

# Generalizing canonical extension to the categorical setting

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# Outline

**Aim:** generalize techniques used in the algebraic study of propositional logics to the setting of first order logics.

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- 1 Propositional logic and algebra
- 2 Duality theory and canonical extension
- 3 'Algebraic' study of first order logics
- 4 Canonical extension in the categorical setting

# Propositional logic and algebra

## Logic

connectives, e.g.,  $\wedge$ ,  $\vee$ ,  $\neg$ ,  $\perp$ ,  $\top$

propositional variables  $P = \{p_0, p_1, \dots\}$

formulas, e.g.,  $p_0 \wedge \neg p_1$

notion of derivability, e.g.,  $p_0 \wedge p_1 \vdash p_0$

## Algebra

operations

generations

terms

inequational theory

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We associate:

**logic  $\mathcal{L}$**   $\longleftrightarrow$  **category of algebras  $\mathcal{V}_{\mathcal{L}}$**

s.t. for all formulas  $\phi, \psi$  in  $\mathcal{L}$ ,

$$\phi \vdash \psi \text{ in } \mathcal{L} \quad \Leftrightarrow \quad \phi \vDash \psi \text{ in } \mathcal{V}_{\mathcal{L}}$$

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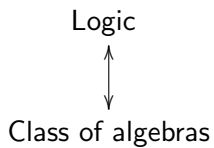
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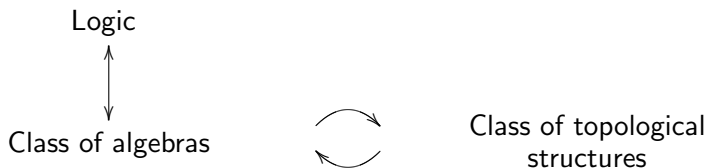
For example: CPL  $\iff$  Boolean algebras

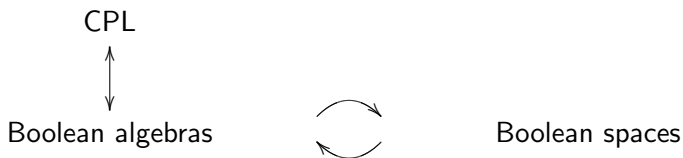
IPL  $\iff$  Heyting algebras

# Logic and duality

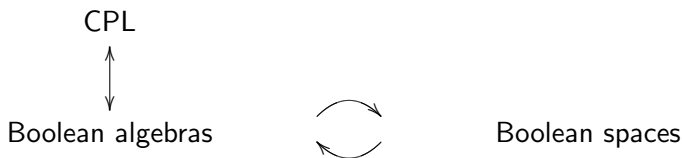


# Logic and duality





**Boolean space:** closed subspace of Cantor space  $2^X$ .



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**Example:** points of the dual space of  $F_{BA}(p, q)$  correspond to rows of the truth table:

p	q
⊥	⊥
⊥	⊤
⊤	⊥
⊤	⊤

**Canonical extension:** algebraic description of topological duality

For a Boolean algebra  $B$  we have

- topological perspective: study  $(X_B, \tau_B)$   
(elements of  $B$  correspond to a basis)
- canonical extension: study  $B \hookrightarrow \mathcal{P}(X_B) = B^\delta$

# Duality and canonical extension

**Canonical extension:** algebraic description of topological duality

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Every Boolean algebra has a unique canonical extension and this assignment is functorial:

$$(-)^\delta: \mathbf{BA} \rightarrow \mathbf{CABA}$$

Similar situation for Heyting algebras.

# Interpolation in propositional logic

Let  $\mathbb{T}$  be a theory in IPC.

**Question:** does  $\mathbb{T}$  have the **interpolation property**, i.e.,

for all formulas  $\phi(p, q)$  and  $\psi(p, r)$  with  $\phi(p, q) \vdash_{\mathbb{T}} \psi(p, r)$ ,  
there exists a formula  $\theta(p)$  s.t.

$$\phi(p, q) \vdash_{\mathbb{T}} \theta(p) \quad \text{and} \quad \theta(p) \vdash_{\mathbb{T}} \psi(p, r).$$

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**Question:** are embeddings stable under pushout in  $\mathcal{V}_{\mathbb{T}}$ ?

