The right choice?

An intuitionistic exploration of Zermelo's Axiom

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Contents

List of Symbols

Chapter 1

Introduction

Imagine a queue of infinitely many prisoners numbered $0, 1, 2, \ldots$. Randomly, each of them is assigned a black or a white hat. Each prisoner can only see the hats of the fellow inmates in front of him (i.e. the hats of the inmates who have a higher number than he has). The guard asks each prisoner in turn to guess the colour of his hat, without the other prisoners being able to hear his reply. If the prisoner answers correctly, he will be released. If not, he has to stay in prison for the rest of his life.
After being given the rules of the game, the prisoners get one hour to determine their strategy. One of them, a classical mathematician accepting the Axiom of Choice, says 'I have a plan that ensures at most finitely many of us guess wrongly'.

The topic of this thesis is the Axiom of Choice, a statement which has led to a huge amount of controversy and discussion since its formulation. The example above is just one of the many peculiar consequences of the Axiom of Choice. Banach and Tarksi explained, for example, how one, using the Axiom, can cut a sphere into finitely many pieces that can be rearranged into two new spheres that are both of the same size as the original one.

At first sight the Axiom of Choice sounds quite plausible, as it states:

for every set A, there exists a function f that assigns to every non-empty subset B of A an element $f(B)$ of B. (AC)

We will refer to this statement as AC.
For instance, for the set $A = \{0, 1\}$, both

$$\{0\} \mapsto 0, \qquad \{1\} \mapsto 1, \qquad \{0,1\} \mapsto 0,$$

and

$$\{0\} \mapsto 0, \qquad \{1\} \mapsto 1, \qquad \{0,1\} \mapsto 1,$$

would do as such a function.
This is a very simple example. Notice, AC states 'for **every** set', which makes it a strong statement, in particular if one has to deal with an infinite set A.

In this thesis we study the Axiom of Choice from Brouwer's intuitionistic point of view. There is much confusion about the role of the Axiom of Choice in constructive mathematics. Some people seem to regard it as a non-constructive principle, while others defend it from a constructive point of view. This confusion is largely caused by the many different, classically equivalent, formulations of the Axiom. At face value, some of these formulations seem plausible in a constructive context, while others seem outright false.
We will show that some forms of the Axiom of Choice are acceptable to the intuitionistic mathematician, while others are highly debatable.
Before giving a more detailed description of the content of this thesis, we briefly sketch the origin and development of the Axiom of Choice. Some familiarity with classical set theory is assumed, but such knowledge is not required for the rest of this essay.

1.1 History

At the end of the 19$^{\text{th}}$ century, Georg Cantor claims that, for all sets X and Y, either X can be embedded into Y or Y can be embedded into X. In order to prove this claim he introduces the Well-Ordering Principle, which states that every set can be well-ordered. He proposes this principle as a 'law of thought', a statement that, according to him, is obvious from the way one should think about sets. Not everyone agrees.
In 1904, searching for a proof of the Well-Ordering Principle, Ernst Zermelo introduces the Axiom of Choice.[1] The formulation of AC by Zermelo makes people aware of the numerous implicit uses of this proposition in earlier arguments. It leads to vehement discussions: should one accept the Axiom of Choice or shouldn't one?
In 1908 Zermelo publishes an axiomatization of set theory. One of his aims is to clarify the role and meaning of AC. Further study and debate leads to a list of nine axioms and axiom schemes for set theory, called the 'Axioms of Zermelo and Fraenkel', abbreviation ZF, which finds wide acceptance.
Due to the work of Fraenkel (1922), Gödel (1938), Mostowski (1945), and Cohen (1963) we now know that the Axiom of Choice is independent of ZF, that is, if ZF is consistent then both ZF+AC and ZF+¬AC are consistent. Cohen used a new method for his proof, called 'forcing', which turned out to be a very fruitful method to obtain independence results [1].

[1] The Axiom of Choice is in fact equivalent to the Well-Ordering Principle.

