

Duality for first order logic

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Outline

- 1 Introduction to logic and duality theory
- 2 Algebraic semantics for classical first order logic:
Boolean hyperdoctrines
- 3 Dual notion of Boolean hyperdoctrines:
Indexed Stone spaces
- 4 Duality for classical first order logic
- 5 Future work

We doing **logic** we analyse mathematical reasoning

Classical Propositional Logic (CPL)

- Propositional variables: p_0, p_1, \dots
- Build formulas using connectives: $\wedge, \vee, \neg, \perp, \top$:

$$p_0 \wedge p_1, p_0 \vee (p_1 \wedge p_2), \neg p_0, \dots$$

- Reasoning rules:

$$\frac{\boxed{\text{DER}}}{\frac{\phi \wedge \psi}{\phi}}$$

$$\frac{\boxed{\text{DER1}} \quad \boxed{\text{DER2}}}{\frac{\phi \quad \psi}{\phi \wedge \psi}}$$

$$\frac{\boxed{\text{DER}}}{\frac{\phi}{\phi \vee \psi}}$$

- Notation: $\phi \vdash \psi$ says ' ψ is derivable from ϕ '.

Questions

- Which formulas are derivable?
- Is it decidable whether a formula is derivable?
- Does the logic have the **interpolation property**, that is, for all formulas $\phi(p, q)$ and $\psi(p, r)$ with $\phi(p, q) \vdash \psi(p, r)$, there exists a formula $\theta(p)$ s.t.

$$\phi(p, q) \vdash \theta(p) \quad \text{and} \quad \theta(p) \vdash \psi(p, r).$$

Algebraizing logic

Relate a logic to a class of algebras:

Logic \longleftrightarrow Class of algebras

Algebraizing logic

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CPL \longleftrightarrow Boolean algebras

Algebraizing logic

Relate a logic to a class of algebras:

CPL \longleftrightarrow Boolean algebras

A **Boolean algebra** is a structure $\mathbb{A} = (A, \wedge, \vee, \neg, 0, 1)$ s.t.

$$\begin{array}{ll} a \vee (b \vee c) = (a \vee b) \vee c & a \wedge (b \wedge c) = (a \wedge b) \wedge c \\ a \vee b = b \vee a & a \wedge b = b \wedge a \\ a \vee (a \wedge b) = a & a \wedge (a \vee b) = a \\ a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c) & a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c) \\ a \vee \neg a = 1 & a \wedge \neg a = 0 \end{array}$$

Algebraizing logic

Relate a logic to a class of algebras:

CPL \longleftrightarrow Boolean algebras

A **Boolean algebra** is a structure $\mathbb{A} = (A, \wedge, \vee, \neg, 0, 1)$ s.t.

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Example: $(\mathcal{P}(X), \cap, \cup, ()^c, \emptyset, X)$

Algebraizing logic

Relate a logic to a class of algebras:

CPL \longrightarrow Boolean algebras

Start from a set of propositional variables P .

Consider $Fm(P)$ and define:

$$\phi \approx \psi \iff \phi \vdash \psi \text{ and } \psi \vdash \phi$$

$(Fm(P)/\approx, \wedge, \vee, \neg, [\perp], [\top])$ is a Boolean algebra, where

$$\begin{aligned} [\phi] \vee [\psi] &= [\phi \vee \psi] \\ [\phi] \wedge [\psi] &= [\phi \wedge \psi] \\ \neg[\phi] &= [\neg\phi] \end{aligned}$$

We call this the **Lindenbaum algebra**.

Algebraizing logic

Interpreting a logic in a Boolean algebra:

Every map $f: P \rightarrow \mathbb{A}$ (**valuation**) extends to a unique homomorphism $\tilde{f}: Fm(P) \rightarrow \mathbb{A}$.

We say ϕ is **valid** in \mathbb{A} if, for every valuation f , $\tilde{f}(\phi) = 1$.

Notation: $\mathbb{A} \models \phi$.

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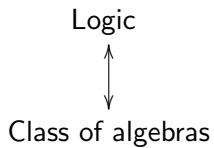
Notation: $\mathbb{A} \models \phi$.