# Interpolation in first order logic

Let  $\mathbb{T}$  be a theory in intuitionistic first order logic.

**Question:** does  $\mathbb{T}$  have the **interpolation property**, i.e.,

for all sentences  $\phi, \psi$  with  $\phi \vdash_{\mathbb{T}} \psi$ , there exists a sentence  $\theta$  s.t.

**1**  $\phi \vdash_{\mathbb{T}} \theta$  and  $\theta \vdash_{\mathbb{T}} \psi$ ;

**2** every relation and function symbol which occurs in  $\theta$  occurs in both  $\phi$  and  $\psi$ .

Open problem for some first order intuitionistic theories, e.g.,

$\mathbb{T} = \text{IFOL} + (\phi \rightarrow \psi) \vee (\psi \rightarrow \phi)$ .

# Algebraic semantics for intuitionistic logic

We start from

Signature:  $\Sigma = (f_0, \dots, f_{k-1}, R_0, \dots, R_{l-1})$

Set of var's:  $X = \{x_0, x_1, \dots\}$

Equality:  $=$

Connectives:  $\wedge, \vee, \rightarrow, \top, \perp, \exists, \forall$

Derivability notion:  $\vdash$  (given by axioms and rules)

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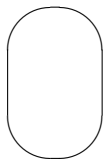
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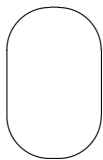
**Problem:** how to define 'algebraic' semantics for first order logic?

- formulas with free variables, e.g.,  $\forall x.R(x, y)$ ;
- quantifiers  $\exists x, \forall x$ .

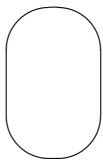
# Algebraic semantics for intuitionistic logic



$\langle \rangle$



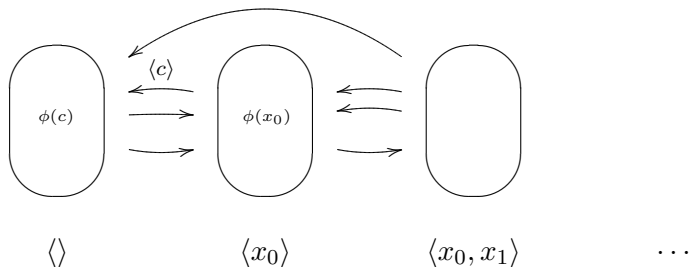
$\langle x_0 \rangle$



$\langle x_0, x_1 \rangle$

$\dots$

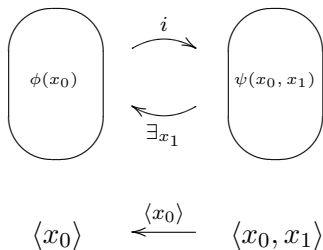
# Algebraic semantics for intuitionistic logic



Substitutions:

$$\begin{aligned} x_0 &\mapsto c \\ \phi(x_0) &\mapsto \phi(c) \end{aligned}$$

**Existential quantification:** related to the inclusion map

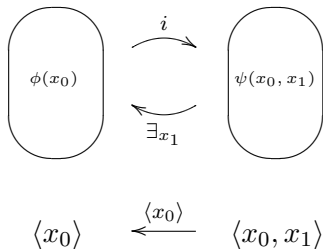


$$\exists x_1 (\psi(x_0, x_1)) \quad \vdash \quad \phi(x_0)$$

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**Existential quantification:** related to the inclusion map



$$\frac{\exists x_1 (\psi(x_0, x_1)) \quad \vdash_{x_0} \quad \phi(x_0)}{\psi(x_0, x_1) \quad \vdash_{x_0, x_1} \quad i(\phi(x_0))}$$

# Algebraic semantics for intuitionistic logic

An **intuitionistic hyperdoctrine** is a functor  $P: \mathbf{B}^{op} \rightarrow \mathbf{HA}$  s.t.

- 1  $\mathbf{B}$  is a category with finite limits;
- 2 for all  $A \xrightarrow{\alpha} B \in \mathbf{B}$ ,  $P(\alpha)$  has a left adjoint  $\exists_{\alpha}$  satisfying Beck-Chevalley and Frobenius and a right adjoint  $\forall_{\alpha}$ .

