

# Constructing the Lindenbaum algebra for a logic step-by-step using duality (extended version)

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## Abstract

We discuss the incremental construction of the Lindenbaum algebra for a modal logic, using discrete duality for Boolean algebras with operators. We work out in detail the case of the modal logic T, as an illustrative example of the method for modal logics with mixed-rank axiomatizations.

## 1 Introduction

In the study of a propositional logic  $\mathcal{L}$ , the following construction is often important: take the set of all formulas in the language of  $\mathcal{L}$ , and partition this set into classes of  $\mathcal{L}$ -equivalent formulas. In many cases, the set of  $\mathcal{L}$ -equivalence classes has a natural algebraic structure, which is called the *Lindenbaum algebra* for the logic  $\mathcal{L}$ .

Algebraic methods are useful to study issues such as term complexity, decidability of logical equivalence, interpolation, and normal forms for a logic, i.e., problems in which one considers formulas whose variables are drawn from a finite set. The oldest instance of the use of algebraic methods in logic goes back to George Boole: the Lindenbaum algebra for *classical propositional logic* (CPL) on  $n$  variables  $\{p_1, \dots, p_n\}$  is a *Boolean algebra*, and it can be shown to be (isomorphic to)  $\mathcal{P}(\mathcal{P}(\{p_1, \dots, p_n\}))$ , which, as we will explain below, is the free Boolean algebra on  $n$  generators. The logical impact of this result is the disjunctive normal form theorem for CPL. However, for logics other than CPL, the situation is often much more complicated, and a simple description of the Lindenbaum algebra is usually not available. For example, the Lindenbaum algebra for *intuitionistic propositional logic* (IPL) on only two variables, i.e., the free Heyting algebra on two generators, is already infinite and non-trivial to describe.

*Modal logics* form another rich class of examples of logics whose Lindenbaum algebras are often infinite and complicated. These logics are based on CPL, enriched with a unary connective ‘ $\diamond$ ’, which is meant to formalize a notion of ‘possibility’. Different axioms for  $\diamond$  yield different modal logics. One may try to gain a better understanding of a particular modal logic through its Lindenbaum algebra. As a representative example, we will mainly concentrate on the Lindenbaum algebra for a

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very simple modal logic called T, and we will also indicate how we expect the described methods may apply to a larger class of logics.

To motivate our methodology, we start from the following well-known definition of the language of propositional logic with one unary modality (see, for instance, [3]).

**Definition 1.** Let  $P$  be a set of **propositional variables**. The set of **propositional formulas in  $P$** ,  $\text{Prop}(P)$ , is defined to be the smallest set which contains  $P$ ,  $\perp$ ,  $\top$ , and is closed under the formation rules: for all  $\varphi, \psi \in \text{Prop}(P)$ ,  $\varphi \wedge \psi$ ,  $\varphi \vee \psi$ ,  $\varphi \rightarrow \psi$  and  $\neg\varphi$  are in  $\text{Prop}(P)$ .

The sets  $\Phi_n(P)$  of **modal formulas in  $P$  of rank at most  $n$**  are defined inductively as follows.

$$\begin{aligned}\Phi_0(P) &:= \text{Prop}(P), \\ \Phi_{n+1}(P) &:= \text{Prop}(P \cup \{\diamond\varphi : \varphi \in \Phi_n(P)\}).\end{aligned}$$

The set of **modal formulas in  $P$**  is defined as  $\Phi(P) := \bigcup_{n \in \mathbb{N}} \Phi_n(P)$ . In algebraic terms,  $\Phi(P)$  is the domain of the **absolutely free algebra** in the signature  $\{\perp, \top, \neg, \diamond, \wedge, \vee, \rightarrow\}$ .

This *step-by-step* construction of the absolutely free algebra (i.e., the algebra describing the modal *language*) is a first example of the construction that we will also use for the Lindenbaum algebra (i.e., the algebra describing a modal *logic*). More precisely, we aim to understand the Lindenbaum algebra  $\mathbb{B}$  for a modal logic  $\mathcal{L}$  in a layered manner: for each  $n \geq 0$ , the  $\mathcal{L}$ -equivalence classes of modal formulas of rank at most  $n$  form a Boolean subalgebra  $\mathbb{B}_n$  of  $\mathbb{B}$ . We thus get a chain of Boolean algebras

$$\mathbb{B}_0 \twoheadrightarrow \mathbb{B}_1 \twoheadrightarrow \mathbb{B}_2 \twoheadrightarrow \dots,$$

and the Lindenbaum algebra is then the direct limit, or colimit, of this chain (see Section 3.4 below). Each of the algebras  $\mathbb{B}_n$  will be finite, and  $\mathbb{B}_n$  embeds into  $\mathbb{B}_{n+1}$  for each  $n$ . These two properties imply that the chain accurately approximates the infinite Lindenbaum algebra  $\mathbb{B}$  by its finite pieces.

In certain well-behaved examples, the finite pieces  $\mathbb{B}_0, \mathbb{B}_1, \dots$  of the Lindenbaum algebra can be described by a uniform process of the following kind: start from a simple  $\mathbb{B}_0$  (usually the Lindenbaum algebra for CPL), and then define  $\mathbb{B}_{n+1}$  from  $\mathbb{B}_n$  in a uniform way. As an immediate application, one then obtains an algorithm for deciding  $\mathcal{L}$ -equivalence: given two formulas  $\varphi, \psi$ , let  $n$  be the maximum of the ranks of  $\varphi$  and  $\psi$  (a formal definition of *rank* will be given below), and check whether  $\varphi$  and  $\psi$  are equal under all interpretations of the propositional variables in the finite algebra  $\mathbb{B}_n$ . Examples of logics for which the step-by-step construction works are (trivially) CPL, but also the basic modal logic K, the modal logic S4, which is closely connected to IPL, and, as we will show in this paper, the modal logic T.

All these examples now beg an interesting question, which one may ask for any given modal logic  $\mathcal{L}$ :

Is there a uniform step-by-step construction of the Lindenbaum algebra for  $\mathcal{L}$ ? ( $Q_{\mathcal{L}}$ )

With the syntactic definition of *rank*, we can describe an important class of examples of logics for which the answer to the question ( $Q_{\mathcal{L}}$ ) is affirmative.

For a modal formula  $\varphi \in \Phi(P)$ , the **rank** of an occurrence of a propositional variable  $p \in P$  is the number of  $\diamond$ 's having that occurrence of  $p$  in its scope. We say that  $\varphi$  is of **rank at most  $n$**  if every occurrence of a variable in  $\varphi$  is of rank less than or equal to  $n$ , and that  $\varphi$  is of **rank exactly  $n$**  or **pure rank  $n$**  if every occurrence of a variable in the formula is of rank equal to  $n$ . We also say, for instance,

“ $\varphi$  is of (mixed) rank 1-2” if every occurrence of a variable is of rank either 1 or 2.

For example, the formulas  $\diamond(p \vee q)$  and  $\diamond p \vee \diamond q$  are of rank exactly 1, whereas the formula  $\diamond p \wedge p$  is not of rank exactly 1, but it is of rank 0-1. The formula  $\diamond \diamond p \wedge \diamond p$  is of rank 1-2.

If all axioms for a logic  $\mathcal{L}$  are of rank exactly 1, then the answer to the question  $(Q_{\mathcal{L}})$  can be shown to be affirmative, using the fact that  $\mathcal{L}$  is in the realm of ‘algebras for a functor’ (cf. [4] and [2] for detailed accounts). As a result, one directly obtains a normal form theorem and an algorithm for deciding  $\mathcal{L}$ -equivalence for all rank 1 logics.

