

Canonical extension of coherent categories

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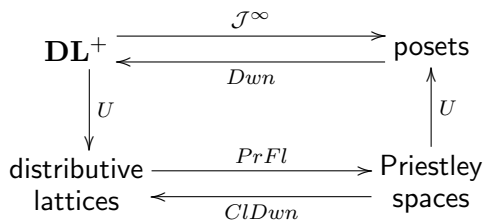
Radboud University Nijmegen

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Outline

- 1 Canonical extension of distributive lattices ($\wedge, \vee, \top, \perp$)
- 2 'Algebraic' semantics for coherent logic ($\wedge, \vee, \top, \perp, \exists$):
 - Polyadic distributive lattices (pDL's)
 - Coherent categories
- 3 Canonical extension of pDL's and coherent categories
- 4 Relation to other constructions (Makkai's topos of types)
- 5 Future work

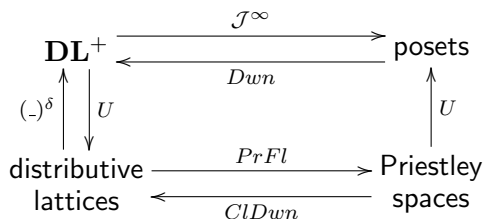
Canonical extension of distributive lattices



DL^+ = completely distributive algebraic lattices

Priestley spaces = totally order-disconnected compact Hausdorff spaces

Canonical extension of distributive lattices



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Canonical extension of distributive lattices

\mathbf{DL}^+ = completely distributive algebraic lattices.

Canonical extension is left adjoint to $\mathbf{DL}^+ \hookrightarrow \mathbf{DL}$.

Universal characterization of canonical extension:

$$\begin{array}{ccc} \mathbf{L} & \xrightarrow{e} & \mathbf{L}^\delta \\ & \searrow f & \downarrow \tilde{f} \\ & & \mathbf{K} \end{array}$$

where $\mathbf{L} \in \mathbf{DL}$ and $\mathbf{K}, \mathbf{L}^\delta \in \mathbf{DL}^+$.

Algebraic semantics for coherent logic

We start from

Signature: $\Sigma = (f_0, \dots, f_{k-1}, R_0, \dots, R_{l-1}, c_0, \dots, c_{m-1})$

Set of var's / sorts: $X = \{x_0, x_1, \dots\} / \{A, B, \dots\}$

Equality: $=$

Connectives: $\wedge, \vee, \top, \perp, \exists$

Derivability notion: \vdash (given by axioms and rules)

Question:

What properties does the logic over Σ have?

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First observation:

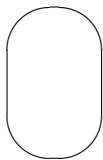
For each $n \in \mathbb{N}$,

$(Fm(x_0, \dots, x_{n-1}) / \vdash_{\top, \perp}, \vdash)$ is a distributive lattice.

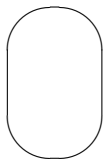
Algebraic semantics for coherent logic



$[\]$



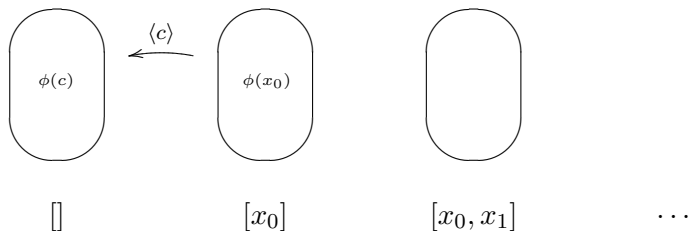
$[x_0]$



$[x_0, x_1]$

\dots

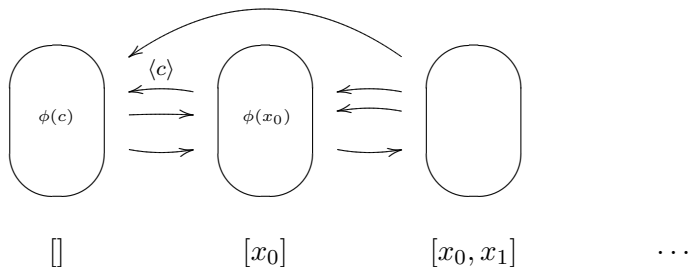
Algebraic semantics for coherent logic



Substitutions:

$$\begin{array}{lcl} x_0 & \mapsto & c \\ \phi(x_0) & \mapsto & \phi(c) \end{array}$$

