

# Semantics and Duality

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Logique, catégories, sémantique

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13 novembre 2010

## Stone duality



At the heart of the logical completeness relation between **syntactic specification** and **state space semantics**

With additional connectives



Algebras and coalgebras for dually equivalent functors

## Logic and topology

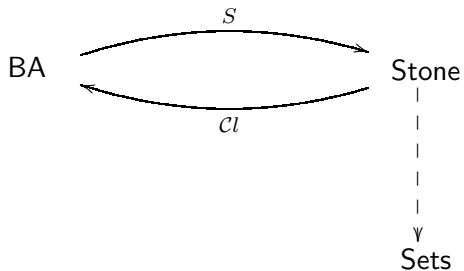
**Claim:** (Most) logicians cannot deal with topological information!

For finitary logics, logicians want first-order structures on both sides

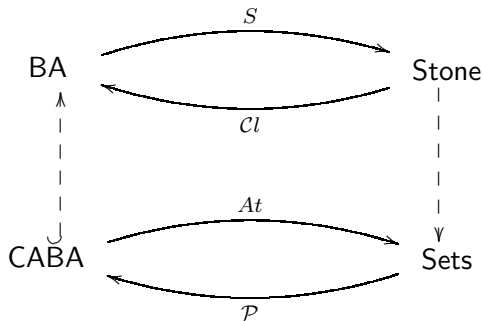
Two solutions are available in the literature:

- ▶ Forget the topology (**discrete relational semantics**)
- ▶ Reduce to order-induced topology case (**Scott's domain theory**)

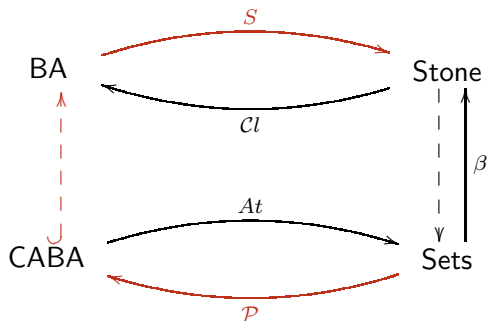
## Forgetting the topology



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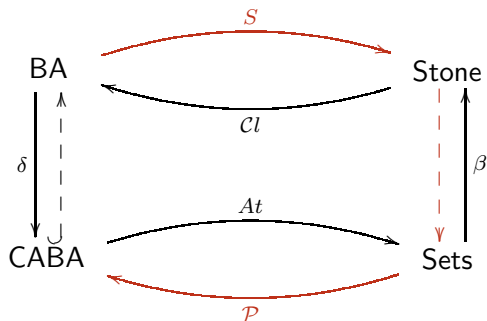


## Forgetting the topology



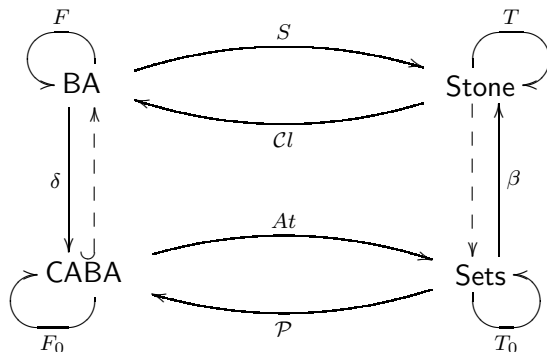
$\beta$  = Stone-Cech compactification  
 = left adjoint to the forgetful functor

## Forgetting the topology



$\delta$  = Jónsson-Tarski Canonical extension  
 = left adjoint to the inclusion functor