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**Example:** powerset functor

$$\mathcal{P}: \mathbf{Set}^{op} \rightarrow \mathbf{HA}$$

$$A \mapsto \mathcal{P}(A)$$

$$A \xrightarrow{f} B \mapsto \mathcal{P}(B) \xrightarrow{f^{-1}} \mathcal{P}(A).$$

# Intuitionistic hyperdoctrines and categories

An **intuitionistic hyperdoctrine** is a functor  $P: \mathbf{B}^{\text{op}} \rightarrow \mathbf{HA}$  s.t.

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A **Heyting category** is a category  $\mathbf{C}$  satisfying

- 1  $\mathbf{C}$  has finite limits;
- 2  $\mathbf{C}$  has stable finite unions;
- 3 pullback morphisms in  $\mathbf{C}$  have stable left and right adjoints.

# Canonical extension of intuitionistic hyperdoctrines

**Recall:** canonical extension of Heyting algebras is a functor

$$\mathbf{HA} \xrightarrow{(-)^\delta} \mathbf{HA}^+$$

## Definition

For an intuitionistic hyperdoctrine  $P: \mathbf{B}^{\text{op}} \rightarrow \mathbf{HA}$  we define:

$$P^\delta: \mathbf{B}^{\text{op}} \xrightarrow{P} \mathbf{HA} \xrightarrow{(-)^\delta} \mathbf{HA}.$$

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## Proposition

For an int. hyperdoctrine  $P$ ,  $P^\delta$  is again an int. hyperdoctrine.

**Proof:** check that, for all  $A \xrightarrow{\alpha} B$  in  $\mathbf{B}$ ,  $P^\delta(\alpha)$  has a left adjoint satisfying BC and Frobenius and a right adjoint.

# Canonical extension of Heyting categories

**Proposition:** there is a 2-categorical adjunction

$$\mathcal{A}: \mathbf{IHyp} \rightleftarrows \mathbf{HCat}: \mathcal{S},$$

where  $\mathcal{A} \dashv \mathcal{S}$  and  $\mathcal{A}(\mathcal{S}(\mathbf{C})) \simeq \mathbf{C}$ .

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## Definition

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## Proposition

For a Heyting algebra  $\mathbf{L}$ ,  $\mathcal{A}(\mathcal{S}(\mathbf{L}))^\delta \simeq \mathbf{L}^\delta$ .

# Canonical extension of Heyting categories

Properties of  $\mathbf{C}^\delta = \mathcal{A}(\mathcal{S}_\mathbf{C}^\delta)$ :

- 1 subobject lattices are in  $\mathbf{HA}^+$
- 2 pullback morphisms are complete lattice homomorphisms

$\mathbf{HCat}^+ =$  Heyting categories satisfying (1) and (2).

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**Universal characterization:**

$$\begin{array}{ccc} \mathbf{C} & \xrightarrow{M_0} & \mathbf{C}^\delta \\ & \searrow M & \downarrow \tilde{M} \\ & & \mathbf{E} \end{array}$$

where  $\mathbf{C} \in \mathbf{HCat}$ ,  $\mathbf{E}, \mathbf{C}^\delta \in \mathbf{HCat}^+$ ,  $M$  a Heyting functor satisfying some extra conditions (related to Esakia Lemma).

## Results and future work

- Canonical extension of coherent categories allows an alternative description of Makkai's topos of types:

$$T(\mathbf{C}) \simeq Sh(\mathcal{S}(\mathbf{C})^\delta)$$

- Study Makkai's results from this perspective

# Results and future work

- Canonical extension of coherent categories allows an alternative description of Makkai's topos of types:

$$T(\mathbf{C}) \simeq Sh(\mathcal{S}(\mathbf{C})^\delta)$$

- Study Makkai's results from this perspective
- Apply our construction in the study of first order logics
- In particular: study interpolation problems for first order logics, e.g. for IFOL +  $(\phi \rightarrow \psi) \vee (\psi \rightarrow \phi)$