Algebraic semantics for coherent logic



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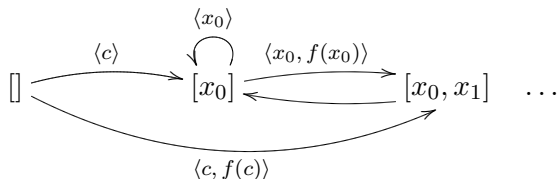
$$\begin{aligned}x_0 &\mapsto c \\ \phi(x_0) &\mapsto \phi(c)\end{aligned}$$

Algebraic semantics for coherent logic

Contexts and substitutions form a category \mathbf{B} :

Objects: natural numbers (contexts) / sorts

Morphism $n \rightarrow m$: m -tuple $\langle t_0, \dots, t_{m-1} \rangle$
s.t. $FV(t_i) \subseteq \{x_0, \dots, x_{n-1}\}$



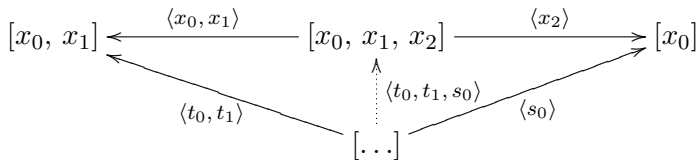
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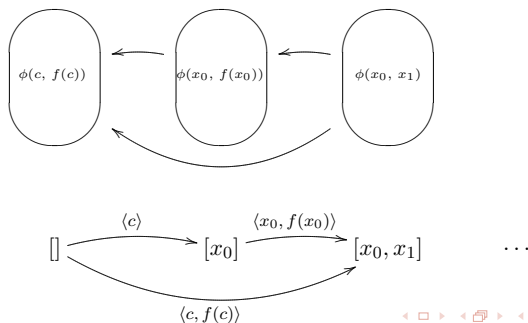
This category has **finite products**:



Algebraic semantics for coherent logic

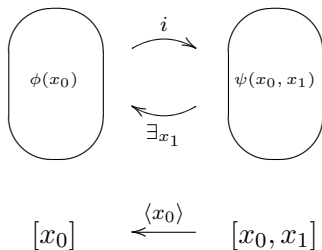
Formulas and substitutions: functor $\mathbf{B}^{op} \rightarrow \mathbf{DL}$

$$\begin{array}{lcl} n & \mapsto & Fm(x_0, \dots, x_{n-1}) \\ n \xrightarrow{\langle t_0, \dots, t_{m-1} \rangle} m & \mapsto & Fm(x_0, \dots, x_{m-1}) \rightarrow Fm(x_0, \dots, x_{n-1}) \\ & & \phi(x_0, \dots, x_{m-1}) \mapsto \phi(t_0, \dots, t_{m-1}) \end{array}$$



Algebraic semantics for coherent logic

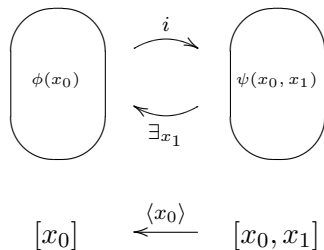
Existential quantification: related to the inclusion map



$$\frac{\exists x_1 (\psi(x_0, x_1)) \vdash \phi(x_0)}{\psi(x_0, x_1) \vdash \phi(x_0)}$$

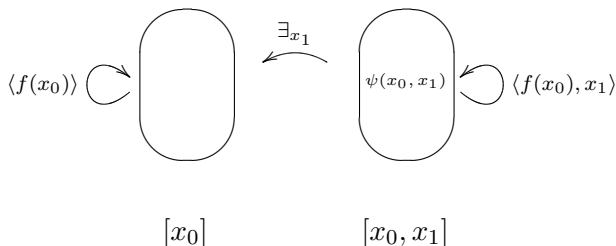
Algebraic semantics for coherent logic

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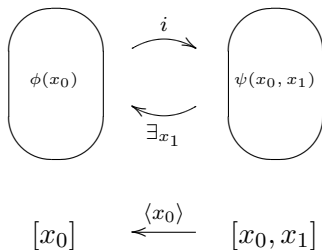
Existential quantification: interaction with substitutions



$$\exists_{x_1}(\psi(x_0, x_1))[f(x_0)/x_0] = \exists_{x_1}(\psi(f(x_0), x_1))$$

(Beck-Chevalley)

Existential quantification: interaction with substitutions



$$\exists_{x_1}[i(\phi(x_0) \wedge \psi(x_0, x_1))] = \phi(x_0) \wedge \exists_{x_1}[\psi(x_0, x_1)]$$

(Frobenius)

Algebraic semantics for coherent logic

A **polyadic distributive lattice** is a functor $P: \mathbf{B}^{\text{op}} \rightarrow \mathbf{DL}$ s.t.