Soundness and completeness theorem:

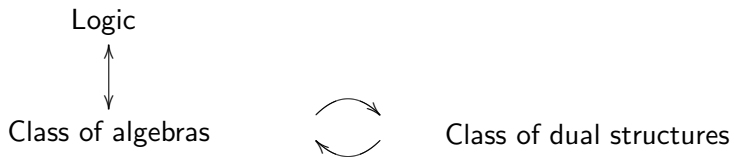
For every formula ϕ ,

$$\vdash_{CPL} \phi \quad \Leftrightarrow \quad \models_{BA} \phi$$

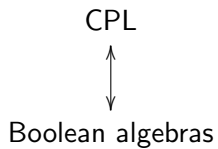
Duality in logic



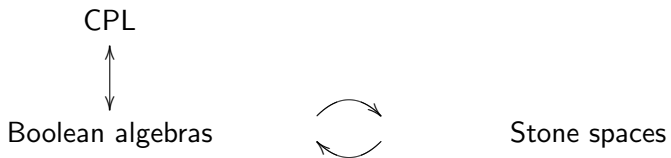
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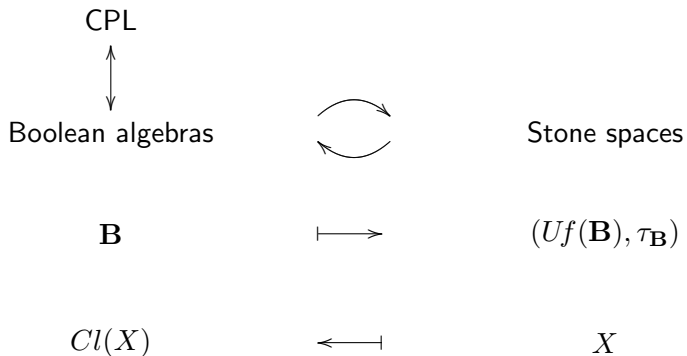
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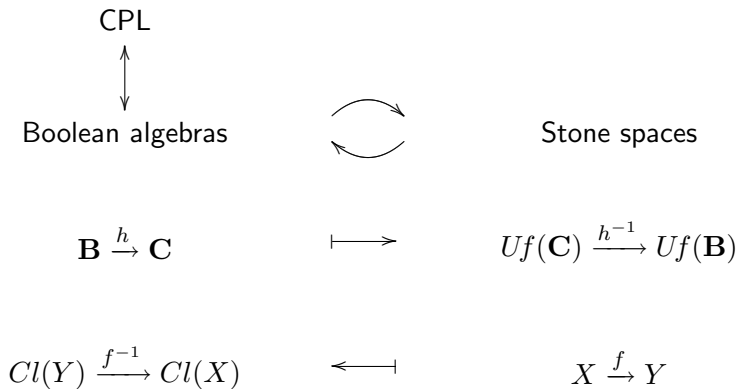
Duality in logic



Duality in logic



Duality in logic



Duality in logic

CPL over a set of variables X



Lindenbaum algebra
of formulas over X



Maps $X \rightarrow 2$
'valuations'

Duality for first order logic

Classical first order logic



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Duality for first order logic

Classical first order logic



Boolean hyperdoctrines



?

1 What are Boolean hyperdoctrines?

Duality for first order logic

Classical first order logic



Boolean hyperdoctrines



Indexed Stone spaces

- 1 What are Boolean hyperdoctrines?
- 2 Identify the dual notion of a Boolean hyperdoctrine.

Algebraic semantics for first order logic

We start from

Signature: $\Sigma = (f_0, \dots, f_{k-1}, R_0, \dots, R_{l-1}, c_0, \dots, c_{m-1})$

Set of variables: $X = \{x_0, x_1, \dots\}$

Build **terms** from the variables using function symbols and constants:

$x_1, f_0(x_0), f_2(f_1(x_0, f_3(c_1))), \dots$

Build **formulas** from the terms using $\wedge, \vee, \neg, \perp, \top, \exists, \forall$ and relation symbols:

$R_0(x_0), R_0(x_1) \wedge R_1(x_0, c_0), \exists_{x_1} R_1(f_0(x_0), x_1), \dots$

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Question:

What properties does the collection of all formulas over Σ have?

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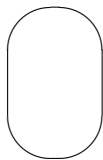
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First observation:

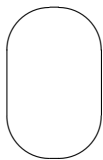
For each $n \in \mathbb{N}$,

$(Fm(x_0, \dots, x_{n-1}), \vdash)$ is a Boolean algebra.

