} \to \mathbf{DL}$, $\mathcal{A}(P)$ is the category with:

objects are pairs (A, a), where $A \in \mathbf{B}$, $a \in P(A)$;

a morphism $(A, a) \to (B, b)$ is an element $f \in P(A \times B)$ which is a functional relation $(A, a) \to (B, b)$.

Canonical extension of coherent hyperdoctrines

Recall: canonical extension for DL's is a functor $\mathbf{DL} \xrightarrow{(\_)^\delta} \mathbf{DL}^+$.

Definition

For a coh. hyperdoctrine $P\colon \mathbf{B}^{op} \to \mathbf{DL}$ we define:

$$P^\delta\colon \mathbf{B}^{op} \xrightarrow{P} \mathbf{DL} \xrightarrow{(\_)^\delta} \mathbf{DL}.$$

Proposition

For a coh. hyperdoctrine P, P^δ is again a coh. hyperdoctrine.

Proof: check that, for all $A \xrightarrow{\alpha} B$ in $\mathbf{B}$, $P^\delta(\alpha)$ has a left adjoint satisfying BC and Frobenius.

Canonical extension of coherent categories

We have:

- adjunction $\mathcal{A}\colon \mathbf{CHyp} \leftrightarrows \mathbf{Coh}\colon \mathcal{S}$, $\mathbf{C} \simeq \mathcal{A}(\mathcal{S}_{\mathbf{C}})$
- for $P \in \mathbf{CHyp}$, $P^{\delta}\colon \mathbf{B}^{op} \xrightarrow{P} \mathbf{DL} \xrightarrow{(\_)^{\delta}} \mathbf{DL}$

Definition

For a coherent category $\mathbf{C}$ we define:

$$\mathbf{C}^{\delta} = \mathcal{A}(\mathcal{S}^{\delta}_{\mathbf{C}})$$

Proposition

For a distributive lattice $\mathbf{L}$, $\mathcal{A}(\mathcal{S}^{\delta}_{\mathbf{L}}) \simeq \mathbf{L}^{\delta}$.

Canonical extension of coherent categories

Properties of $\mathbf{C}^\delta = \mathcal{A}(\mathcal{S}_{\mathbf{C}}^\delta)$:

1. subobject lattices are in $\mathbf{DL}^+$
2. pullback morphisms are complete lattice homomorphisms

$\mathbf{Coh}^+$ = coherent categories satisfying (1) and (2).

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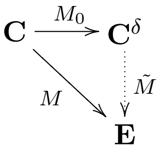
Universal characterization:



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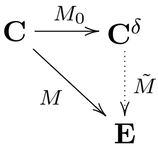

where $\mathbf{C} \in \mathbf{Coh}$, $\mathbf{E}, \mathbf{C}^\delta \in \mathbf{Coh}^+$, M a coherent functor satisfying:

for all $A \xrightarrow{\alpha} B$ in $\mathbf{C}$, ρ (prime) filter in $\mathcal{S}_C(A)$,

$\exists_{M(\alpha)}(\bigwedge\{M(U) \mid U \in \rho\}) \cong \bigwedge\{\exists_{M(\alpha)}(M(U)) \mid U \in \rho\}$.

Topos of types

Topos of types was introduced by Makkai in 1979 as:

- 'a reasonable codification of the 'discrete' (non topological) syntactical structure of types of the theory'
- a tool to prove representation theorems
- 'conceptual tool meant to enable us to formulate precisely certain natural intuitive questions'

Some later work by: Magnan & Reyes and Butz.

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Alternative construction:

The functor $\mathcal{S}^{\delta}_{\mathbf{C}} \colon \mathbf{C}^{op} \to \mathbf{DL}$ is an internal frame in $\mathbf{Set}^{\mathbf{C}^{op}} = \widehat{\mathbf{C}}$.

Then $Sh_{\widehat{C}}(\mathcal{S}^{\delta}_{\mathbf{C}}) \simeq T(\mathbf{C})$ = topos of types of $\mathbf{C}$.

Topos of types and the class of models

For a distributive lattice L,

$$\begin{aligned} \text{prime ideals of } L &= \text{lattice homomorphisms } L \to \mathbf{2} \\ &= \text{`models of } L\text{'}. \end{aligned}$$

$L^\delta = Up(Mod(L))$

Categorical analogue:

$Mod(\mathbf{C})$ = coherent functors $M \colon \mathbf{C} \to \mathbf{Set}$.

Study: $\mathbf{Set}^{Mod(\mathbf{C})}$.

We have to restrict to an appropriate subcategory $\mathcal{K}$ of $Mod(\mathbf{C})$.

Question: How does $\mathbf{Set}^{\mathcal{K}}$ relate to $T(\mathbf{C}) = Sh_{\hat{\mathbf{C}}}(\mathcal{S}^\delta_{\mathbf{C}})$?