1.2 Overview

This thesis is written from an intuitionistic point of view. A basic introduction to intuitionism is given in Chapter 2.
The statement AC does not make immediate sense to the intuitionistic mathematician. The concept 'for every set' is far too general and we prefer to first consider some special sets A, like the set $\mathbb{N}$ of the natural numbers and Baire space $\mathcal{N}$, to see what becomes of AC in those cases.
Some forms of the Axiom of Choice are acceptable to us, as these follow from the way we think about the objects of Baire space. We will explain and justify those in Section 2.5 and we will use these forms in the rest of the thesis.
In the intuitionistic study of sets, the equality relation deserves special treatment. On the set of the natural numbers, the equality relation is decidable, but on Baire space it is not. We have tried to find a characterization of both spreads and fans (two typically intuitionistic notions, explained in Chapter 2) with a decidable equality. The result is presented in Chapter 3.

A second difficulty in AC is how to understand the phrase 'every non-empty subset of A'. Again, the notion of an arbitrary subset of a given set is too general. In intuitionistic mathematics not all subsets are of the same kind. In the example at the beginning, we have only defined the function for the decidable subsets of $\{0, 1\}$. The statement becomes stronger if we consider the enumerable subsets of A or an even broader notion of subset. We want it to be clear which collection of (sub)sets we are working with, and therefore we start from the classically equivalent statement:

> *for every family $\mathcal{F}$ of non-empty sets, there exists a function $f : \mathcal{F} \to \bigcup \mathcal{F}$ such that, for each $S \in \mathcal{F}$, $f(S) \in S$.*

Given a family $\mathcal{F}$, a function satisfying the above condition is called a *choice function on $\mathcal{F}$*. Inspired by this formulation of the Axiom of Choice, we consider various families of non-empty sets and ask whether one can define a choice function on them. This is the topic of Chapter 5.
Before we go into the existence and non-existence of choice functions, we have to gain a better insight into some of the families of sets we wish to consider, in particular the collection of the closed subsets of $\mathcal{N}$ and the collection of the open subsets of $\mathcal{N}$. As we explain in Chapter 4, the intuitionistic mathematician has to distinguish between several definitions of the notions 'open subset of $\mathcal{N}$' and 'closed subset of $\mathcal{N}$', which would be equivalent from a classical point of view.

In classical mathematics, the Axiom of Choice has a large number of equivalent formulations [2]. In the last chapters we study two of these and see what becomes of them if we try to interpret them intuitionistically. In Chapter 6 we start from the statement:

> *every equivalence relation has a set of representatives.*

We try to find out if, for several kinds of equivalence relations on the natural numbers, there exists a set of representatives and in what sense.

We also return to the problem of the prisoners and their hats formulated at the beginning of this introduction and explain the classical strategy of the prisoner that claims to be able to get most of them free. We will see that he uses the 'fact' that there exists a set of representatives for a certain equivalence relation on Cantor space $\mathcal{C}$. This equivalence relation appears to be connected with the famous equivalence relation defined by Vitali on the set $\mathbb{R}$ of the real numbers to construct a non-Lebesgue measurable subset of $\mathbb{R}$. Considering the relation on $\mathcal{C}$ from an intuitionistic point of view, we prove some theorems to the effect that it fails to have an easily obtainable set of representatives.
In Chapter 7, we consider the following classical equivalent of AC:

every vector space has a basis.

We study $\mathbb{R}$ as a vectorspace over $\mathbb{Q}$ and investigate the existence of a basis for $\mathbb{R}$ over $\mathbb{Q}$, a so-called *Hamel basis*. We show that, as in classical set theory, there is a close connection between Hamel bases and Vitali sets. Using this connection we prove some positive results showing there is no easily obtainable Hamel basis.

Chapter 2

Prerequisites

This chapter provides a short introduction to intuitionistic mathematics. Some important basic concepts of intuitionism are explained and I hope that by going through the examples the reader gets some feeling for a number of typically intuitionistic principles. Definitions of more specific concepts used in this thesis will be given along the way.