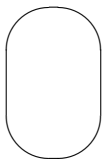
Algebraic semantics for first order logic



$[\]$



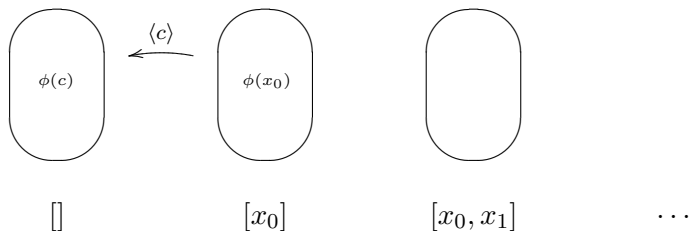
$[x_0]$



$[x_0, x_1]$

\dots

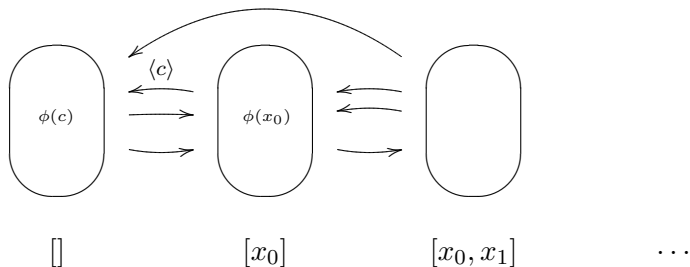
Algebraic semantics for first order logic



Substitutions:

$$\begin{array}{lcl} x_0 & \mapsto & c \\ \phi(x_0) & \mapsto & \phi(c) \end{array}$$

Algebraic semantics for first order logic



Substitutions:

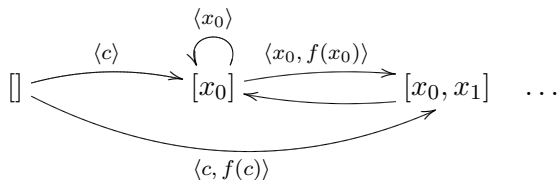
$$\begin{aligned} x_0 &\mapsto c \\ \phi(x_0) &\mapsto \phi(c) \end{aligned}$$

Algebraic semantics for first order logic

Contexts and substitutions form a category **B**:

Objects: natural numbers (contexts)

Morphism $n \rightarrow m$: m -tuple $\langle t_0, \dots, t_{m-1} \rangle$
s.t. $FV(t_i) \subseteq \{x_0, \dots, x_{n-1}\}$



Algebraic semantics for first order logic

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This category has **finite products**:

$$[x_0, x_1] \xleftarrow{\langle x_0, x_1 \rangle} [x_0, x_1, x_2] \xrightarrow{\langle x_2 \rangle} [x_0]$$

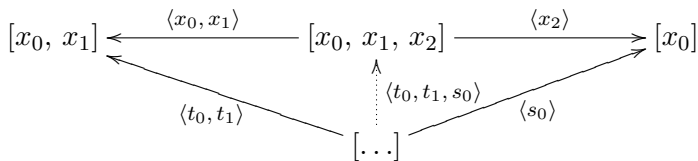
Algebraic semantics for first order logic

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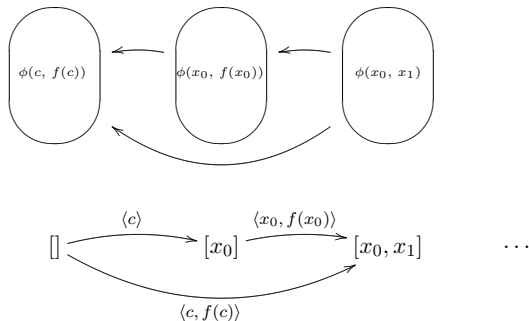
This category has **finite products**:



Algebraic semantics for first order logic

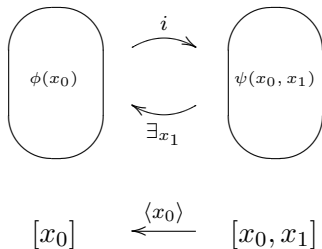
Formulas and substitutions: functor $\mathbf{B}^{op} \rightarrow \mathbf{BA}$

$$\begin{array}{lcl} n & \mapsto & Fm(x_0, \dots, x_{n-1}) \\ n \xrightarrow{\langle t_0, \dots, t_{m-1} \rangle} m & \mapsto & Fm(x_0, \dots, x_{m-1}) \rightarrow Fm(x_0, \dots, x_{n-1}) \\ & & \phi(x_0, \dots, x_{m-1}) \mapsto \phi(t_0, \dots, t_{m-1}) \end{array}$$



