Duality for first order logic

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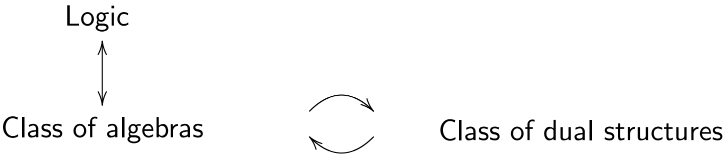
Outline

1. Duality in logic
2. Algebraic semantics for classical first order logic:

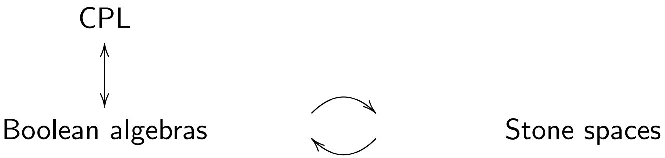
   Boolean hyperdoctrines
3. Dual notion of a Boolean hyperdoctrine:

   Indexed Stone spaces
4. Duality for classical first order logic
5. Future work

Duality in logic



Duality in logic



Duality in logic

CPL

$\updownarrow$

Boolean algebras $\rightleftarrows$ Stone spaces

$$\mathbf{B} \longmapsto (Uf(\mathbf{B}), \tau_{\mathbf{B}})$$

$$Cl(X) \longleftarrow\!\!\mid X$$

Duality in logic

CPL

$\updownarrow$

Boolean algebras $\rightleftarrows$ Stone spaces

$\mathbf{B} \xrightarrow{h} \mathbf{C} \quad \longmapsto \quad Uf(\mathbf{C}) \xrightarrow{h^{-1}} Uf(\mathbf{B})$

$Cl(Y) \xrightarrow{f^{-1}} Cl(X) \quad \longleftarrow\!\shortmid \quad X \xrightarrow{f} Y$

Duality in logic

CPL over a set of variables X

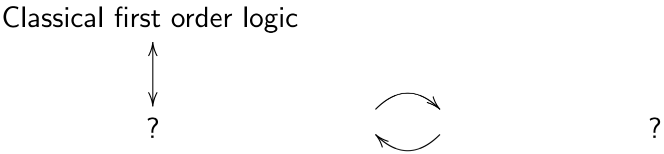
$\updownarrow$

Lindenbaum algebra
of formulas over X

$\rightleftarrows$

Maps $X \to 2$
'valuations'

Duality for first order logic



Duality for first order logic

Classical first order logic

↕

Boolean hyperdoctrines ⇄ ?

1. What are Boolean hyperdoctrines?

Duality for first order logic

Classical first order logic

Boolean hyperdoctrines

Indexed Stone spaces

1. What are Boolean hyperdoctrines?
2. Identify the dual notion of a Boolean hyperdoctrine.

Algebraic semantics for first order logic

We start from

Signature:	$\Sigma = (f_0, \dots, f_{k-1}, R_0, \dots, R_{l-1}, c_0, \dots, c_{m-1})$
Set of variables:	$X = \{x_0, x_1, \dots\}$

Question:

What properties does the collection of all formulas over Σ have?

Algebraic semantics for first order logic

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Question:

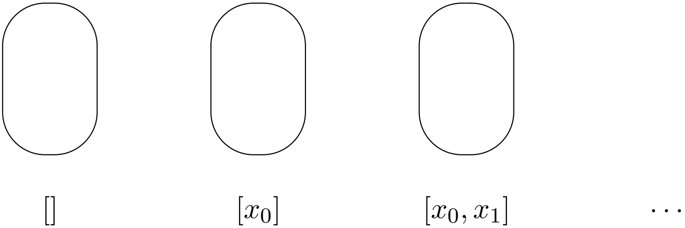
What properties does the collection of all formulas over Σ have?

First observation:

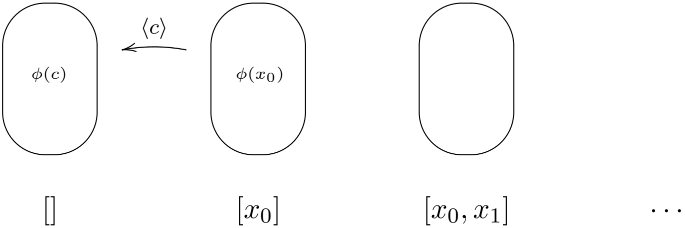
For each $n \in \mathbb{N}$,

$(Fm(x_0, \ldots, x_{n-1}), \vdash)$ is a Boolean algebra.