On the other side of the spectrum, there exist modal logics in which  $\mathcal{L}$ -equivalence is undecidable (cf. Chapter 6 of [3]), so that, in particular, one can not hope to have an affirmative answer to the question  $(Q_{\mathcal{L}})$  for such logics. As far as we know, the most general class for which a positive answer to the question  $(Q_{\mathcal{L}})$  has been given is the class of logics with axioms of rank exactly 1. However, [6], [8] and [1] have widened the scope of the uniform step-by-step approach to a few particular logics that are not axiomatizable by pure rank 1 axioms, namely IPL and S4. In this paper, we will discuss another example of a rank 0-1 logic which admits a step-by-step approach, namely the modal logic T, the modal logic of reflexive frames. For the line of research outlined above, it is important to properly understand the example of the modal logic T, because it falls ‘in between’ the basic modal logic K and the logic S4, so that it is on the one hand simpler than S4, but on the other hand already needs much of the same machinery as is needed for S4, and falls outside the realm of pure rank 1 logics. In the final section of [2], the authors already briefly indicate how to see the modal logic T as a first step outside pure rank 1 axioms. We will work this out in detail here.

A key feature of the approach to answering the question  $(Q_{\mathcal{L}})$  is the use of *duality* to obtain a concrete, set-theoretic, *dual description* of the finite algebras in the approximating sequence. Important questions about the chain of algebras and homomorphisms can then be answered more quickly by looking at the dual chain of sets and functions. Most crucially, for the step-by-step construction of the Lindenbaum algebra to be useful in applications, one needs the homomorphisms  $\mathbb{B}_n \rightarrow \mathbb{B}_{n+1}$  in the chain to be injective. Using the dual description, this amounts to checking that certain functions are surjective (cf. Proposition 10 below). One thus reduces an abstract, formulaic question about algebra to a concrete, spatial question about sets. As such, this approach is a typical example of the use of duality in algebra and logic.

The outline of this paper is as follows. We will start by reviewing some preliminaries on universal algebra and on duality for finite Boolean algebras in Section 2, which we will need, respectively, to justify the step-by-step construction of the Lindenbaum algebra, and its dual description. We will discuss the construction and its dual description in Section 3, which forms the heart of this paper. Finally, we will point at some possible applications and future questions in our concluding Section 4.

## 2 Preliminaries

### 2.1 The Lindenbaum algebra as a free algebra for a variety

In this subsection, we quickly review the necessary background from universal algebra which justifies the step-by-step construction of the Lindenbaum algebra. We will not go into much detail, and will assume a basic understanding of the algebraic perspective on logic, cf. [3] for more details.

From the perspective of algebraic logic, a logic which can be defined by equations naturally gives rise to an equationally defined class of algebras, i.e., a *variety*. For example, CPL corresponds to the

variety of Boolean algebras, and IPL corresponds to the variety of Heyting algebras. The basic modal logic  $\mathbf{K}$  is defined by adding an operator  $\diamond$  to the language of CPL (as in Definition 1 above), and adding as axioms the logical equivalences

$$\diamond \perp \leftrightarrow \perp \tag{1}$$

$$\diamond(a \vee b) \leftrightarrow \diamond a \vee \diamond b. \tag{2}$$

Modal logics also naturally correspond to varieties of algebras, which are called ‘modal algebras’.

**Definition 2.** A **modal algebra** is a pair  $(\mathbb{B}, \diamond^{\mathbb{B}})$ , where  $\mathbb{B}$  is a Boolean algebra, and  $\diamond^{\mathbb{B}} : \mathbb{B} \rightarrow \mathbb{B}$  is a unary operation preserving  $\perp$  and binary joins. A **homomorphism**  $f : (\mathbb{B}, \diamond^{\mathbb{B}}) \rightarrow (\mathbb{C}, \diamond^{\mathbb{C}})$  of modal algebras is a Boolean algebra homomorphism  $f : \mathbb{B} \rightarrow \mathbb{C}$  such that  $f \circ \diamond^{\mathbb{B}} = \diamond^{\mathbb{C}} \circ f$ .

Let  $\mathcal{L}$  be a modal logic which contains the basic modal logic  $\mathbf{K}$ . The **variety of  $\mathcal{L}$ -algebras**,  $\mathcal{V}_{\mathcal{L}}$ , is defined to be the class of modal algebras in which all  $\mathcal{L}$ -equivalent formulas are equal under all interpretations of the variables. In particular,  $\mathcal{V}_{\mathbf{K}}$  is the variety of all modal algebras.

Recall that we defined the Lindenbaum algebra for a logic  $\mathcal{L}$  as the quotient of the absolutely free algebra  $\Phi(X)$  by the relation of  $\mathcal{L}$ -equivalence. The important fact from universal algebra that we need is that the Lindenbaum algebra for the logic  $\mathcal{L}$  is exactly the so-called **free algebra for the variety  $\mathcal{V}_{\mathcal{L}}$** .

**Proposition 3.** *Let  $\mathbb{B}$  be the Lindenbaum algebra on a set of propositional variables  $P$  for a modal logic  $\mathcal{L}$ . Then  $\mathbb{B}$  is the free  $\mathcal{V}_{\mathcal{L}}$ -algebra over  $P$ , i.e.,  $\mathbb{B}$  is a  $P$ -generated algebra in  $\mathcal{V}_{\mathcal{L}}$  with the following **universal mapping property**: For any  $\mathbb{A} \in \mathcal{V}_{\mathcal{L}}$  and any function  $f : P \rightarrow \mathbb{A}$ , there exists a (necessarily unique) modal algebra homomorphism  $\bar{f} : \mathbb{B} \rightarrow \mathbb{A}$  which extends  $f$ .*

*Proof.* Cf. any reference on universal algebra or algebraic logic, for example [5], Theorem 10.10.  $\square$

For us, the importance of this Proposition is that in order to construct the Lindenbaum algebra for a logic  $\mathcal{L}$  on  $n$  propositional variables, we may now construct an algebra  $\mathbb{B}$  and show that it is the free  $\mathcal{V}_{\mathcal{L}}$ -algebra on  $n$  generators. As we pointed out in the introduction, logical problems, such as deciding logical equivalence, then become equivalent to algebraic problems, i.e., deciding equality in the algebra  $\mathbb{B}$ .

## 2.2 Duality for finite Boolean algebras

The dual, set-theoretic, description of the chain of Boolean algebras that we will build relies on so-called discrete duality for finite Boolean algebras. The results in this section have been around since [11], and there is no claim of originality.

Let us first introduce the following notation for the categories that are involved.

Category	Objects	Morphisms
$\mathbf{BA}_{<\omega}$	finite Boolean algebras	Boolean algebra homomorphisms
$\mathbf{BA}_{<\omega}^{\vee}$	finite Boolean algebras	$\vee$ -semilattice homomorphisms
$\mathbf{Set}_{<\omega}$	finite sets	functions
$\mathbf{Set}_{<\omega}^{\text{Rel}}$	finite sets	relations

The following is the basic Stone duality result in the finite case. We will sketch the proof and, while doing so, introduce notation that we will use later.

**Theorem 4.** *The categories  $\mathbf{BA}_{<\omega}^\vee$  and  $\mathbf{Set}_{<\omega}^{\text{Rel}}$  are dually equivalent. The dual equivalence restricts to a dual equivalence between the categories  $\mathbf{BA}_{<\omega}$  and  $\mathbf{Set}_{<\omega}$ .*

*Proof (Sketch).* Given a finite set  $X$ , the power set algebra  $\mathbb{P}(X)$  is the Boolean algebra based on the power set  $\mathcal{P}(X)$  of  $X$ . Given a Boolean algebra  $\mathbb{B}$ , its dual is defined to be the set of atoms<sup>1</sup> of  $\mathbb{B}$ . An atom  $x$  defines a homomorphism  $h_x : \mathbb{B} \rightarrow \mathbf{2}$  by putting  $h_x(a) := \top$  iff  $x \leq a$ , and conversely, if  $h : \mathbb{B} \rightarrow \mathbf{2}$  is a homomorphism, then  $\bigwedge \{b \in \mathbb{B} : h(b) = \top\}$  is an atom. We denote the set of atoms of  $\mathbb{B}$  by  $\text{At}(\mathbb{B})$ .

