

# Finitely generated free algebras and duality

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Lunteren, November 2010

# Outline

**Aim:** describe finitely generated free algebras concretely

- 1 Logical motivation to study free algebras
- 2 Construction of the free  $K$  modal algebra
- 3 Recent and future research

# Motivation

**Example:** Classical Propositional Logic (CPL)

Signature:  $\Sigma = (\wedge, \vee, \neg, 0, 1)$

Consider  $Fm(X)$  and define:

$$\phi \approx \psi \quad \Leftrightarrow \quad \phi \vdash \psi \text{ and } \psi \vdash \phi$$

$(Fm(X)/\approx, \wedge, \vee, \neg, [0], [1])$  is a Boolean algebra, where

$$\begin{aligned} [\phi] \vee [\psi] &= [\phi \vee \psi] \\ [\phi] \wedge [\psi] &= [\phi \wedge \psi] \\ \neg[\phi] &= [\neg\phi] \end{aligned}$$

This is the free Boolean algebra over  $X$  and it contains all information about the logic.

# Examples

Study finitely generated free algebras:

- in general free algebras are hard to describe concretely (word problem)
- in many situations only finitely many variables play a role, e.g.
  - term rewriting
  - decidability of logical equivalence
  - normal forms for formulas of a logic

## Examples:

CPL: Boolean algebras  $(\wedge, \vee, \neg, 0, 1)$

Free BA generated by a finite set  $X$ :  $\wp(\wp(X))$   
(disjunctive normal form)

IPL: Heyting algebras  $(\wedge, \vee, \rightarrow, 0, 1)$

Free HA on 2 generators is infinite and complicated

ML: Modal algebras  $(\wedge, \vee, \neg, 0, 1, \diamond)$

Free algebra in general infinite and complicated

# Free modal algebras

$\mathcal{M}$  = class of modal algebras

$X$  = finite set

We may approximate  $\mathbb{F}_{\mathcal{M}}(X)$  by finite BA's:

$$\mathbf{B}_0 \begin{array}{c} \hookrightarrow \\ \xrightarrow{\diamond} \end{array} \mathbf{B}_1 \begin{array}{c} \hookrightarrow \\ \xrightarrow{\diamond} \end{array} \mathbf{B}_2 \begin{array}{c} \hookrightarrow \\ \xrightarrow{\diamond} \end{array} \dots$$

where  $\mathbf{B}_n$  = elements of  $\mathbb{F}_{\mathcal{M}}(X)$  of rank  $\leq n$ .

E.g.:  $\diamond(x_1) \wedge x_2 \in \mathbf{B}_1$

$\diamond(x_1 \vee \diamond(x_2 \wedge \diamond(x_1))) \in \mathbf{B}_3$

# Free rank 1 modal algebras

$\mathcal{K}$  = class of modal algebras satisfying:

(1)  $\Diamond 0 = 0$

(2)  $\Diamond(a \vee b) = \Diamond a \vee \Diamond b$

Approximate  $\mathbb{F}_{\mathcal{K}}(X)$  by

$$\mathbf{B}_0 \hookrightarrow \mathbf{B}_1 \hookrightarrow \mathbf{B}_2 \hookrightarrow \dots$$

# Free rank 1 modal algebras

$\mathcal{K}$  = class of modal algebras satisfying

$$(1) \ \diamond 0 = 0$$

$$(2) \ \diamond(a \vee b) = \diamond a \vee \diamond b$$

Approximate  $\mathbb{F}_{\mathcal{K}}(X)$  by

$$\mathbf{B}_0 \hookrightarrow \mathbf{B}_0 + H(\mathbf{B}_0) \hookrightarrow \mathbf{B}_0 + H(\mathbf{B}_1) \hookrightarrow \dots$$

where  $\mathbf{B}_0 = \mathbb{F}_{BA}(X)$

$$H(\mathbf{B}) = \mathbb{F}_{BA}(\diamond \mathbf{B}) /_{(1)+(2)}$$

$$\diamond \mathbf{B} = \{\diamond b \mid b \in \mathbf{B}\}$$

# Intermezzo on Duality Theory

**finite Boolean algebras**



**finite sets**

$\mathbf{B}$



$At(\mathbf{B})$

$\wp(X)$



$X$

# Intermezzo on Duality Theory

**finite Boolean algebras**



**finite sets**

$$\mathbf{B} \xrightarrow{h} \mathbf{C}$$



$$At(\mathbf{B}) \xleftarrow{h^b} At(\mathbf{C})$$

$$\wp(X) \xleftarrow{f^{-1}} \wp(Y)$$



$$X \xrightarrow{f} Y$$

# Free rank 1 modal algebras

**Aim:** compute  $\mathbb{F}_{BA}(\diamond \mathbf{B}) / (1)+(2)$  where

(1)  $\diamond 0 = 0$

(2)  $\diamond(a \vee b) = \diamond a \vee \diamond b$

$$\mathbb{F}_{BA}(\diamond \mathbf{B}) \twoheadrightarrow \mathbb{F}_{BA}(\diamond \mathbf{B}) / (1) \twoheadrightarrow \mathbb{F}_{BA}(\diamond \mathbf{B}) / (1)+(2)$$

$$At(\mathbb{F}_{BA}(\diamond \mathbf{B})) \longleftarrow X_1 \longleftarrow X_2$$