Algebraic semantics for first order logic



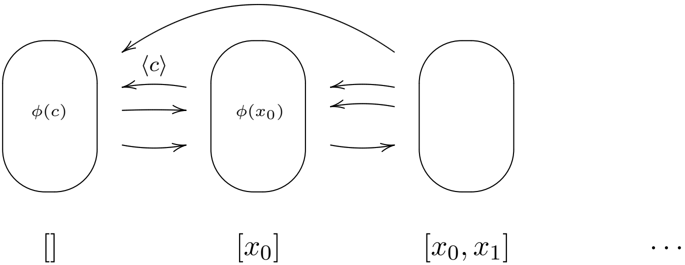
Algebraic semantics for first order logic



Substitutions:

$$\begin{array}{ccc} x_0 & \mapsto & c \\ \phi(x_0) & \mapsto & \phi(c) \end{array}$$

Algebraic semantics for first order logic



Substitutions:

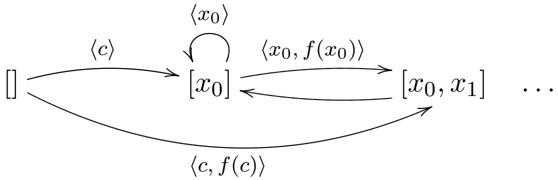
$$\begin{array}{ccc} x_0 & \mapsto & c \\ \phi(x_0) & \mapsto & \phi(c) \end{array}$$

Algebraic semantics for first order logic

Contexts and substitutions form category $\mathbf{B}$:

Objects: natural numbers (contexts)

Morphism $n \to m$: m-tuple $\langle t_0, \ldots, t_{m-1} \rangle$ s.t. $FV(t_i) \subseteq \{x_0, \ldots, x_{n-1}\}$



Algebraic semantics for first order logic

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This category has **finite products**:

$$[x_0, x_1] \xleftarrow{\langle x_0, x_1 \rangle} [x_0, x_1, x_2] \xrightarrow{\langle x_2 \rangle} [x_0]$$

Algebraic semantics for first order logic

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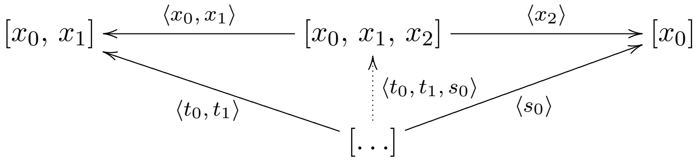
This category has **finite products**:

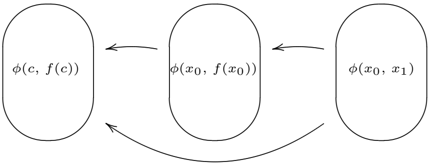


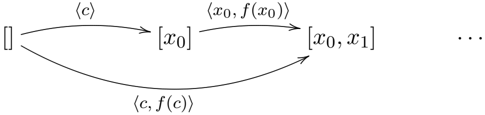
Algebraic semantics for first order logic

Formulas and substitutions: functor $\mathbf{B}^{op} \to \mathbf{BA}$

$$n \mapsto Fm(x_0, \dots, x_{n-1})$$

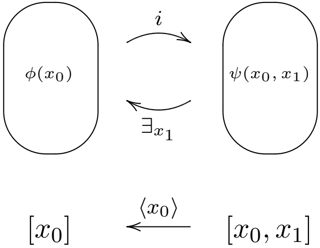
$$n \xrightarrow{\langle t_0, \dots, t_{m-1} \rangle} m \mapsto \begin{array}{rcl} Fm(x_0, \dots, x_{m-1}) & \to & Fm(x_0, \dots, x_{n-1}) \\ \phi(x_0, \dots, x_{m-1}) & \mapsto & \phi(t_0, \dots, t_{m-1}) \end{array}$$





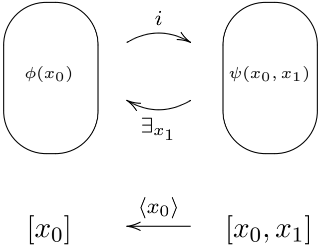
Algebraic semantics for first order logic