Given a relation  $R : X \rightarrow Y$  between finite sets, we have a Boolean algebra hemimorphism  $\diamond_R : \mathbb{P}(Y) \rightarrow \mathbb{P}(X)$ , defined by

$$\diamond_R(U) := \{x \in X \mid \exists y \in U : x R y\}.$$

Given a join-preserving function  $h : \mathbb{B} \rightarrow \mathbb{C}$  between finite Boolean algebras, we define a relation  $R_h : \text{At}(\mathbb{C}) \rightarrow \text{At}(\mathbb{B})$  by

$$x_C R_h x_B \iff x_C \leq h(x_B).$$

For any set  $X$ , we have the natural bijection  $\varphi_X : X \rightarrow \text{At}(\mathbb{P}(X))$  which sends  $x$  to  $\{x\}$ . For any finite Boolean algebra, we have the natural isomorphism  $\psi_{\mathbb{B}} : \mathbb{B} \rightarrow \mathbb{P}(\text{At}(\mathbb{B}))$  which sends  $b$  to  $\{x \in \text{At}(\mathbb{B}) : x \leq b\}$ .

It is straightforward to show that the assignments  $\text{At}$  and  $\mathbb{P}$  given above define functors, and that  $\varphi$  and  $\psi$  are natural transformations in the sense of category theory.

For the second statement, it suffices to note that, given a join-preserving function  $h : \mathbb{B} \rightarrow \mathbb{C}$ , the relation  $R_h$  is the graph of some function  $\text{At}(\mathbb{C}) \rightarrow \text{At}(\mathbb{B})$  if and only if  $h$  is also meet-preserving. In that case, the function of which  $R_h$  is the graph can be given explicitly by restricting the lower adjoint  $h^b$  of  $h$  to the set of atoms  $\text{At}(\mathbb{C})$ .  $\square$

There are several consequences of the dual equivalence  $\mathbf{BA}_{<\omega} \rightleftarrows \mathbf{Set}_{<\omega}$  that we will often use. Firstly, it allows us to describe coproducts of Boolean algebras using products of sets, as follows.

**Proposition 5.** *Let  $\mathbb{B}$  and  $\mathbb{C}$  be finite Boolean algebras. The coproduct  $\mathbb{B} + \mathbb{C}$  is isomorphic to  $\mathbb{P}(\text{At}(\mathbb{B}) \times \text{At}(\mathbb{C}))$ . The coproduct injection  $\mathbb{B} \rightarrow \mathbb{B} + \mathbb{C}$  corresponds under the duality to the projection function  $\text{At}(\mathbb{B}) \times \text{At}(\mathbb{C}) \rightarrow \text{At}(\mathbb{B})$ , and similarly for the injection  $\mathbb{C} \rightarrow \mathbb{B} + \mathbb{C}$ .*

*Proof.* By duality  $\text{At}(\mathbb{B} + \mathbb{C}) \cong \text{At}(\mathbb{B}) \times \text{At}(\mathbb{C})$ , so that the first statement readily follows by applying  $\mathbb{P}$  to both sides. For the second part, it is not hard to show, again using duality, that the co-cone under  $\mathbb{B}$  and  $\mathbb{C}$  given by the duals of the projection functions has the desired universal property.  $\square$

The second consequence is that a surjective homomorphism (i.e., epimorphism)  $\pi : \mathbb{B} \rightarrow \mathbb{C}$  of finite Boolean algebras corresponds to an injective function (i.e., monomorphism)  $i_\pi : \text{At}(\mathbb{C}) \rightarrow \text{At}(\mathbb{B})$  of sets. So the atoms of a quotient of  $\mathbb{B}$  may be identified with a subset of the atoms of  $\mathbb{B}$ . We can be a bit more specific, introducing some notation: if  $h : \mathbb{B} \rightarrow \mathbb{C}$  is a BA homomorphism, recall that its kernel is the set  $\ker(h) := \{(b, b') \in B^2 : h(b) = h(b')\}$ , and that atoms  $x_B \in \text{At}(\mathbb{B})$  may be identified with homomorphisms  $h_{x_B} : \mathbb{B} \rightarrow \mathbf{2}$ .

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<sup>1</sup>An **atom** of  $\mathbb{B}$  is a non-zero element  $a \in \mathbb{B}$  which does not have any non-zero elements below it.

**Proposition 6.** *Let  $\pi : \mathbb{B} \rightarrow \mathbb{C}$  be a surjective homomorphism, and  $i_\pi : \text{At}(\mathbb{C}) \rightarrow \text{At}(\mathbb{B})$  the injective function dual to  $\pi$ . Then*

$$\begin{aligned} i_\pi[\text{At}(\mathbb{C})] &= \{x_B \in \text{At}(\mathbb{B}) : \ker(\pi) \subseteq \ker(h_{x_B})\} \\ &= \{x_B \in \text{At}(\mathbb{B}) : \forall b, b' \in B : \pi(b) = \pi(b') \rightarrow (x_B \leq b \leftrightarrow x_B \leq b')\}. \end{aligned}$$

*In particular, since  $i_\pi$  is injective,  $\text{At}(\mathbb{C})$  may be identified with the set  $\{x_B \in \text{At}(\mathbb{B}) : \forall b, b' \in B : \pi(b) = \pi(b') \rightarrow (x_B \leq b \leftrightarrow x_B \leq b')\}$ .*

*Proof.* We recall from the proof of Theorem 4 that the function dual to  $\pi$  is its lower adjoint  $\pi^b$ , restricted to  $\text{At}(\mathbb{C})$ .

The equality from the first to second line follows by writing out the definitions. We show that  $i_\pi[\text{At}(\mathbb{C})]$  is equal to the set in the second line.

Suppose  $x_B = i_\pi(x_C) = \pi^b(x_C)$  for some atom  $x_C$  of  $\mathbb{C}$ , and let  $b, b' \in B$  such that  $\pi(b) = \pi(b')$ . Then  $x_B \leq b$  iff  $\pi^b(x_C) \leq b$  iff  $x_C \leq \pi(b)$  iff  $x_C \leq \pi(b')$  iff  $x_B \leq b'$ .

Now suppose that  $x_B$  is in the set on the second line. We claim that  $x_B = \pi^b(\pi(x_B))$ . The inequality  $\pi^b(\pi(x_B)) \leq x_B$  holds by adjunction. For the other inequality, note that  $\pi\pi^b\pi(x_B) = \pi(x_B)$ , so that  $x_B \leq \pi^b\pi(x_B)$  iff  $x_B \leq x_B$ , because  $x_B$  is in the set on the second line. Since  $\pi(x_B) = \bigvee \{y \in \text{At}(\mathbb{C}) : y \leq \pi(x_B)\}$ ,  $\pi^b$  preserves joins, and  $x_B$  is join-irreducible, we can pick  $y \in \text{At}(\mathbb{C})$  with  $y \leq \pi(x_B)$  such that  $x_B = \pi(y)$ .  $\square$

The duality can also be used to give concrete descriptions of the free Boolean algebra  $F_{\text{BA}}(P)$ . Recall that the universal property defining the free Boolean algebra says that for any Boolean algebra  $\mathbb{B}$ , if  $f : P \rightarrow |\mathbb{B}|$  is a function, there is a unique Boolean algebra homomorphism  $\bar{f} : F_{\text{BA}}(P) \rightarrow \mathbb{B}$  such that the diagram

$$\begin{array}{ccc} P & \xrightarrow{i} & F_{\text{BA}}(P) \\ \downarrow f & & \searrow \bar{f} \\ \mathbb{B} & & \end{array}$$

commutes.

The following proposition underlies the well-known fact that  $F_{\text{BA}}(P)$  is the Boolean algebra with  $2^{|P|}$  elements.