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Evaluation functor

$$
\begin{array}{rcl}
ev\colon \mathbf{C} & \to & \mathbf{Set}^{\mathcal{K}} \\
A & \mapsto & \begin{array}{rcl} ev(A)\colon \mathcal{K} & \to & \mathbf{Set} \\ M & \mapsto & M(A) \end{array}
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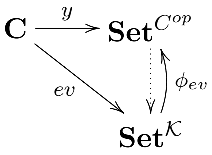
Gives a geometric morphism $\phi_{ev}\colon \mathbf{Set}^{\mathcal{K}} \to \mathbf{Set}^{C^{op}}$:



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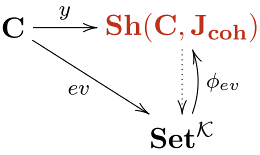
$$\begin{array}{rcl} ev\colon \mathbf{C} & \to & \mathbf{Set}^{\mathcal{K}} \\ A & \mapsto & \begin{array}{rcl} ev(A)\colon \mathcal{K} & \to & \mathbf{Set} \\ M & \mapsto & M(A) \end{array} \end{array}$$

Gives a geometric morphism $\phi_{ev}\colon \mathbf{Set}^{\mathcal{K}} \to \mathbf{Sh}(\mathbf{C}, \mathbf{J_{coh}})$:



Topos of types and the class of models

$\mathcal{K}$ appropriate subcategory of $Mod(\mathbf{C})$.

Question: How does $\mathbf{Set}^{\mathcal{K}}$ relate to $T(\mathbf{C}) = Sh_{\hat{\mathbf{C}}}(\mathcal{S}^{\delta}_{\mathbf{C}})$?

Evaluation functor
$$\begin{array}{rcl} ev\colon \mathbf{C} & \to & \mathbf{Set}^{\mathcal{K}} \\ A & \mapsto & \begin{array}{rcl} ev(A)\colon \mathcal{K} & \to & \mathbf{Set} \\ M & \mapsto & M(A) \end{array} \end{array}$$

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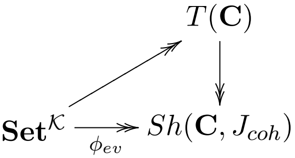


Topos of types and the class of models

Claim: the topos of types yields the hyper-connected localic factorization of $\mathbf{Set}^{\mathcal{K}} \xrightarrow{\phi_{ev}} Sh(\mathbf{C}, J_{coh})$.

Description of the factorization:

$$\begin{array}{ccc} & & Sh((\phi_{ev})_*(\Omega_{Set^{\mathcal{K}}})) \\ & \nearrow & \downarrow \\ \mathbf{Set}^{\mathcal{K}} & \xrightarrow[\phi_{ev}]{} & Sh(\mathbf{C}, J_{coh}) \end{array}$$

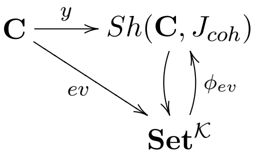
$T(\mathbf{C}) = Sh(S^{\delta}_{\mathbf{C}})$

To prove: $S^{\delta}_{\mathbf{C}} \cong (\phi_{ev})_*(\Omega_{Set^{\mathcal{K}}})$ in $Sh(\mathbf{C}, J_{coh})$.

Topos of types and the class of models

To prove: $S^{\delta}_{\mathbf{C}} \cong (\phi_{ev})_*(\Omega_{Set^{\mathcal{K}}})$ in $Sh(\mathbf{C}, J_{coh})$

Recall:



Hence, for $A \in \mathbf{C}$,

$$\begin{aligned} (\phi_{ev})_*(\Omega_{Set^{\mathcal{K}}})(A) &= Hom_{\mathbf{Set}^{\mathcal{K}}}(ev(A), \Omega_{Set^{\mathcal{K}}}) \\ &= Sub(ev(A)). \end{aligned}$$

Let $\sigma_A \colon S^{\delta}_{\mathbf{C}}(A) \to (\phi_{ev})_*(\Omega_{Set^{\mathcal{K}}})(A)$ be the unique map given by:

Topos of types and the class of models

To prove: $S^{\delta}_{\mathbf{C}} \cong (\phi_{ev})_*(\Omega_{Set^{\mathcal{K}}})$ in $Sh(\mathbf{C}, J_{coh})$

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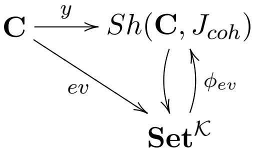

Hence, for $A \in \mathbf{C}$,

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\begin{aligned}
(\phi_{ev})_*(\Omega_{Set^{\mathcal{K}}})(A) &= Hom_{\mathbf{Set}^{\mathcal{K}}}(ev(A), \Omega_{Set^{\mathcal{K}}}) \\
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\end{aligned}
$$

Let $\sigma_A : S^{\delta}_{\mathbf{C}}(A) \to (\phi_{ev})_*(\Omega_{Set^{\mathcal{K}}})(A)$ be the unique map given by:

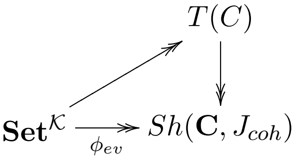
$$
\begin{aligned}
Sub_{\mathbf{C}}(A) &\to Sub(ev(A)) \\
U &\mapsto ev(U).
\end{aligned}
$$

Future work

We have: notion of canonical extension for coherent categories

We would like to:

- Study the following diagram (where $\mathcal{K} \subseteq Mod(\mathbf{C})$):



- Apply the developed theory in the study of first order logics
- In particular: study interpolation problems for first order logics, e.g. for IPL $+ (\phi \to \psi) \vee (\psi \to \phi)$