Existential quantification: related to the inclusion map



Algebraic semantics for first order logic

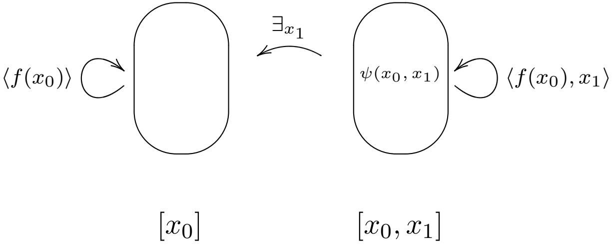
Existential quantification: related to the inclusion map



$$\frac{\exists_{x_1}(\psi(x_0,x_1)) \quad \vdash_{x_0} \quad \phi(x_0)}{\psi(x_0,x_1) \quad \vdash_{x_0,x_1} \quad i(\phi(x_0))}$$

Algebraic semantics for first order logic

Quantification: interaction with substitutions



$$\exists_{x_1}(\psi(x_0, x_1))[f(x_0)/x_0] \;=\; \exists_{x_1}(\psi(f(x_0), x_1))$$

(Beck-Chevalley)

Algebraic semantics for first order logic

A **Boolean hyperdoctrine** is a functor $\mathcal{F}\colon \mathbf{B}^{\mathbf{op}} \to \mathbf{BA}$ s.t.

1. $\mathbf{B}$ is a category with finite products;
2. for all $I, J \in \mathbf{B}$, $\mathcal{F}(\pi_{I,J})\colon \mathcal{F}(I) \to \mathcal{F}(I \times J)$ has a left adjoint $\exists_{I,J}$ such that, for all $I \xrightarrow{u} K$ in $\mathbf{B}$,

$$\begin{array}{ccc} \mathcal{F}(K \times J) & \xrightarrow{\exists_{K,J}} & \mathcal{F}(K) \\ {\scriptstyle \mathcal{F}(u\times \mathrm{id})}\downarrow & & \downarrow{\scriptstyle \mathcal{F}(u)} \\ \mathcal{F}(I \times J) & \xrightarrow[\exists_{I,J}]{} & \mathcal{F}(I) \end{array}$$

commutes.

Algebraic semantics for first order logic

Examples of Boolean hyperdoctrines:

- **Syntactic hyperdoctrine**

  $\mathbf{B}$ = contexts and substitutions

  $$\begin{array}{rccc} \mathcal{F}\colon & \mathbf{B}^{op} & \to & \mathbf{BA} \\ & n & \mapsto & Fm(x_0, \ldots, x_{n-1}) \end{array}$$

- **Subset hyperdoctrine**

  $\mathbf{B} = \mathbf{Set}$

  $$\begin{array}{rccc} \mathcal{P}\colon & \mathbf{B}^{op} & \to & \mathbf{BA} \\ & A & \mapsto & \text{powerset of } A \end{array}$$

Duality for first order logic

$$\begin{array}{cccc} \mathbf{B} & \mathbf{BA} & \underset{Cl}{\overset{Uf}{\rightleftarrows}} & \textbf{Stone spaces} \\ I & \mathcal{F}(I) & & Uf(\mathcal{F}(I)) \\ u\big\downarrow & \big\uparrow \mathcal{F}(u) & & \big\downarrow \mathcal{F}(u)^{-1} \\ J & \mathcal{F}(J) & & Uf(\mathcal{F}(J)) \end{array}$$

Duality for first order logic

$$\mathbf{B} \qquad \mathbf{BA} \underset{\mathcal{C}l}{\overset{Uf}{\rightleftarrows}} \textbf{Stone spaces}$$

$$\begin{array}{ccc} I & \mathcal{F}(I) & Uf(\mathcal{F}(I)) \\ u \downarrow & \uparrow \mathcal{F}(u) & \downarrow \mathcal{F}(u)^{-1} \\ J & \mathcal{F}(J) & Uf(\mathcal{F}(J)) \end{array}$$

This gives us a **dual equivalence** between:

Functors $\mathcal{F}\colon \mathbf{B}^{\mathrm{op}} \to \mathbf{BA}$ Functors $\mathcal{G}\colon \mathbf{B} \to \mathbf{StSp}$

$$\mathcal{F} \longmapsto Uf \circ \mathcal{F}$$

$$\mathcal{C}l \circ \mathcal{G} \longleftarrow\!\shortmid \mathcal{G}$$

Duality for first order logic

$$\mathcal{F}\colon \mathbf{B}^{\mathbf{op}} \to \mathbf{BA} \qquad\qquad \mathcal{G}\colon \mathbf{B} \to \mathbf{StSp}$$

$\mathcal{F}(\pi_{I,J})$ has a left adjoint $\exists_{I,J}$

for all $I \xrightarrow{u} K$,

$$\begin{array}{ccc}
\mathcal{F}(K \times J) & \xrightarrow{\exists_{K,J}} & \mathcal{F}(K) \\
{\scriptstyle \mathcal{F}(u\times \mathrm{id})}\downarrow & & \downarrow{\scriptstyle \mathcal{F}(u)} \\
\mathcal{F}(I \times J) & \xrightarrow[\exists_{I,J}]{} & \mathcal{F}(I)
\end{array}$$

commutes.