**Proposition 7.** *There is a bijective function between the sets  $\text{At}(F_{\text{BA}}(P))$  and  $\mathcal{P}(P)$ . It can be given explicitly by sending an atom  $x$  of  $F_{\text{BA}}(P)$  to the set  $\{p \in P : x \leq p\}$ , and, conversely, sending  $A \subseteq P$  to  $z(A) := \bigwedge_{p \in A} p \wedge \bigwedge_{q \notin A} \neg q$ .*

*Proof.* The atoms of  $F_{\text{BA}}(P)$  correspond to homomorphisms  $F_{\text{BA}}(P) \rightarrow \mathbf{2}$ , and by the universal property they must be in a one-to-one correspondence with functions  $P \rightarrow \mathbf{2}$ , i.e., elements of  $\mathcal{P}(P)$ .

The explicit bijection can be calculated by going through this chain of correspondences; starting with an atom  $x$  of  $F_{\text{BA}}(P)$ , the associated homomorphism  $h_x : F_{\text{BA}}(P) \rightarrow \mathbf{2}$  sends  $a$  to  $\top$  iff  $x \leq a$ . Restricting this homomorphism to  $P$  we get the function which sends  $p$  to  $\top$  iff  $x \leq p$ , which is the indicator function of the set  $\{p \in P : x \leq p\}$ .

Conversely, starting with a subset  $A$  of  $P$ , let  $\overline{I}_A : F_{\text{BA}}(P) \rightarrow \mathbf{2}$  be the unique BA homomorphism which extends the indicator function  $I_A : P \rightarrow \mathbf{2}$ . This homomorphism represents the atom  $\bigwedge\{a \in F_{\text{BA}}(P) : \overline{I}_A(a) = \top\}$ , which can be shown to be equal to  $z(A) := \bigwedge_{p \in A} p \wedge \bigwedge_{q \notin A} \neg q$ .  $\square$

### 3 Construction of the free $\top$ -algebra

We will concentrate on the construction of the finitely generated free algebras for the variety  $\mathcal{V}_\top \subseteq \mathcal{V}_K$ , consisting of the modal algebras  $(\mathbb{B}, \diamond)$  satisfying the additional equation

$$a \leq \diamond a \tag{T}$$

(Note that any inequality is indeed equivalent to an equation, since in any lattice we have  $a \leq b$  iff  $a \wedge b = a$ .)

#### 3.1 A chain of finite Boolean algebras

Let  $P$  be a finite set of proposition letters. We are going to construct a chain diagram in the category of Boolean algebras, whose colimit will be the Boolean algebra underlying the free modal algebra for  $\mathcal{V}_\top$  over  $P$ .

Let  $\mathbb{B}_0 := F_{\text{BA}}(P)$ , the free Boolean algebra over  $P$ .

For  $n \geq 0$ , assume  $\mathbb{B}_n$  has been defined by induction. Denote by  $\blacklozenge B_n$  the *set* of symbols  $\{\blacklozenge b : b \in B_n\}$ . Let  $\mathbb{C}_{n+1} := \mathbb{B}_n + F_{\text{BA}}(\blacklozenge B_n)$ . We emphasize that in the second summand, we are taking the free Boolean algebra over the *set*  $\blacklozenge B_n$  underlying the Boolean algebra  $\mathbb{B}_n$ , forgetting all the structure of  $\mathbb{B}_n$  for this part.

Denote the coproduct injection of the first coordinate by  $j_n : \mathbb{B}_n \rightarrow \mathbb{C}_{n+1}$ , which is a BA homomorphism. Further let  $\blacklozenge_n : B_n \rightarrow C_{n+1}$  be the composition of the universal arrow  $B_n \rightarrow F_{\text{BA}}(\blacklozenge B_n)$ , followed by coproduct injection  $F_{\text{BA}}(\blacklozenge B_n) \rightarrow \mathbb{C}_{n+1}$  of the second coordinate. We emphasize again that, so far,  $\blacklozenge_n$  is only a function between the domains of the algebras.

Consider the following equations, for  $a_n, b_n \in B_n, a_{n-1} \in \mathbb{B}_{n-1}$ :

$$\blacklozenge \perp = \perp \tag{1_n}$$

$$\blacklozenge(a_n \vee b_n) = \blacklozenge a_n \vee \blacklozenge b_n \tag{2_n}$$

$$j_n a_n \leq \blacklozenge a_n \tag{T_n}$$

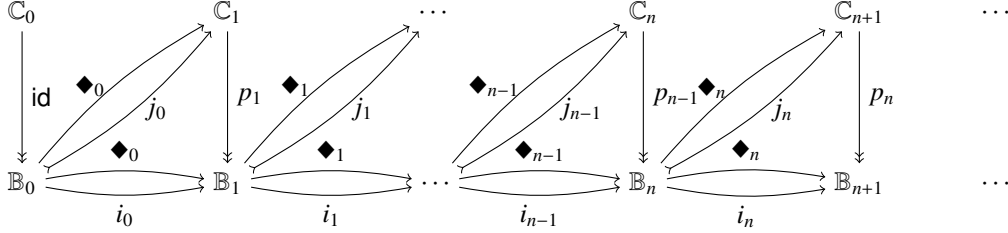
$$j_n \blacklozenge_{n-1} a_{n-1} = \blacklozenge_{n-1} a_{n-1} \tag{W_n}$$

Let us write  $\mathcal{E}_n$  for the set of instances of the equations  $(1_n), (2_n), (T_n), (W_n)$ . We may view  $\mathcal{E}_n$  as the set of those pairs  $(c_{n+1}, d_{n+1}) \in C_{n+1} \times C_{n+1}$  such that “ $c_{n+1} = d_{n+1}$ ” is an instance of one of these equations<sup>2</sup>. Then  $\mathcal{E}_n$  generates a congruence relation  $\approx_n$  on  $\mathbb{C}_{n+1}$ . We define  $\mathbb{B}_{n+1}$  to be the quotient  $\mathbb{C}_{n+1}/\approx_n$  and  $p_n : \mathbb{C}_{n+1} \rightarrow \mathbb{B}_{n+1}$  the canonical projection, for which  $\ker(p_n) = \approx_n$ . We define  $i_n := p_n \circ j_n : \mathbb{B}_n \rightarrow \mathbb{B}_{n+1}$ , and, by a slight abuse of notation, also denote the function  $p_n \circ \blacklozenge_n$  by  $\blacklozenge_n$ . Note that  $i_n$  is a BA homomorphism, and moreover that  $\blacklozenge_n$  is now a join-preserving function, because equations  $(1_n)$  and  $(2_n)$  hold in  $\mathbb{B}_{n+1}$ . Intuitively, the axiom  $(T_n)$  is included to ensure that

<sup>2</sup>For the equation  $(W_n)$  to make sense when  $n = 0$ , we put  $\mathbb{B}_{-1} := \mathbf{2}$ , and  $i_{-1} = \blacklozenge_{-1}$  the homomorphism  $\mathbb{B}_{-1} \rightarrow \mathbb{B}_0$ , and note that  $(W_n)$  then only says  $\blacklozenge_0 \top = \top$  and  $\blacklozenge_0 \perp = \perp$ , which already follows from equations  $(1_0)$  and  $(T_0)$ .

we are really building an algebra in the variety  $\mathcal{V}_\top$ . Axiom  $(W_n)$  is needed so that we will end up having a well-defined operator  $\diamond$  on the colimit of the chain of Boolean algebras. This argument will be made precise in Subsection 3.4 below.