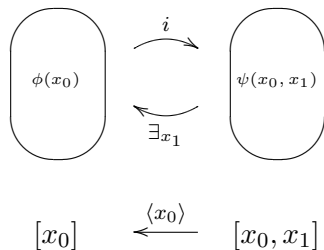
Algebraic semantics for first order logic

Existential quantification: related to the inclusion map



Algebraic semantics for first order logic

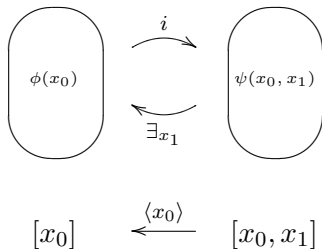
Existential quantification: related to the inclusion map



$$\exists_{x_1}(\psi(x_0, x_1)) \vdash_{x_0} \phi(x_0) \quad \Leftrightarrow \quad \psi(x_0, x_1) \vdash_{x_0, x_1} i(\phi(x_0))$$

Algebraic semantics for first order logic

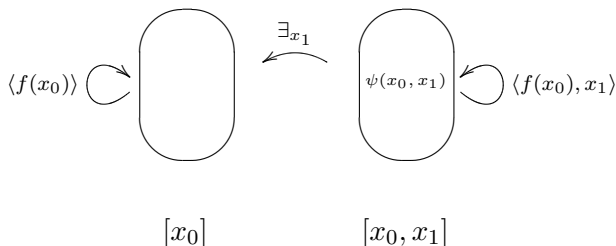
Existential quantification: related to the inclusion map



$$\exists_{x_1}(\psi(x_0, x_1)) \leq_{x_0} \phi(x_0) \quad \Leftrightarrow \quad \psi(x_0, x_1) \leq_{x_0, x_1} i(\phi(x_0))$$

Algebraic semantics for first order logic

Quantification: interaction with substitutions



$$\exists_{x_1}(\psi(x_0, x_1))[f(x_0)/x_0] = \exists_{x_1}(\psi(f(x_0), x_1))$$

(Beck-Chevalley)

Algebraic semantics for first order logic

A **Boolean hyperdoctrine** is a functor $\mathcal{F}: \mathbf{B}^{\text{op}} \rightarrow \mathbf{BA}$ s.t.

- 1** \mathbf{B} is a category with finite products;
- 2** for all $I, J \in \mathbf{B}$, $\mathcal{F}(\pi_{I,J}): \mathcal{F}(I) \rightarrow \mathcal{F}(I \times J)$ has a left adjoint $\exists_{I,J}$ such that, for all $I \xrightarrow{u} K$ in \mathbf{B} ,

$$\begin{array}{ccc} \mathcal{F}(K \times J) & \xrightarrow{\exists_{K,J}} & \mathcal{F}(K) \\ \mathcal{F}(u \times \text{id}) \downarrow & & \downarrow \mathcal{F}(u) \\ \mathcal{F}(I \times J) & \xrightarrow{\exists_{I,J}} & \mathcal{F}(I) \end{array}$$

commutes.

Algebraic semantics for first order logic

Examples of Boolean hyperdoctrines:

■ Syntactic hyperdoctrine

\mathbf{B} = contexts and substitutions

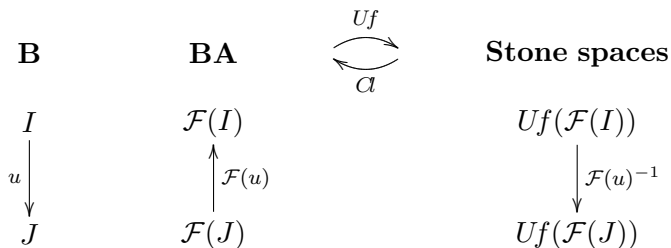
$$\begin{aligned} \mathcal{F}: \mathbf{B}^{op} &\rightarrow \mathbf{BA} \\ n &\mapsto \mathit{Fm}(x_0, \dots, x_{n-1}) \end{aligned}$$

■ Subset hyperdoctrine

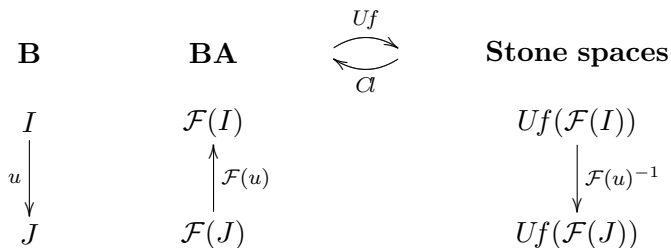
\mathbf{B} = Set

$$\begin{aligned} \mathcal{P}: \mathbf{B}^{op} &\rightarrow \mathbf{BA} \\ A &\mapsto \text{powerset of } A \end{aligned}$$