Duality for first order logic

$$\mathcal{F}\colon \mathbf{B}^{\mathbf{op}} \to \mathbf{BA} \qquad\qquad \mathcal{G}\colon \mathbf{B} \to \mathbf{StSp}$$

$\mathcal{F}(\pi_{I,J})$ has a left adjoint $\exists_{I,J}$

$\mathcal{G}(\pi_{I,J})$ is an open map

for all $I \xrightarrow{u} K$,

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commutes.

Duality for first order logic

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\end{array}$$

commutes.

$$\mathcal{G}\colon \mathbf{B} \to \mathbf{StSp}$$

$\mathcal{G}(\pi_{I,J})$ is an open map

for all $I \xrightarrow{u} K$,

$$\begin{array}{ccc}
z \in \ \mathcal{G}(I \times J) & \xrightarrow{\mathcal{G}(\pi_{I,J})} & \mathcal{G}(I) \ \ni y \\
{\scriptstyle \mathcal{G}(u\times\mathrm{id})}\downarrow & & \downarrow{\scriptstyle \mathcal{G}(u)} \\
x \in \ \mathcal{G}(K \times J) & \xrightarrow[\mathcal{G}(\pi_{K,J})]{} & \mathcal{G}(K)
\end{array}$$

$\mathcal{G}(u)(x) = \mathcal{G}(\pi_{K,J})(y)$ implies there exists $z \in \mathcal{G}(I \times J)$ s.t.

$$\begin{aligned}
\mathcal{G}(\pi_{I,J})(z) &= x \\
\mathcal{G}(u \times \mathrm{id})(z) &= y.
\end{aligned}$$

Duality for first order logic

Boolean hyperdoctrines

Functors $\mathcal{F}\colon \mathbf{B}^{\mathbf{op}} \to \mathbf{BA}$ s.t.

1. $\mathbf{B}$ has finite products;
2. $\mathcal{F}(\pi_{I,J})$ has a left adjoint $\exists_{I,J}$ and for all $I \xrightarrow{u} K$,

$$\begin{array}{ccc} \mathcal{F}(K \times J) & \xrightarrow{\exists_{K,J}} & \mathcal{F}(K) \\ {\scriptstyle \mathcal{F}(u\times \mathrm{id})}\downarrow & & \downarrow{\scriptstyle \mathcal{F}(u)} \\ \mathcal{F}(I \times J) & \xrightarrow[\exists_{I,J}]{} & \mathcal{F}(I) \end{array}$$

commutes.

Indexed Stone spaces

Functors $\mathcal{G}\colon \mathbf{B} \to \mathbf{StSp}$ s.t.

1. $\mathbf{B}$ has finite products;
2. $\mathcal{G}(\pi_{I,J})$ is an open map and for all $I \xrightarrow{u} K$,

$$\begin{array}{ccc} \mathcal{G}(I \times J) & \xrightarrow{\mathcal{G}(\pi_{K,J})} & \mathcal{G}(I) \\ {\scriptstyle \mathcal{G}(u\times \mathrm{id})}\downarrow & & \downarrow{\scriptstyle \mathcal{G}(u)} \\ \mathcal{G}(K \times J) & \xrightarrow[\mathcal{G}(\pi_{I,J})]{} & \mathcal{G}(K) \end{array}$$

is epicartesian.

Duality for first order logic

Duality theorem for classical first order logic:

The category of Boolean hyperdoctrines and the category of indexed Stone spaces are dually equivalent.

Boolean hyperdoctrines		**Indexed Stone spaces**
$\mathcal{F}$	$\longmapsto$	$Uf \circ \mathcal{F}$
$Cl \circ \mathcal{G}$	$\longleftarrow\!\!\!\!\dashv$	$\mathcal{G}$

Future work

Having a duality for classical first order logic we would like to:

1. Describe dual structures for non-classical first order logics.
2. Obtain information about these first order logics via studying their dual structures.
3. In particular: study the interpolation property dually.