# Free rank 1 modal algebras

**Step 0:**  $\wp(\mathbf{B}) \cong \text{At}(\mathbb{F}_{BA}(\diamond\mathbf{B}))$

$$A \in \wp(\mathbf{B}) \quad \leftrightarrow \quad \bigwedge_{a \in A} \diamond a \wedge \bigwedge_{a \notin A} \neg \diamond a = j_A \in \text{At}(\mathbb{F}_{BA}(\diamond\mathbf{B}))$$

**Lemma:** for  $a \in \mathbf{B}$  and  $A \in \wp(\mathbf{B})$ ,

$$j_A \leq \diamond a \quad \Leftrightarrow \quad a \in A$$

$\begin{array}{ccccc} \mathbb{F}_{BA}(\diamond\mathbf{B}) & \twoheadrightarrow & \mathbb{F}_{BA}(\diamond\mathbf{B})/_{(1)} & \twoheadrightarrow & \mathbb{F}_{BA}(\diamond\mathbf{B})/_{(1)+(2)} \\ \text{At}(\mathbb{F}_{BA}(\diamond\mathbf{B})) & \longleftarrow & \supset X_1 & \longleftarrow & \supset X_2 \end{array}$
--

# Free rank 1 modal algebras

**Step 1:**  $\diamond 0 = 0$

$X_1$  = collection of  $A \in \wp(\mathbf{B}) \cong At(\mathbb{F}_{BA}(\diamond \mathbf{B}))$  s.t.

$$j_A \leq \diamond 0 \quad \Leftrightarrow \quad j_A \leq 0$$

$$\begin{aligned} X_1 &= \{A \in \wp(\mathbf{B}) \mid j_A \not\leq \diamond 0\} \\ &= \{A \in \wp(\mathbf{B}) \mid 0 \notin A\} \end{aligned}$$

$\mathbb{F}_{BA}(\diamond \mathbf{B})$	$\twoheadrightarrow$	$\mathbb{F}_{BA}(\diamond \mathbf{B})/_{(1)}$	$\twoheadrightarrow$	$\mathbb{F}_{BA}(\diamond \mathbf{B})/_{(1)+(2)}$
$\wp(\mathbf{B})$	$\longleftarrow$	$X_1$	$\longleftarrow$	$X_2$

# Free rank 1 modal algebras

**Step 2:**  $\diamond(a \vee b) = \diamond a \vee \diamond b$

$X_2$  = collection of  $A \in X_1$  s.t.

$$j_A \leq \diamond(a \vee b) \quad \Leftrightarrow \quad j_A \leq \diamond a \vee \diamond b$$

$$j_A \leq \diamond(a \vee b) \quad \Leftrightarrow \quad j_A \leq \diamond a \text{ or } j_A \leq \diamond b$$

$$a \vee b \in A \quad \Leftrightarrow \quad a \in A \text{ or } b \in A$$

$\begin{array}{ccccc} \mathbb{F}_{BA}(\diamond \mathbf{B}) & \twoheadrightarrow & \mathbb{F}_{BA}(\diamond \mathbf{B}) / (1) & \twoheadrightarrow & \mathbb{F}_{BA}(\diamond \mathbf{B}) / (1)+(2) \\ \wp(\mathbf{B}) & \longleftarrow & X_1 & \longleftarrow & X_2 \end{array}$
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# Free rank 1 modal algebras

**Step 2:**  $\diamond(a \vee b) = \diamond a \vee \diamond b$

$X_2$  = collection of  $A \in X_1$  s.t.

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$$j_A \leq \diamond(a \vee b) \quad \Leftrightarrow \quad j_A \leq \diamond a \text{ or } j_A \leq \diamond b$$

$$a \vee b \in A \quad \Leftrightarrow \quad a \in A \text{ or } b \in A$$

$$X_2 \cong \wp(\text{At}(\mathbf{B}))$$

$$A \mapsto A \cap \text{At}(\mathbf{B})$$

$$\begin{array}{ccccc} \mathbb{F}_{BA}(\diamond \mathbf{B}) & \twoheadrightarrow & \mathbb{F}_{BA}(\diamond \mathbf{B}) / (1) & \twoheadrightarrow & \mathbb{F}_{BA}(\diamond \mathbf{B}) / (1)+(2) \\ \wp(\mathbf{B}) & \longleftarrow & \curvearrowright X_1 & \longleftarrow & \curvearrowright X_2 \end{array}$$