## Additional connectives

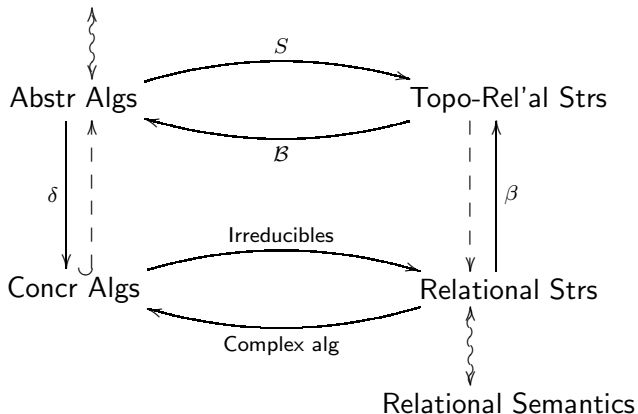


**Problem:** Given  $F$  (and/or  $T$ ), how to find an  $F_0$  (and/or  $T_0$ )?

(Jónsson-Tarski) via canonical extension

## Duality for logic

Deductive systems



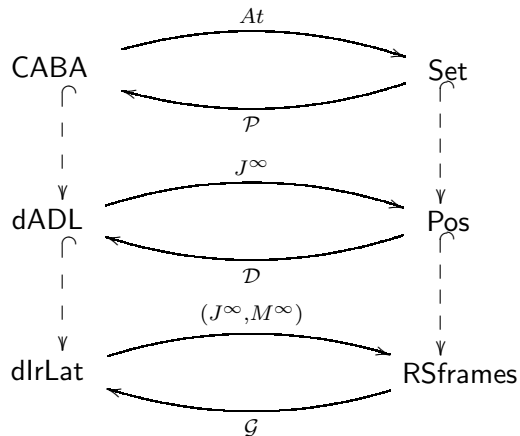
## Some applications relevant to this meeting

- ▶ **Substructural logics:** with or without lattice operations; with or without associativity, commutativity, contraction, weakening, Grishin's dually residuated operations, interaction axioms, other linear logic operators (negation, bang, etc) – **(fully modular completeness results)**  
[ongoing work with Chernilovskaya, Coumans, and van Rooijen]
- ▶ Duality between some not-necessarily-étale quantales and corresponding groupoids  
[Palmigiano and Re, *Studia Logica* (2010) **95**:125–137]
- ▶ Duality for first-order logic  
[ongoing work by Dion Coumans – **2nd half of this talk**]

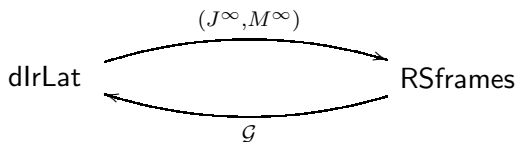
## Substructural logics

- ▶ Algebraic models are partially ordered residuated monoids  $(A, \leq, \cdot, \rightarrow, \leftarrow)$  possibly with additional operations such as  $\wedge, \vee, \perp, \top, 0, 1, \oplus, \otimes, \oslash, \sim$ , and !
- ▶ **(Canonicity)** All the standard axioms considered for these operations lift to canonical extensions of algebras satisfying them
- ▶ **(Correspondence)** On concrete algebras, these axioms correspond to first-order properties of the dual relational structures

## Generalised sets



## RSframes



$\text{dlrLat}$ : complete lattices

completely join-generated by completely join-irreducibles

completely meet-generated by completely meet-irreducibles

with complete lattice homomorphisms

$\text{RSframes}$ :  $(X, Y, \leq)$  with  $\leq \subseteq X \times Y$

$(x \neq x' \Rightarrow \uparrow x \neq \uparrow x')$  and  $(y \neq y' \Rightarrow \downarrow y \neq \downarrow y')$

$\forall x \exists y [ x \not\leq y \text{ and } \forall x' ( \uparrow x \not\subseteq \uparrow x' \Rightarrow x' \leq y ) ]$

$\forall y \exists x [ x \not\leq y \text{ and } \forall y' ( \downarrow y \not\subseteq \downarrow y' \Rightarrow x \leq y' ) ]$

## Residuated family of operations

$(\cdot, /, \backslash)$   $\iff R \subseteq X \times X \times Y$  satisfying  
 $R[x, x', \_]$ ,  $R[x, \_, y]$ ,  $R[\_, x, y]$   
are Galois-closed sets  
( $R[x, x', \_]$  is  $x \cdot x'$  described in  $\mathcal{P}(Y)$ )

commutativity  $\iff \forall x, x' \quad R[x, x', \_] = R[x', x, \_]$

associativity  $\iff \forall x, x', x'' \quad (x \cdot x') \cdot x'' = x \cdot (x' \cdot x'')$

contraction  $\iff \forall x \quad (x \leq x \cdot x)$

weakening  $\iff \forall x, x' \quad (x \cdot x' \leq x)$

Uniform and modular representation theory for basic substructural hierarchy [Dunn, G, Palmigiano - 2005]

## Lambek-Grishin

$(\oplus, \otimes, \circ)$   $\iff$   $S \subseteq Y \times Y \times X$  satisfying  
 $S[y, y', -], S[y, -, x], S[-, y, x]$   
 are Galois-closed sets

$\forall A, B, C ((A \cdot B) \otimes C \leq A \cdot (B \otimes C))$  holds in the lattice

if and only if

$\forall x, x', y ((x \cdot x') \otimes y \leq x \cdot (x' \otimes y))$  holds in the corresponding frame

[Chernilovskaya, G, van Rooijen - ongoing work]

## Some linear logic operations

linear negation  $\rightsquigarrow$   $h : X \rightarrow Y$  bijection with  
 $x \leq h(x') \iff x' \leq h(x)$