2.1 Basic sets

We study two basic sets. The first is $\mathbb{N}$, the set of the natural numbers. $\mathbb{N}$ has a so-called decidable equality. This means we can decide for all m, n in $\mathbb{N}$ whether $m = n$ or $m \neq n$.
This is not so for the second fundamental set, $\mathcal{N}$, the collection of all infinite sequences of natural numbers, or, one could say, the collection of all functions from $\mathbb{N}$ to $\mathbb{N}$. $\mathcal{N}$ is called *Baire space*. The elements of $\mathcal{N}$ are thought of as being constructed step by step. One defines an element α of $\mathcal{N}$ by giving, one by one, its values $\alpha(0)$, $\alpha(1)$, $\alpha(2), \ldots$. Such a sequence may be given by an explicit algorithm (for example 'pick 0 in each step'), but, even then, it might be imagined to be the result of choosing randomly in each step.

Definition 2.1. For each $m \in \mathbb{N}$, we define an element $\underline{m}$ of $\mathcal{N}$ by

$$\underline{m}(n) \underset{\text{D}}{=} m \quad \text{for all } n \text{ in } \mathbb{N}.$$

For example, $\underline{0}$ is the sequence consisting of only zeros.

For $\alpha, \beta \in \mathcal{N}$, we define

$$\alpha = \beta \ \leftrightarrows\ \forall n\,[\,\alpha(n) = \beta(n)\,].$$

Saying $\neg\,(\alpha = \beta)$ is saying that the assumption $\forall n\,[\,\alpha(n) = \beta(n)\,]$ leads to a contradiction. This is quite weak. We only know that α and β are not everywhere the same, but we may not be able to indicate a place where they differ. Therefore, we also define a stronger relation on $\mathcal{N}$.

Definition 2.2. We say: α is *apart from* β, notation $\alpha \# \beta$, if and only if there exists $n \in \mathbb{N}$ such that $\alpha(n) \neq \beta(n)$.

In intuitionistic mathematics the phrase 'there exists' is interpreted in a constructive way. That is, for any property A of natural numbers, the statement $\exists n \in \mathbb{N} [A(n)]$ means that we are able to explicitly find a natural number n_0 satisfying $A(n_0)$.

We cannot decide for each $\alpha, \beta \in \mathcal{N}$,

$$\alpha = \beta \vee \alpha \# \beta.$$

Let us first explain how we, following Brouwer, interpret the logical symbol $\vee$. For any two mathematical statements A and B, $A \vee B$ means that we can pick one of the two statements (A or B) and give a proof of this statement.
Now consider for instance $\alpha \in \mathcal{N}$ defined by:

$\alpha(n) =_{\text{D}} 0$ if there is no block of 99 nines in the first n digits of the decimal expansion of π,

$=_{\text{D}} 1$ if there is a block of 99 nines in the first n digits of the decimal expansion of π.

Note that

$$\forall n [\alpha(n) = 0] \quad \leftrightarrows \quad \text{there is no block of 99 nines in the decimal expansion of } \pi,$$

and

$$\exists n [\alpha(n) \neq 0] \quad \leftrightarrows \quad \text{there is a block of 99 nines in the decimal expansion of } \pi.$$

As, at this moment, we do not know whether or not there exists a block of 99 nines in the decimal expansion of π, we cannot prove $\alpha = \underline{0}$, nor $\alpha \# \underline{0}$. Hence we cannot decide $\alpha = \underline{0} \vee \alpha \# \underline{0}$.

We will use the example of the 99 nines in the decimal expansion of π more often, and therefore, we introduce some abbreviations. For each natural number n we define

$n < k_{99} =_{\text{D}}$ there is no block of 99 nines in the first n digits of the decimal expansion of π.

$n = k_{99} =_{\text{D}}$ n is the first position in the decimal expansion of π where a block of 99 nines is completed.

$n > k_{99} =_{\text{D}}$ there is a block of 99 nines in the first $n - 1$ digits of the decimal expansion of π.

Note that we do not give a definition of k_{99} on its own. Nevertheless, we will occasionally talk about k_{99} as if it exists in its own right. We could, for example, say 'k_{99} exists', when we mean: $\exists n [n = k_{99}]$.