We thus get the following chains of (finite) Boolean algebras



### 3.2 The dual chain of finite sets

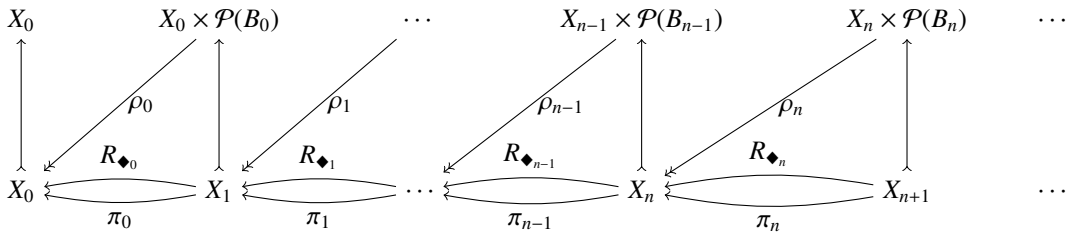
We will now explicitly calculate what the sets, functions and relations dual to the Boolean algebras and morphisms in the above-defined chain are.

Since  $\mathbb{B}_0 = F_{\text{BA}}(P)$ , the set  $X_0$  can be identified with  $\mathcal{P}(P)$ , by Proposition 7. In the rest of this subsection, we will calculate an explicit description of the set  $X_{n+1} = \text{At}(\mathbb{B}_{n+1})$ , assuming the sets  $X_m$  for  $m \leq n$  have been described.

First of all, we note that  $\text{At}(\mathbb{C}_{n+1})$  is equal to  $X_n \times \mathcal{P}(B_n)$ , using Propositions 5 and 7. We write an arbitrary atom of  $\mathbb{C}_{n+1}$  as  $x_C = (x, A)$ , where  $x \in X_n$  and  $A \subseteq B_n$ .

By Proposition 5, the dual of  $j_n : \mathbb{B}_n \rightarrow \mathbb{B}_n + F_{\text{BA}}(\diamond B_n)$  is given by the projection onto the first coordinate,  $\rho_n : X_n \times \mathcal{P}(B_n) \rightarrow X_n$ . Its restriction to  $X_{n+1}$  is dual to the map  $i_n : \mathbb{B}_n \rightarrow \mathbb{B}_{n+1}$ , and we will denote it by  $\pi_n$ .

Summing up, we now have the following diagram of finite sets, functions and relations, the relation  $R_{\diamond_n}$  being dual to the join-preserving function  $\diamond_n : \mathbb{B}_n \rightarrow \mathbb{B}_{n+1}$ , recalling Theorem 4.



The following lemma will give concrete conditions under which certain inequalities in  $\mathbb{C}_{n+1}$  hold.

**Lemma 8.** *Let  $\mathbb{B}$  be a Boolean algebra,  $X$  its set of atoms,  $\mathbb{C} := \mathbb{B} + F_{\text{BA}}(\diamond B)$ , and  $j$  and  $\diamond$  the obvious maps  $\mathbb{B} \rightarrow \mathbb{C}$ . For any atom  $(x, A) \in X \times \mathcal{P}(B)$  of  $\mathbb{C}$  and  $b \in \mathbb{B}$ , we have*

1.  $(x, A) \leq j(b)$  in  $\mathbb{C} \iff x \leq b$  in  $\mathbb{B}$ ,
2.  $(x, A) \leq \diamond b$  in  $\mathbb{C} \iff b \in A$ .

*Proof.* 1. If  $(x, A) \leq j(b)$  then  $(x, A) \leq j(y)$  for some atom  $y \leq b$ , using that atoms are join-prime and that  $j(b) = \bigvee \{j(y) : y \leq b\}$ .

For atoms  $y \in B$ , we have  $j(y) = \bigvee \{(y, S) : S \subseteq B\}$ , using the dual characterization of  $j$  as the projection onto the first coordinate. Therefore, since  $(x, A)$  is join-prime,  $(x, A) = (y, S)$  for some  $S \subseteq B$ , and in particular  $x = y \leq b$ . Conversely, if  $x \leq b$ , then  $(x, A) \leq j(x) \leq j(b)$ .

2. Since  $\blacklozenge b$  is in the second coordinate of the coproduct, by an argument similar to (1), we get that  $(x, A) \leq \blacklozenge b$  in  $\mathbb{C}$  iff  $z(A) \leq \blacklozenge b$  in  $F_{\text{BA}}(\blacklozenge B)$ , where we write  $z(A)$  for the atom of  $F_{\text{BA}}(\blacklozenge B)$  that the set  $A$  represents:  $z(A) := \bigwedge_{a \in A} \blacklozenge a \wedge \bigwedge_{a' \notin A} \neg \blacklozenge a'$ .

Suppose  $z(A) \leq \blacklozenge b$ . Let  $I_A : B \rightarrow 2$  be the characteristic function of the set  $A \subseteq B$ , i.e.,  $I_A(a) = \top$  iff  $\blacklozenge a \in A$ . By the universal property of  $F_{\text{BA}}(\blacklozenge B)$ , there is a unique BA homomorphism  $\overline{I}_A : F_{\text{BA}}(\blacklozenge B) \rightarrow 2$  such that  $\overline{I}_A \circ \blacklozenge = I_A$ . Then  $\overline{I}_A(z(A)) = \bigwedge_{a \in A} I_A(a) \wedge \bigwedge_{a' \notin A} \neg I_A(a') = \top$ , so  $I_A(b) = \overline{I}_A(\blacklozenge b) \geq \overline{I}_A(z(A)) = \top$ . We conclude that  $b \in A$ .

Conversely, if  $b \in A$ , then obviously  $z(A) = \bigwedge_{a \in A} \blacklozenge a \wedge \bigwedge_{a' \notin A} \neg \blacklozenge a' \leq \blacklozenge b$ .  $\square$

Now, by Proposition 6, the quotient  $\mathbb{B}_{n+1}$  of  $\mathbb{C}_{n+1}$  has as its dual the set of those atoms  $x_C \in \text{At}(\mathbb{C}_{n+1})$  for which  $\approx_n$  is contained in  $\ker(x_C)$ . Concretely, this happens if and only if the atom  $x_C$  ‘‘satisfies’’ all the inequalities in  $\mathcal{E}_n$ . Here, we say an atom  $x_C$  *satisfies* the formal inequality ‘‘ $s(x_1, \dots, x_i) \leq t(x_1, \dots, x_i)$ ’’ if, for all  $a_1, \dots, a_i \in \mathbb{B}_n$ , we have  $x_C \leq s(a_1, \dots, a_i)$  implies  $x_C \leq t(a_1, \dots, a_i)$ . Let  $(x, A)$  be an atom of  $\mathbb{C}_{n+1} = \mathbb{B}_n + F_{\text{BA}}(\blacklozenge B_n)$ . We will repeatedly apply Lemma 8 to give conditions under which  $(x, A)$  satisfies the equations in  $\mathcal{E}_n$ .

- $(1_n) : \blacklozenge \perp = \perp$ .

Since we have  $(x, A) \not\leq \perp$ , we need that  $(x, A) \not\leq \blacklozenge \perp$ . By Lemma 8, this happens iff

$$\perp \notin A \tag{1_n^\partial}$$

- $(2_n) : \blacklozenge(a \vee b) = \blacklozenge a \vee \blacklozenge b$ .

We need to have  $(x, A) \leq \blacklozenge(a \vee b)$  iff  $(x, A) \leq \blacklozenge a \vee \blacklozenge b$ . Since  $(x, A)$  is an atom, we have  $(x, A) \leq \blacklozenge a \vee \blacklozenge b$  iff  $(x, A) \leq \blacklozenge a$  or  $(x, A) \leq \blacklozenge b$ , which happens iff  $a \in A$  or  $b \in A$ . We thus get the following condition on  $A$ :

$$\forall a, b \in B_n : a \vee b \in A \iff a \in A \text{ or } b \in A \tag{2_n^\partial}$$

- $(T_n) : j_n a \leq \blacklozenge a$ .