$! : L \rightarrow L$  satisfying

(i)  $!$  is order-preserving

(ii)  $!!A = !A \leq A$

(iii)  $!(A \wedge B) = !A \cdot !B$

(iv)  $!\top = 1$

corresponds to a family  $\mathcal{F}$  of Galois-closed subsets of  $X$  with  
 $F \cdot F = F \leq 1$  [Note that there is always a largest such]

[Coumans, G, van Rooijen - ongoing work]

## Conclusions

Duality theory for lattices leads to two-sorted structures doubly determining the elements of the algebra (or logic)

The generalised powerset construction is Galois-closed sets

Linear negation ensures we can take  $X = Y$

The exponential is given by idempotent points

It would be nice to see how close we can get to coherent spaces just on the basis of duality

## Duality for first order logic

Duality theory is a powerful tool in the study of propositional logic.

**Aim:** generalise those techniques to first order logic.

1. Algebraic semantics for classical first order logic:  
Boolean hyperdoctrines
2. Dual notion of a Boolean hyperdoctrine:  
Indexed Stone spaces
3. Interpolation for classical first order logic

## Algebraic semantics for first order logic

A **Boolean hyperdoctrine** is a functor  $\mathcal{F}: \mathbf{B}^{\text{op}} \rightarrow \mathbf{BA}$  s.t.

1.  $\mathbf{B}$  is a category with finite products;
2. for all  $I, J \in \mathbf{B}$ ,  $\mathcal{F}(\pi_{I,J}): \mathcal{F}(I) \rightarrow \mathcal{F}(I \times J)$  has a left adjoint  $\exists_{I,J}$  satisfying Beck-Chevalley, i.e. for all  $I \xrightarrow{u} K$  in  $\mathbf{B}$ ,

$$\begin{array}{ccc}
 \mathcal{F}(K \times J) & \xrightarrow{\exists_{K,J}} & \mathcal{F}(K) \\
 \mathcal{F}(u \times \text{id}) \downarrow & & \downarrow \mathcal{F}(u) \\
 \mathcal{F}(I \times J) & \xrightarrow{\exists_{I,J}} & \mathcal{F}(I)
 \end{array}$$

3. for all  $I, J \in \mathbf{B}$ ,  $\mathcal{F}(\delta_{I,J}): \mathcal{F}((I \times J) \times J) \rightarrow \mathcal{F}(I \times J)$  has a left adjoint  $E_{qI,J}$  satisfying Beck-Chevalley.

## Algebraic semantics for first order logic

Examples of Boolean hyperdoctrines:

► **Syntactic hyperdoctrine**

For a signature  $\Sigma$  and a set of variables  $X$ , define:

$\mathbf{B}$  = contexts and substitutions

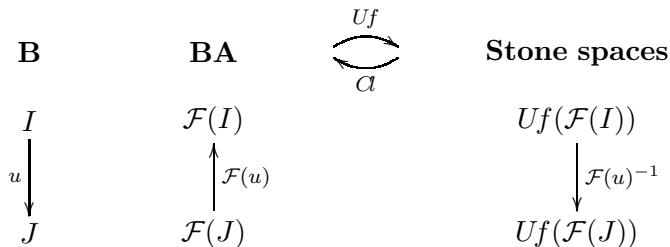
$$\mathcal{F}_\Sigma: [x_0, \dots, x_{n-1}] \mapsto Fm(x_0, \dots, x_{n-1})/\approx$$

► **Subset hyperdoctrine**

$\mathbf{B}$  = **Set**

$$\mathcal{P}: A \mapsto \wp(A)$$

## Duality for first order logic



This gives us a **dual equivalence** between:

Functors  $\mathcal{F}: \mathbf{B}^{\text{op}} \rightarrow \mathbf{BA}$

Functors  $\mathcal{G}: \mathbf{B} \rightarrow \mathbf{StSp}$

$$\begin{array}{ccc} \mathcal{F} & \xrightarrow{\quad} & Uf \circ \mathcal{F} \\ \mathcal{A} \circ \mathcal{G} & \xleftarrow{\quad} & \mathcal{G} \end{array}$$

## Duality for first order logic

### Boolean hyperdoctrines

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and for all  $I \xrightarrow{u} K$ ,

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### Indexed Stone spaces

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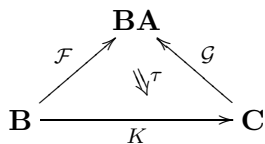
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 \end{array}$$

is a quasi pullback.