2.2 Special subsets of $\mathbb{N}$

Let A be a subset of $\mathbb{N}$ and n a natural number. We cannot decide in general

$$n \in A \;\vee\; n \notin A. \tag{2.1}$$

One could define, for example,

$$A \underset{\mathrm{D}}{=} \{n \in \mathbb{N} \,|\, n = 0 \;\wedge\; \text{the Riemann Hypothesis}\}.$$

As we can neither prove nor reject the Riemann Hypothesis (at this moment), we cannot decide $0 \in A \vee 0 \notin A$.
A subset A of $\mathbb{N}$ for which we are able to decide (2.1) for all natural numbers n is called a decidable subset of $\mathbb{N}$. In fact, we will use the following definition of the notion 'decidable subset of $\mathbb{N}$', which is slightly stronger.

Definition 2.3. For any $\alpha \in \mathcal{N}$, we define:

$$D_\alpha \underset{\mathrm{D}}{=} \{n{\in}\mathbb{N} \,|\, \alpha(n) = 1\}.$$

Let $X \subseteq \mathbb{N}$.
We say that α *decides* X if and only if X coincides with D_α.
X is a *decidable subset* of $\mathbb{N}$ if and only if some $\alpha \in \mathcal{N}$ decides X.

Two other special collections of subsets of $\mathbb{N}$ are the collection of the enumerable subsets of $\mathbb{N}$ and the collection of the co-enumerable subsets of $\mathbb{N}$. These are defined as follows.

Definition 2.4. For any $\alpha \in \mathcal{N}$, we define:

$$E_\alpha \underset{\mathrm{D}}{=} \{n{\in}\mathbb{N} \,|\, \exists k{\in}\mathbb{N}\,[\, n = \alpha(k) - 1\,]\},$$
$$CE_\alpha \underset{\mathrm{D}}{=} \mathbb{N} \setminus E_\alpha = \{n{\in}\mathbb{N} \,|\, \neg(n \in E_\alpha)\,\}.$$

Let $X \subseteq \mathbb{N}$.
We say that α *enumerates* (resp. *co-enumerates*) X if and only if X coincides with E_α (CE_α).
X is a *enumerable subset* (resp. *co-enumerable subset*) of $\mathbb{N}$ if and only if some $\alpha \in \mathcal{N}$ enumerates (co-enumerates) X.

Not every enumerable subset of $\mathbb{N}$ is decidable. For instance, the set

$$A \underset{\mathrm{D}}{=} \{n \in \mathbb{N} \,|\, n = 1 \;\wedge\; k_{99}\,\text{exists}\}$$

is not decidable. As we do not know whether or not there exists a block of 99 nines in the decimal expansion of π, we are not able to decide whether the natural number 1 is in A or not.
A is enumerable for we can define $\alpha \in \mathcal{N}$ by

$$\begin{aligned} \alpha(k) &\underset{\mathrm{D}}{=} 0 && \text{if}\, k < k_{99}, \\ &\underset{\mathrm{D}}{=} 2 && \text{if}\, k \geq k_{99}. \end{aligned}$$

Then $A = E_\alpha$.
We will say some more about this topic in Section 6.3.1.

2.3 Spreads

A typical intuitionistic concept is the notion of a spread. In order to define spreads we first introduce some other concepts. To start with, $\mathbb{N}^*$ is the set of all finite sequences of natural numbers. We define a bijective function $\langle\,\rangle : \mathbb{N}^* \to \mathbb{N}$ by defining, for all $k \in \mathbb{N}$, for all $s_0, \ldots, s_{k-1} \in \mathbb{N}$,

$$\langle s_0, \ldots, s_{k-1} \rangle \underset{\mathrm{D}}{=} p_0^{s_0} \cdot \ldots \cdot p_{k-2}^{s_{k-2}} \cdot p_{k-1}^{s_{k-1}+1} - 1 \tag{2.2}$$

where p_n is the n^{th} prime number and by definition the empty product is 1 (whence $\langle\,\rangle = 1 - 1 = 0$).
Using this function, we may view $\mathbb{N}^*$ and $\mathbb{N}$ as the same set.
We define a function $lg : \mathbb{N}^* \to \mathbb{N}$ by

$$lg(\,\langle s_0, \ldots, s_{k-1} \rangle\,) \underset{\mathrm{D}}{=} k.$$

We also define, for each finite sequence s and all $i < lg(s)$,

$$\langle s_0, \ldots, s_{k-1} \rangle(i) \underset{\mathrm{D}}{=} s_i.$$

We define two concatenation functions, both denoted by $*$. The first one is between finite sequences and the second one is between finite sequences and infinite sequences.