1 (Contexts & substitutions)

\mathbf{B} is a category with finite products;

2 (Existential quantification)

for all $I, J \in \mathbf{B}$, $P(\pi): P(I) \rightarrow P(I \times J)$ has a left adjoint \exists_π satisfying Beck-Chevalley and Frobenius;

3 (Equality)

for all $I, J \in \mathbf{B}$, $P(\delta): P(I \times I \times J) \rightarrow P(I \times J)$ has a left adjoint \exists_δ satisfying Beck-Chevalley and Frobenius,

(where $\delta = \langle \pi_1, \pi_1, \pi_2 \rangle: I \times J \rightarrow I \times I \times J$).

Algebraic semantics for coherent logic

Examples of polyadic distributive lattices (pDL's):

■ Syntactic pDL

\mathbf{B} = contexts and substitutions

$$\begin{aligned} \mathcal{F}: \mathbf{B}^{op} &\rightarrow \mathbf{DL} \\ n &\mapsto Fm(x_0, \dots, x_{n-1}) / \vdash \cap \dashv \end{aligned}$$

■ Powerset pDL

\mathbf{B} = Set

$$\begin{aligned} \mathcal{P}: \mathbf{B}^{op} &\rightarrow \mathbf{DL} \\ A &\mapsto \mathcal{P}(A) \\ A \xrightarrow{f} B &\mapsto \mathcal{P}(B) \xrightarrow{f^{-1}} \mathcal{P}(A). \end{aligned}$$

Polyadic distr. lattices and coherent categories

Polyadic distr. lattices

Functor $P: \mathbf{B}^{op} \rightarrow \mathbf{DL}$ s.t.

- \mathbf{B} has finite products;
- $P(\pi)$ and $P(\delta)$ have left adjoints satisfying BC and Frobenius.

Coherent categories

Category \mathbf{C} s.t.

- \mathbf{C} has finite limits;
- \mathbf{C} has stable finite unions;
- \mathbf{C} has stable images.

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Proposition

There is an adjunction $\mathcal{A}: \mathbf{pDL} \rightleftarrows \mathbf{Coh}: \mathcal{S}, \mathcal{A} \dashv \mathcal{S}$.

For $\mathbf{C} \in \mathbf{Coh}$, $\mathcal{S}(\mathbf{C}) = \mathcal{S}_{\mathbf{C}}: \mathbf{C}^{op} \rightarrow \mathbf{DL}$
 $A \mapsto \text{Sub}_{\mathbf{C}}(A)$

and $\mathcal{A}(\mathcal{S}(\mathbf{C})) \simeq \mathbf{C}$.

Canonical extension of pDL's

Recall: canonical extension for DL's is a functor $\mathbf{DL} \xrightarrow{(-)^\delta} \mathbf{DL}^+$.

Definition

For a pDL $P: \mathbf{B}^{op} \rightarrow \mathbf{DL}$ we define:

$$P^\delta: \mathbf{B}^{op} \xrightarrow{P} \mathbf{DL} \xrightarrow{(-)^\delta} \mathbf{DL}.$$

Proposition

For a pDL P , P^δ is again a pDL.

Proof: check that $P^\delta(\pi)$ and $P^\delta(\delta)$ have left adjoints satisfying BC and Frobenius.

Canonical extension of coherent categories

We have:

- adjunction $\mathcal{A}: \mathbf{pDL} \rightleftarrows \mathbf{Coh}: \mathcal{S}, \mathbf{C} \simeq \mathcal{A}(\mathcal{S}_{\mathbf{C}})$
- for a pDL P , $P^{\delta}: \mathbf{B}^{op} \xrightarrow{P} \mathbf{DL} \xrightarrow{(-)^{\delta}} \mathbf{DL}$

Definition

For a coherent category \mathbf{C} we define:

$$\mathbf{C}^{\delta} = \mathcal{A}(\mathcal{S}_{\mathbf{C}}^{\delta})$$

Proposition

For a distributive lattice \mathbf{L} , $\mathcal{A}(\mathcal{S}_{\mathbf{L}}^{\delta}) \simeq \mathbf{L}^{\delta}$.

Canonical extension of coherent categories

Properties of $\mathbf{C}^\delta = \mathcal{A}(\mathcal{S}_\mathbf{C}^\delta)$:

- 1 subobject lattices are in \mathbf{DL}^+
- 2 pullback morphisms are complete lattice homomorphisms

\mathbf{Coh}^+ = coherent categories satisfying (1) and (2).