# Free rank 1 modal algebras

We have shown

$$X_2 = At(\mathbb{F}_{BA}(\diamond \mathbf{B}) / (1)+(2)) \cong \wp(At(\mathbf{B})).$$

Hence, the functor

$$\mathbf{B} \mapsto \mathbb{F}_{BA}(\diamond \mathbf{B}) / (1)+(2)$$

is given dually by

$$X \mapsto \wp(X)$$

$$\begin{array}{ccccc} \mathbb{F}_{BA}(\diamond \mathbf{B}) & \twoheadrightarrow & \mathbb{F}_{BA}(\diamond \mathbf{B}) / (1) & \twoheadrightarrow & \mathbb{F}_{BA}(\diamond \mathbf{B}) / (1)+(2) \\ \wp(\mathbf{B}) & \longleftarrow & X_1 & \longleftarrow & X_2 \end{array}$$

# Free rank 1 modal algebras

**Aim:** compute the free  $K$  modal algebra  $\mathbf{F}_{\mathcal{K}}(X)$

$$\mathbf{B}_0 \hookrightarrow \mathbf{B}_0 + H(\mathbf{B}_0) \hookrightarrow \mathbf{B}_0 + H(\mathbf{B}_1) \hookrightarrow \dots$$

$$X_0 \longleftarrow X_0 \times \wp(X_0) \longleftarrow X_0 \times \wp(X_1) \longleftarrow \dots$$

Define a  $\diamond$  on  $\mathbf{B}^\omega = \varinjlim \mathbf{B}_n$  by

$$\diamond[a \in \mathbf{B}_n] = [\diamond a \in \mathbf{B}_{n+1}]$$

$(\mathbf{B}^\omega, \diamond)$  is the free  $K$  modal algebra over  $X$ .

# Recent and future research

**Aim:** generalise this construction

$\mathcal{S}4$  = class of  $K$  modal logics satisfying:

$$(3) a \leq \diamond a$$

$$(4) \diamond \diamond a = \diamond a$$

Approximate  $\mathbb{F}_{\mathcal{S}4}(X)$  by:

$$\mathbf{B}_0 \begin{array}{c} \hookrightarrow \\ \xrightarrow{\diamond} \end{array} \mathbf{B}_1 \begin{array}{c} \hookrightarrow \\ \xrightarrow{\diamond} \end{array} \mathbf{B}_2 \hookrightarrow \dots$$



# Recent and future research

**Aim:** Generalise this construction

$\mathcal{S}_4$  = class of  $K$  modal logics satisfying

$$(3) a \leq \diamond a$$

$$(4) \diamond \diamond a = \diamond a$$

Approximate  $\mathbb{F}_{\mathcal{S}_4}(X)$  by:

$$\begin{array}{ccccccc} (\mathbf{B}_0, \diamond_0) & \xleftarrow{\quad} & (\mathbf{B}_1, \diamond_1) & \xleftarrow{\quad} & (\mathbf{B}_2, \diamond_2) & \hookrightarrow & \dots \\ & \xrightarrow{\quad} & & \xrightarrow{\quad} & & & \\ & \xrightarrow{\quad \diamond \quad} & & \xrightarrow{\quad \diamond \quad} & & & \end{array}$$
  
$$(X_0, \preceq_0) \longleftarrow (X_1, \preceq_1) \longleftarrow (X_2, \preceq_2) \longleftarrow \dots$$

# Recent and future research

**Example:** Construction of the free Heyting algebra

Approximate  $\mathbb{F}_{HA}(X)$  by

$$\mathbf{D}_0 \hookrightarrow \mathbf{D}_1 \hookrightarrow \mathbf{D}_2 \hookrightarrow \dots$$
$$(X_0, \leq_0) \longleftarrow (X_1, \leq_1) \longleftarrow (X_2, \leq_2) \longleftarrow \dots$$

$$X_0 = J(\mathbb{F}_{DL}(X)) = \wp(X)$$

$$X_1 = \wp_r(X_0)$$

$$X_{n+1} = \left\{ \tau \in \wp_r(X_n) \mid \forall T \in \tau \forall S \in X_n \right. \\ \left. S \leq T \Rightarrow \exists T' \in \tau (T' \leq T \text{ and } \text{root}(S) = \text{root}(T')) \right\}$$

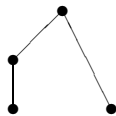
# Construction of the free Heyting algebra on 1 generator



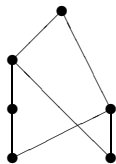
$X_0$



$X_1$

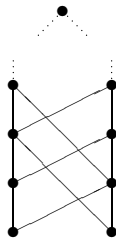


$X_2$



$X_3$

...



$X_\omega$