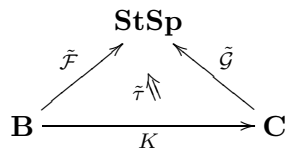
# Duality for first order logic

## Boolean hyperdoctrines



$$\tau: \mathcal{F} \rightarrow \mathcal{G} \circ K$$

## Indexed Stone spaces



$$\tilde{\mathcal{F}} = Uf \circ \mathcal{F} \quad \tilde{\mathcal{G}} = Uf \circ \mathcal{G}$$

$$\tilde{\tau}: \tilde{\mathcal{G}} \circ K \rightarrow \tilde{\mathcal{F}}$$

## Interpolation in first order logic

A logic has the **interpolation property** iff:

For all sentences  $\phi$  and  $\psi$  with  $\phi \vdash \psi$ , there exists a sentence  $\theta$  s.t.

1.  $\phi \vdash \theta$  and  $\theta \vdash \psi$ ;
2. every relation, function and constant symbol which occurs in  $\theta$  occurs in both  $\phi$  and  $\psi$ .

## Interpolation in first order logic

$$\Sigma_1 = \{R, S\} \qquad \Sigma_2 = \{R, T\}$$

$$\Sigma_0 = \{R\} \qquad \Sigma = \{R, S, T\}$$

Let  $\phi \in \mathcal{L}_1, \psi \in \mathcal{L}_2$  such that, for all  $\theta \in \mathcal{L}_0$ ,

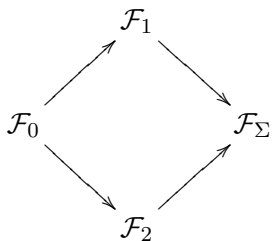
$$\phi \vdash \theta \quad \text{implies} \quad \theta \not\vdash \psi$$

To prove:  $\phi \not\vdash \psi$ .

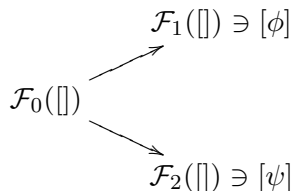
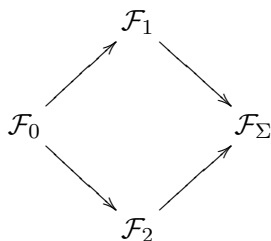
**Aim:** Find a morphism  $\tau: \mathcal{F}_\Sigma \rightarrow \mathcal{F}$  s.t.

$$\tau([\phi]) \not\leq \tau([\psi]).$$

# Interpolation in first order logic



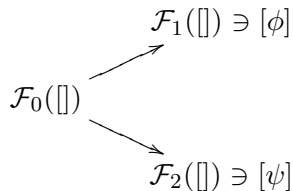
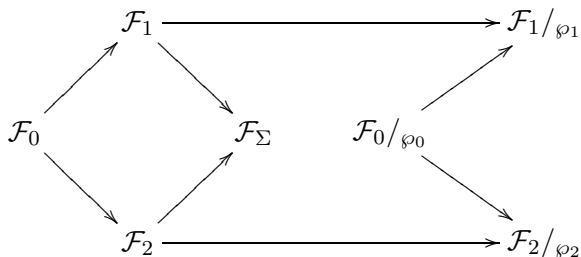
## Interpolation in first order logic



Let  $\wp_1 \subseteq \mathcal{F}_1(\square)$ ,  $\wp_2 \subseteq \mathcal{F}_2(\square)$  s.t.

1.  $[\phi] \in \wp_1$  and  $[\psi] \notin \wp_2$
2.  $\wp_1 \cap \mathcal{F}_0(\square) = \wp_2 \cap \mathcal{F}_0(\square)$

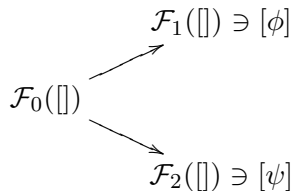
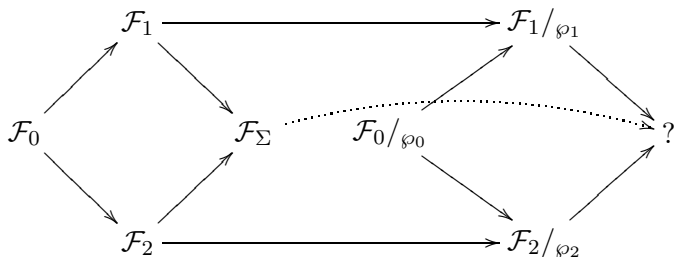
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## Future work

Having a basic set up of the duality for classical first order logic we would like to:

1. Describe dual structures for non-classical first order logics.
2. Study problems which have been solved in the propositional case using duality, e.g. the interpolation property.
3. Understand how this set-up relates to the internal duality described by Joyal & Tierney.