Using Lemma 8 again, we have  $(x, A) \leq j_n a$  iff  $x \leq a$  and  $(x, A) \leq \blacklozenge a$  iff  $a \in A$ . We thus get the condition

$$\forall a \in B_n : x \leq a \implies a \in A. \tag{T_n^\partial}$$

- $(W_n) : j_n \blacklozenge_{n-1} a_{n-1} = \blacklozenge_{n-1} a_{n-1}$

Note that if this equation is satisfied for all atoms  $y_{n-1}$  of  $B_{n-1}$ , then it is satisfied for all  $a_{n-1} \in B_{n-1}$ , since all the operations involved preserve  $\vee$ .

Note also that

$$\text{At}(\mathbb{B}_{n+1}) \subseteq X_n \times \mathcal{P}(B_n) \subseteq (X_{n-1} \times \mathcal{P}(B_{n-1})) \times \mathcal{P}(B_n),$$

so we write a potential element of  $\text{At}(\mathbb{B}_{n+1})$  as  $((x, A), \mathcal{A})$ , where  $x \in X_{n-1}$ ,  $A \subseteq B_{n-1}$  and  $\mathcal{A} \subseteq B_n$ .

Now we can calculate, by the same methods as above,

$$\begin{aligned} ((x, A), \mathcal{A}) \leq j_n \blacklozenge_{n-1} y_{n-1} &\iff (x, A) \leq \blacklozenge_{n-1} y_{n-1} \\ &\iff y_{n-1} \in A. \end{aligned}$$

For the correspondent of  $((x, A), \mathcal{A}) \leq \blacklozenge_n i_{n-1} y_{n-1}$ , we first write

$$i_{n-1} y_{n-1} = \bigvee \{(z, B) \in \text{At}(\mathbb{B}_n) : (z, B) \leq i_{n-1} y_{n-1}\},$$

and now calculate, using that  $\blacklozenge_n$  preserves finite joins and atoms are join-prime:

$$\begin{aligned} ((x, A), \mathcal{A}) \leq \blacklozenge_n i_{n-1} y_{n-1} &\iff \exists (z, B) \in \text{At}(\mathbb{B}_n) : ((x, A), \mathcal{A}) \leq \blacklozenge_n (z, B) \text{ and } (z, B) \leq i_{n-1} y_{n-1} \\ &\iff \exists (z, B) \in \text{At}(\mathbb{B}_n) : (z, B) \in \mathcal{A} \text{ and } y_{n-1} \leq z \\ &\iff \exists B \subseteq \mathbb{B}_{n-1} : (y_{n-1}, B) \in \mathcal{A}. \end{aligned}$$

We conclude that  $((x, A), \mathcal{A})$  satisfies  $(W_n)$  iff

$$\forall y_{n-1} \in \text{At}(\mathbb{B}_{n-1}) [(\exists B \subseteq \mathbb{B}_{n-1} : (y_{n-1}, B) \in \mathcal{A}) \iff y_{n-1} \in A]$$

or, more concisely, iff

$$\rho_{n-1}[\mathcal{A}] = A \cap \text{At}(\mathbb{B}_{n-1}), \quad (W_n^\partial)$$

where  $\rho_{n-1}$  is the projection  $X_{n-1} \times \mathcal{P}(\mathbb{B}_{n-1}) \rightarrow X_{n-1}$  onto the first coordinate.

Now that we have the conditions  $(1_n^\partial)$ ,  $(2_n^\partial)$ ,  $(T_n^\partial)$  and  $(W_n^\partial)$  for the atom  $x_C = (x, A) \in \text{At}(\mathbb{C}_{n+1})$  to be an atom of  $\mathbb{B}_{n+1}$ , we can give a more concise description.

**Lemma 9.** *Let  $A \subseteq \mathbb{B}_n$ . Then  $A$  satisfies  $(1_n^\partial)$  and  $(2_n^\partial)$  iff  $A = \uparrow(A \cap \text{At}(\mathbb{B}_n))$ .*

*Proof.* For the left-to-right direction, take  $a \in A$ . Then  $a = \bigvee \{x \in \text{At}(\mathbb{B}_n) : x \leq a\}$ , where this join is not empty, by  $(1_n^\partial)$ . Using that  $\mathbb{B}_n$  is finite, repeatedly applying the left-to-right direction of  $(2_n^\partial)$  now yields some  $x \in A$  such that  $a \geq x$ . For the other inclusion, note that if  $a \in A$  and  $a' \geq a$ , then  $a' = a \vee a' \in A$  by the right-to-left direction of  $(2_n^\partial)$ . So  $A$  is an upset, and therefore in particular it must contain  $\uparrow(A \cap \text{At}(\mathbb{B}_n))$ .

For the right-to-left direction, note that  $\perp$  is not greater than or equal to any atom, which proves  $(1_n^\partial)$ , and that  $a \vee b \in A$  iff there is an  $x \in \text{At}(\mathbb{B}_n)$  such that  $x \leq a \vee b$ , iff there is an  $x \in \text{At}(\mathbb{B}_n)$  such that  $x \leq a$  or  $x \leq b$ , iff  $a \in A$  or  $b \in A$ , proving  $(2_n^\partial)$ .  $\square$

Note that condition  $(T_n^\partial)$  always implies that  $x \in A$ , but in case  $A$  is an upset, the condition “ $x \in A$ ” is even *equivalent* to condition  $(T_n^\partial)$ . We conclude that the pairs  $(x, A)$  satisfying  $(1_n^\partial)$ ,  $(2_n^\partial)$  and  $(T_n^\partial)$  are exactly those for which  $A = \uparrow(A \cap \text{At}(\mathbb{B}_n))$  and  $x \in A$ . Thus, by sending  $(x, A)$  to  $(x, A \cap \text{At}(\mathbb{B}_n))$ , we may identify the atoms of  $\mathbb{B}_{n+1}$  with certain pairs  $(x, T)$ , where  $T \subseteq \text{At}(\mathbb{B}_n) = X_n$  and  $x \in T$ .

For  $n = 0$ , in fact all such pairs are atoms of  $\mathbb{B}_1$ , since the equation  $(W_0)$  is vacuously true. So we get

$$X_1 \cong \{(x, T) : x \in X_0, T \subseteq X_0, x \in T\}. \quad (1)$$

For  $n \geq 1$ , we need to restrict to those atoms of  $\mathbb{C}_{n+1}$  which also satisfy  $(W_n^j)$ . We then get

$$X_{n+1} \cong \{((x, T), \mathcal{A}) : (x, T) \in X_n, \mathcal{A} \subseteq X_n, (x, T) \in \mathcal{A}, T = \rho_{n-1}[\mathcal{A}]\}. \quad (2)$$

We can now also calculate what the dual of the join-preserving function  $\blacklozenge_n : \mathbb{B}_n \rightarrow \mathbb{B}_{n+1}$  is. By Stone duality, it will be the relation  $R_{\blacklozenge_n} : X_{n+1} \rightarrow X_n$ , given by

$$(x, T) R_{\blacklozenge_n} x' \iff (x, T) \leq \blacklozenge_n x'.$$

Applying Lemma 8 to the set  $A = \uparrow T$  and using the fact that the atoms of  $\mathbb{B}_n$  form an antichain, we now easily see that  $(x, T) R_{\blacklozenge_n} x'$  iff  $x' \in T$ .

### 3.3 Application of duality: injectivity of the chain

We can use the dual description of the chain of Boolean algebras to give an easy proof of the following fact.