Duality for first order logic



Duality for first order logic



This gives us a **dual equivalence** between:

Functors $\mathcal{F}: \mathbf{B}^{\text{op}} \rightarrow \mathbf{BA}$

Functors $\mathcal{G}: \mathbf{B} \rightarrow \mathbf{StSp}$

$$\begin{array}{ccc} \mathcal{F} & \xrightarrow{\quad} & Uf \circ \mathcal{F} \\ \mathcal{A} \circ \mathcal{G} & \xleftarrow{\quad} & \mathcal{G} \end{array}$$

Duality for first order logic

$$\mathcal{F}: \mathbf{B}^{\text{op}} \rightarrow \mathbf{BA}$$

$$\mathcal{G}: \mathbf{B} \rightarrow \mathbf{StSp}$$

$\mathcal{F}(\pi_{I,J})$ has a left adjoint $\exists_{I,J}$

for all $I \xrightarrow{u} K$,

$$\begin{array}{ccc} \mathcal{F}(K \times J) & \xrightarrow{\exists_{K,J}} & \mathcal{F}(K) \\ \mathcal{F}(u \times \text{id}) \downarrow & & \downarrow \mathcal{F}(u) \\ \mathcal{F}(I \times J) & \xrightarrow{\exists_{I,J}} & \mathcal{F}(I) \end{array}$$

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Duality for first order logic

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commutes.

$$\mathcal{G}: \mathbf{B} \rightarrow \mathbf{StSp}$$

$\mathcal{G}(\pi_{I,J})$ is an open map

Duality for first order logic

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commutes.

$$\mathcal{G}: \mathbf{B} \rightarrow \mathbf{StSp}$$

$\mathcal{G}(\pi_{I,J})$ is an open map

for all $I \xrightarrow{u} K$,

$$\begin{array}{ccc} z \in \mathcal{G}(I \times J) & \xrightarrow{\mathcal{G}(\pi_{I,J})} & \mathcal{G}(I) \ni y \\ \mathcal{G}(u \times \text{id}) \downarrow & & \downarrow \mathcal{G}(u) \\ x \in \mathcal{G}(K \times J) & \xrightarrow{\mathcal{G}(\pi_{K,J})} & \mathcal{G}(K) \end{array}$$

$\mathcal{G}(u)(x) = \mathcal{G}(\pi_{K,J})(y)$ implies
there exists $z \in \mathcal{G}(I \times J)$ s.t.

$$\begin{aligned} \mathcal{G}(\pi_{I,J})(z) &= x \\ \mathcal{G}(u \times \text{id})(z) &= y. \end{aligned}$$

Duality for first order logic

Boolean hyperdoctrines

Functors $\mathcal{F}: \mathbf{B}^{\text{op}} \rightarrow \mathbf{BA}$ s.t.

- \mathbf{B} has finite products;
- $\mathcal{F}(\pi_{I,J})$ has a left adjoint $\exists_{I,J}$ and for all $I \xrightarrow{u} K$,

$$\begin{array}{ccc} \mathcal{F}(K \times J) & \xrightarrow{\exists_{K,J}} & \mathcal{F}(K) \\ \mathcal{F}(u \times \text{id}) \downarrow & & \downarrow \mathcal{F}(u) \\ \mathcal{F}(I \times J) & \xrightarrow{\exists_{I,J}} & \mathcal{F}(I) \end{array}$$

commutes.

Indexed Stone spaces

Functors $\mathcal{G}: \mathbf{B} \rightarrow \mathbf{StSp}$ s.t.

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- $\mathcal{G}(\pi_{I,J})$ is an open map and for all $I \xrightarrow{u} K$,

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is epicartesian.

Duality for first order logic

Duality theorem for classical first order logic:

The category of Boolean hyperdoctrines and the category of indexed Stone spaces are dually equivalent.

Boolean hyperdoctrines

$$\mathcal{F}$$
$$\mathcal{A} \circ \mathcal{G}$$

$$\dashv \longrightarrow$$
$$\longleftarrow \dashv$$

Indexed Stone spaces

$$Uf \circ \mathcal{F}$$
$$\mathcal{G}$$

Future work

Having a duality for classical first order logic we would like to:

- 1 Describe dual structures for non-classical first order logics.
- 2 Obtain information about these first order logics via studying their dual structures.
- 3 In particular: study the interpolation property dually.