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Universal characterization:

$$\begin{array}{ccc} \mathbf{C} & \xrightarrow{M_0} & \mathbf{C}^\delta \\ & \searrow M & \downarrow \tilde{M} \\ & & \mathbf{E} \end{array}$$

where $\mathbf{C} \in \mathbf{Coh}$, $\mathbf{E}, \mathbf{C}^\delta \in \mathbf{Coh}^+$, M a coherent functor satisfying:

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where $\mathbf{C} \in \mathbf{Coh}$, $\mathbf{E}, \mathbf{C}^\delta \in \mathbf{Coh}^+$, M a coherent functor satisfying:

for all $A \xrightarrow{\alpha} B$ in \mathbf{C} , ρ (prime) filter in $\mathcal{S}_C(A)$,

$$\exists_{M(\alpha)}(\bigwedge\{M(U) \mid U \in \rho\}) = \bigwedge\{\exists_{M(\alpha)}(M(U)) \mid U \in \rho\}.$$

Topos of types

Note: $\mathcal{S}_{\mathbf{C}}^{\delta}: \mathbf{C}^{op} \rightarrow \mathbf{DL}^{+}$ is an internal frame in $\mathbf{Set}^{\mathbf{C}^{op}} = \widehat{\mathbf{C}}$.

Then $Sh_{\widehat{\mathbf{C}}}(\mathcal{S}_{\mathbf{C}}^{\delta}) \simeq T(\mathbf{C}) =$ **topos of types** of \mathbf{C} .

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Topos of types was introduced by Makkai in 1979 as:

- 'a reasonable codification of the 'discrete' (non topological) syntactical structure of types of the theory'
- a tool to prove representation theorems
- 'conceptual tool meant to enable us to formulate precisely certain natural intuitive questions'

Some later work by: Magnan & Reyes and Butz.

Topos of types and the class of models

For a distributive lattice \mathbf{L} ,

$$\begin{aligned}\text{prime filters of } \mathbf{L} &= \text{lattice homomorphisms } \mathbf{L} \rightarrow \mathbf{2} \\ &= \text{'models of } \mathbf{L}\text{'}. \end{aligned}$$

$$\mathbf{L}^\delta = \mathcal{D}(\text{Mod}(\mathbf{L}))$$

Categorical analogue:

$$\text{Mod}(\mathbf{C}) = \text{coherent functors } M: \mathbf{C} \rightarrow \mathbf{Set}.$$

$$\text{Study: } \mathbf{Set}^{\text{Mod}(\mathbf{C})}.$$

We have to restrict to an appropriate subcategory \mathcal{K} of $\text{Mod}(\mathbf{C})$.

Question: How does $\mathbf{Set}^{\mathcal{K}}$ relate to $T(\mathbf{C}) = \text{Sh}_{\hat{\mathbf{C}}}(\mathcal{S}_{\mathbf{C}}^\delta)$?

Topos of types and the class of models

\mathcal{K} appropriate subcategory of $Mod(\mathbf{C})$.

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Evaluation functor $ev: \mathbf{C} \rightarrow \mathbf{Set}^{\mathcal{K}}$

$$A \mapsto ev(A): \mathcal{K} \rightarrow \mathbf{Set}$$

$$M \mapsto M(A)$$

Gives a geometric morphism $\phi_{ev}: \mathbf{Set}^{\mathcal{K}} \rightarrow \mathbf{Set}^{C^{op}}$

$$\begin{array}{ccc} & & T(\mathbf{C}) \\ & \nearrow & \downarrow \\ \mathbf{Set}^{\mathcal{K}} & \xrightarrow{\phi_{ev}} & \mathbf{Set}^{C^{op}} \end{array}$$

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Makkai: $T(\mathbf{C}) \simeq$ functors in $\mathbf{Set}^{\mathcal{K}}$ with finite support property

Future work

We have: notion of canonical extension for coherent categories

We would like to:

- Study the following diagram (where $\mathcal{K} \subseteq \text{Mod}(\mathbf{C})$):

$$\begin{array}{ccc} & & T(\mathbf{C}) \\ & \nearrow & \downarrow \\ \text{Set}^{\mathcal{K}} & \xrightarrow{\phi_{ev}} & \text{Sh}(\mathbf{C}, J_{coh}) \end{array}$$

- Generalize to Heyting categories and study addition of axioms
- Apply the developed theory in the study of first order logics
- In particular: study interpolation problems for first order logics, e.g. for IPL + $(\phi \rightarrow \psi) \vee (\psi \rightarrow \phi)$