**Proposition 10.** *For each  $n$ , the map  $i_n : \mathbb{B}_n \rightarrow \mathbb{B}_{n+1}$  is injective.*

*Proof.* The statement is, by duality, equivalent to: for each  $n$ , the map  $\pi_n : X_{n+1} \rightarrow X_n$  is surjective. We will prove this by induction. Recall that  $\pi_n$  is the restriction of the projection map  $X_n \times \mathcal{P}(B_n) \rightarrow X_n$  to  $X_{n+1}$ . As  $X_1 = \{(x, T) \in X_0 \times \mathcal{P}(X_0) \mid x \in T\}$ , for each  $x \in X_0$ ,  $(x, X_0) \in X_1$ . Hence,  $\pi_0$  is surjective. Now suppose, for some  $n \geq 1$ , that  $\pi_{n-1} : X_n \rightarrow X_{n-1}$  is surjective. We will show that  $\pi_n$  is surjective. Let  $(x, T) \in X_n$  be arbitrary. Define  $\mathcal{A} := \{(y, S) \in X_n \mid y \in T\}$ . We will show  $((x, T), \mathcal{A}) \in X_{n+1}$ , which is enough, because  $\pi_n((x, T), \mathcal{A}) = (x, T)$ . We have shown in (2):

$$X_{n+1} = \{((x, T), \mathcal{A}) : (x, T) \in X_n, \mathcal{A} \subseteq X_n, (x, T) \in \mathcal{A}, T = \rho_{n-1}[\mathcal{A}]\}.$$

By assumption,  $(x, T) \in X_n$ , and by definition,  $\mathcal{A} \subseteq X_n$ . Since  $(x, T) \in X_n$ , we have  $x \in T$ , whence  $(x, T) \in \mathcal{A}$ , by the definition of  $\mathcal{A}$ . Furthermore, for all  $(y, S) \in \mathcal{A}$ , by definition,  $y \in T$  and therefore  $\rho_{n-1}[\mathcal{A}] \subseteq T$ . Finally, as  $T \subseteq X_{n-1}$ , by the induction hypothesis, for each  $y \in T$  there exists  $S \subseteq X_{n-1}$  such that  $(y, S) \in X_n$ . Hence  $T \subseteq \rho_{n-1}[\mathcal{A}]$ , and we conclude  $((x, T), \mathcal{A}) \in X_{n+1}$ .  $\square$

### 3.4 Colimit of the chain is the free algebra

The variety  $\mathcal{V}_\top$  consists, by definition, of algebras from the variety  $\mathcal{V}_{\text{BA}}$  of Boolean algebras, equipped with an additional operation  $f = \diamond$  which satisfies certain equations. We will now prove that the colimit of the chain defined in Subsection 3.1 is indeed the free algebra for  $\mathcal{V}_\top$  over the finite set of variables  $P$ . We thus obtain the finitely generated free algebra for the variety  $\mathcal{V}_\top$  as a colimit of a countable increasing chain of finite algebras in  $\mathcal{V}$ . This construction already played a crucial role in [7], [1] and [2]. The proof we give here could be extended to a more general setting of varieties of modal algebras, but to ease the notation we choose to give the proof for the specific case of the variety  $\mathcal{V}_\top$ .

Let  $(\mathbb{B}_n \xrightarrow{k_n} \mathbb{B})_{n \in \mathbb{N}}$  be the colimit in the category **BA** of the chain diagram that we constructed in Subsection 3.1:

$$\mathbb{B}_0 \xrightarrow{i_0} \mathbb{B}_1 \xrightarrow{i_1} \dots \quad \mathbb{B}_n \xrightarrow{i_n} \mathbb{B}_{n+1} \xrightarrow{i_{n+1}} \dots$$

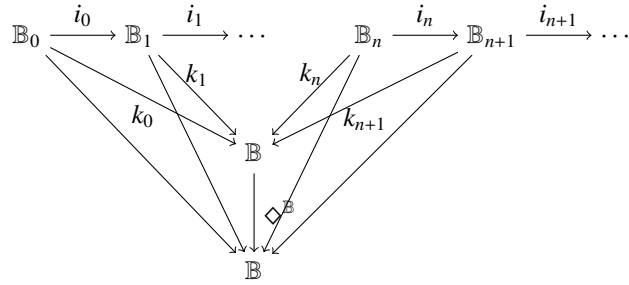
By a theorem of Manes [10], the colimit in  $\mathbf{BA}$  is given by lifting the colimit in  $\mathbf{Set}$ . Concretely, the underlying set of  $\mathbb{B}$  can be described by taking the disjoint union  $\bigsqcup_{n \in \mathbb{N}} B_n$ , and quotienting it by the equivalence relation  $\sim$ , which is defined to be the smallest equivalence relation containing all pairs  $\langle b_n, i_n(b_n) \rangle$ , for  $n \in \mathbb{N}$ ,  $b_n \in B_n$ . The Boolean algebra operations are then well-defined, and the function  $k_n$  is the inclusion of  $B_n$  into  $\bigsqcup_{n \in \mathbb{N}} B_n$ , followed by taking the class under  $\sim$ .

Now, the functions  $k_{n+1} \circ \diamond_n : B_n \rightarrow \mathbb{B}$  form a cone under the diagram of which  $k_n : B_n \rightarrow \mathbb{B}$  is the colimit:

$$k_{n+2} \circ \diamond_{n+1} \circ i_n = k_{n+2} \circ i_{n+1} \circ \diamond_n = k_{n+1} \circ \diamond_n,$$

using that  $\diamond_{n+1} \circ i_n = i_{n+1} \circ \diamond_n$  for any  $n$ , since the equation  $(4_{n+1})$  occurs in the set of equations  $\mathcal{E}_{n+1}$ , and is therefore satisfied in  $\mathbb{B}_{n+2}$ .

So, by the universal property of the colimit, there exists a (unique) function, which we will denote by  $\diamond^{\mathbb{B}}$ , from  $\mathbb{B} \rightarrow \mathbb{B}$ , such that  $\diamond^{\mathbb{B}} \circ k_n = k_{n+1} \circ \diamond_n$ . In a diagram, this looks as follows:



Concretely, the function  $\diamond^{\mathbb{B}}$  on  $\mathbb{B}$  may be defined, for  $b \in \mathbb{B}$ , by taking some  $n \in \mathbb{N}$  and  $b_n \in B_n$  such that  $k_n b_n = b$ , and then put  $\diamond^{\mathbb{B}} b := k_{n+1} \diamond_n b_n$ . Using that  $\diamond_{n+1} \circ i_n = i_{n+1} \circ \diamond_n$  for any  $n$ , it is not hard to see directly that this function  $\diamond^{\mathbb{B}}$  is well-defined.

We thus get an algebra  $(\mathbb{B}, \diamond)$  in the modal signature. To see that  $(\mathbb{B}, \diamond)$  is indeed the free  $\mathcal{V}_{\mathcal{T}}$ -algebra over  $P$ , we need to show the following two things:

1.  $(\mathbb{B}, \diamond)$  has the universal mapping property for algebras in  $\mathcal{V}_{\mathcal{T}}$ ,
2.  $(\mathbb{B}, \diamond)$  is in the variety  $\mathcal{V}_{\mathcal{T}}$ .

It may be possible to prove these things with a general category-theoretic argument (similar to the one used to define  $\diamond$  on  $\mathbb{B}$ ), but we will give a more algebraic proof for now.

Note that we have a map  $f : P \rightarrow \mathbb{B}_0 = F_{\mathbf{BA}}(P)$ , which we compose with  $k_0$  to get a map  $g : P \rightarrow \mathbb{B}$ . Note that, by an easy induction argument,  $\mathbb{B}$  is generated by  $g[P]$ .

Now, let  $(\mathbb{A}, \diamond)$  be an algebra in  $\mathcal{V}_{\mathcal{T}}$ , and  $h : P \rightarrow \mathbb{A}$  a function. We have to prove that there is a modal algebra homomorphism  $\bar{h} : (\mathbb{B}, \diamond^{\mathbb{B}}) \rightarrow (\mathbb{A}, \diamond^{\mathbb{A}})$  such that  $\bar{h} \circ g = h$ .

The map  $h : P \rightarrow \mathbb{A}$  gives rise to a cone under the diagram  $(\mathbb{B}_n \xrightarrow{i_n} \mathbb{B}_{n+1})_{n \in \mathbb{N}}$ , as follows.

Let  $h_0 : \mathbb{B}_0 \rightarrow \mathbb{A}$  be the unique BA homomorphism such that  $h_0 \circ f = h$ , from the free property of  $F_{\mathbf{BA}}(P)$ . We now define  $h_{n+1} : \mathbb{B}_{n+1} \rightarrow \mathbb{A}$  as follows. First define  $h'_{n+1} : \mathbb{B}_n + F_{\mathbf{BA}}(\diamond_n B_n) \rightarrow \mathbb{A}$  to be the coproduct map whose components are given by  $h_n : \mathbb{B}_n \rightarrow \mathbb{A}$  and the unique Boolean algebra homomorphism  $l_n : F_{\mathbf{BA}}(\diamond_n B_n) \rightarrow \mathbb{A}$  satisfying  $l_n \circ \diamond_n = \diamond^{\mathbb{A}} \circ h_n$ .

We now show that  $\approx_n \subseteq \ker h'_{n+1}$ , where we recall that  $\approx_n$  is the smallest congruence containing the equations  $\mathcal{E}_n$ . Hence, since  $\ker h'_{n+1}$  is a congruence, it is enough to show that elements which are equated by an equation in  $\mathcal{E}_n$  are sent to the same element by  $h'_{n+1}$ , i.e., that the equations in  $\mathcal{E}_n$  are “satisfied” in  $\ker h'_{n+1}$ . Showing that  $(1_n)$ ,  $(2_n)$  and  $(T_n)$  are satisfied in  $\ker h'_{n+1}$  is a routine argument, using that  $\mathbb{A}$  is in the variety  $\mathcal{V}_T$ . To give one example of this argument, we show that  $(T_n)$  is satisfied in  $\ker h'_{n+1}$ . For  $a \in \mathbb{B}_n$ , we have

$$\begin{aligned} h'_{n+1}(j_n(a)) &= h_n(a) \\ &\leq \diamond^{\mathbb{A}} h_n(a) \quad (\mathbb{A} \in \mathcal{V}_T) \\ &= l_n(\blacklozenge_n a) \\ &= h'_{n+1}(\blacklozenge_n a). \end{aligned}$$

The proofs that  $(1_n)$  and  $(2_n)$  are satisfied in  $\ker h'_{n+1}$  are similar.

To show that also  $(W_n)$  is satisfied in  $\ker h'_{n+1}$ , we use, by induction, the definition of the previous  $h_n$ . Let  $n \geq 1$  and  $a \in \mathbb{B}_{n-1}$ . Then

$$\begin{aligned} h'_{n+1}(j_n(\blacklozenge_{n-1} a)) &= h_n(\blacklozenge_{n-1} a) \\ &= \diamond^{\mathbb{A}} h_{n-1}(a) \quad (\text{definition } h_n) \\ &= \diamond^{\mathbb{A}} h_n(j_{n-1}(a)) \quad (\text{definition } h_n) \\ &= h'_{n+1}(\blacklozenge_n j_{n-1}(a)). \end{aligned}$$

We have now shown that  $\approx_n \subseteq \ker h'_{n+1}$ . Then  $h'_{n+1}$  factors through  $\mathbb{B}_{n+1} := (\mathbb{B}_n + F_{\text{BA}}(\blacklozenge_n \mathbb{B}_n)) / \approx_n$ , we call this factorisation  $h_{n+1} : \mathbb{B}_{n+1} \rightarrow \mathbb{A}$ . Note that it satisfies  $h_{n+1} i_n = h_n$  and  $h_{n+1} \blacklozenge_n = \diamond^{\mathbb{A}} h_n$ .

By the universal property of the colimit, there is a unique BA homomorphism  $\bar{h} : \mathbb{B} \rightarrow \mathbb{A}$  such that  $\bar{h} k_n = h_n$  for all  $n$ . We then also have, for all  $n \in \mathbb{N}$ ,

$$\bar{h} \diamond^{\mathbb{B}} k_n = \bar{h} k_{n+1} \blacklozenge_n = h_{n+1} \blacklozenge_n = \diamond^{\mathbb{A}} h_n = \diamond^{\mathbb{A}} \bar{h} k_n,$$

so by the uniqueness part of the universal property of the colimit, we conclude  $\bar{h} \diamond^{\mathbb{B}} = \diamond^{\mathbb{A}} \bar{h}$ . So  $\bar{h}$  is a homomorphism of modal algebras. Furthermore,  $\bar{h} g = \bar{h} k_0 f = h_0 f = h$ .

To see that  $(\mathbb{B}, \diamond^{\mathbb{B}})$  is in the variety  $\mathcal{V}_T$ , it suffices to show that  $(\mathbb{B}, \diamond^{\mathbb{B}})$  satisfies the equations (1), (2) and (T). First note that, by equation  $(1_n)$  for  $n = 1$ , we have

$$\diamond^{\mathbb{B}} \perp = k_1 \blacklozenge_1 \perp = k_1 \perp = \perp.$$

Now let  $a, b \in \mathbb{B}$ . We may determine  $n \in \mathbb{N}$  and  $a_n, b_n \in \mathbb{B}_n$  s.t.  $a = k_n(a_n)$  and  $b = k_n(b_n)$ . Then, by equation  $(2_n)$  and the definition of  $\diamond^{\mathbb{B}}$ ,

$$\begin{aligned} \diamond^{\mathbb{B}}(a \vee b) &= \diamond^{\mathbb{B}}(k_n(a_n) \vee k_n(b_n)) = \diamond^{\mathbb{B}}(k_n(a_n \vee b_n)) = k_{n+1}(\blacklozenge_n(a_n \vee b_n)) = k_{n+1}(\blacklozenge_n a_n \vee \blacklozenge_n b_n) \\ &= k_{n+1}(\blacklozenge_n a_n) \vee k_{n+1}(\blacklozenge_n b_n) = \diamond^{\mathbb{B}} k_n(a_n) \vee \diamond^{\mathbb{B}} k_n(b_n) = \diamond^{\mathbb{B}} a \vee \diamond^{\mathbb{B}} b. \end{aligned}$$

And, finally, by equation  $(T_n)$ ,

$$a = k_n(a_n) = k_{n+1} i_n(a_n) \leq k_{n+1} \blacklozenge_n(a_n) = \diamond^{\mathbb{B}} a.$$

Hence  $\mathbb{B} \in \mathcal{V}_T$ .

## 4 Conclusion

In this paper, we showed how to construct the Lindenbaum algebra for the modal logic  $T$  via a uniform step-by-step construction. The two main tools we used were *universal algebra*, in particular the fact that the Lindenbaum algebra for the logic  $T$  on  $n$  propositional variables is exactly the  $n$ -generated free algebra in the variety  $\mathcal{V}_T$ , and *Stone duality* for finite Boolean algebras, which enabled us to give a concrete set-theoretic description of the chain of algebras.

As indicated in the introduction, the actual use of uniform step-by-step constructions lies in the fact that it immediately gives normal forms: any formula in the logic of rank at most  $n$  is equivalent to the disjunction of atoms which are below it in the algebra  $\mathbb{B}_n$ . An interesting application of the theory in this paper would be to obtain normal forms for the modal logic  $T$ . Along the same lines, one could try implementing the uniform construction to obtain an actual algorithm for deciding logical equivalence in the modal logic  $T$ .

The idea of step-by-step constructing the free algebra originated from the case of algebras for a functor, where one can also describe the process of ‘defining the next algebra in the chain’ by means of a functor. For many logics, such as  $T$  and  $S4$ , the algebras for the logic are *not* algebras for a functor. However, it may still be possible to describe the process of building the chain by repeatedly applying a functor, which would then probably not be based on the category  $\mathbf{BA}_{<\omega}$  itself, but on a category of certain diagrams in  $\mathbf{BA}_{<\omega}$ .

A more fundamental open question in this line of research is whether we can give syntactic conditions on the axioms defining a modal logic  $\mathcal{L}$  which ensure that the answer to the question  $(Q_{\mathcal{L}})$  from the Introduction is affirmative. This will be the next direction to pursue in this research project.

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