Definition 2.5. Let $\alpha \in \mathcal{N}$ and $r, s \in \mathbb{N}^*$, say $r = \langle r_0, \ldots r_m \rangle$, $s = \langle s_0, \ldots, s_n \rangle$.

$$r * s \underset{\mathrm{D}}{=} \langle r_0, \ldots, r_m, s_0, \ldots, s_n \rangle,$$
$$r * \alpha \underset{\mathrm{D}}{=} (r_0, \ldots, r_m, \alpha(0),\ \alpha(1),\ \alpha(2), \ldots).$$

The collection of all finite sequences of natural numbers may be ordered as follows.

Definition 2.6. Let $a, b \in \mathbb{N}^*$.
We say a *extends to* b, notation $a \sqsubseteq b$, if and only if

$$\exists c \in \mathbb{N}^*\, [\, b = a * c\,].$$

We say a is a *proper initial segment* of b, notation $a \sqsubset b$, if and only if

$$a \sqsubseteq b \wedge a \neq b.$$

Definition 2.7. For all $\alpha \in \mathcal{N}$, for all $m \in \mathbb{N}$,

$$\overline{\alpha}(m) \underset{\mathrm{D}}{=} \langle \alpha(0), \ldots, \alpha(m-1) \rangle.$$

In case there is no confusion possible, we leave out the parentheses and write $\overline{\alpha}m$. We call $\overline{\alpha}m$ an *initial segment* of α.

Similary, for all $s \in \mathbb{N}^*$, for all $m \leq lg(s)$,

$$\overline{s}(m) \underset{\mathrm{D}}{=} \langle\, s(0), \ldots,\ s(m-1)\,\rangle.$$

Now we are ready to define the notion of a spread.

Definition 2.8. Let $\sigma : \mathbb{N}^* \to \{0, 1\}$.
We say σ is a *spread law* if and only if:

(L1) $\sigma(\langle\,\rangle) = 0$,

(L2) for all $s \in \mathbb{N}^*$, $\sigma(s) = 0 \leftrightarrows \exists n\,[\,\sigma(s * \langle\, n\,\rangle) = 0\,]$.

For all $\alpha \in \mathcal{N}$, we say: α is *admitted* by σ, notation $\alpha \in \sigma$, if and only if for all $n \in \mathbb{N}$, $\sigma(\overline{\alpha}n) = 0$. The set of all sequences that are admitted by a spread law σ is called a *spread.* We write σ for both the spread and the spread law.

One can think of $\mathcal{N}$ as a tree with the finite sequences of natural numbers as nodes. In this tree there is a path from the finite sequence a to the finite sequence b if and only if a extends to b. The elements of $\mathcal{N}$ are all the infinite paths one may construct by going upwards in this tree.
For a given spread σ, the spread law tells you, in each node, which branches you can walk if you want to construct an infinite sequence that is an element of σ. Due to property L2 there are no dead ends. Property L1 guarantees that σ contains at least one element.

Cantor space, $\mathcal{C}$, is the collection of all infinite sequences of natural numbers assuming no other values than zero and one. This is a spread, as we can define the appropriate spread law σ by:

$$\sigma(s) = 0 \leftrightarrows \forall i < lg(s)\,[\,s(i) = 0 \vee s(i) = 1\,]$$

2.4 Brouwer's Continuity Principle

Let $R \subseteq \mathcal{N} \times \mathbb{N}$ be a relation such that, for all α in $\mathcal{N}$, we can find a natural number n satisfying $R(\alpha, n)$. Then Brouwer's Continuity Principle (CP) states that, for all α in $\mathcal{N}$, we can already determine a natural number n satisfying $R(\alpha, n)$ after knowing only a finite part of α, i.e.

$$\forall \alpha{\in}\mathcal{N}\,\exists m{\in}\mathbb{N}\,\exists n{\in}\mathbb{N}\,\forall \beta{\in}\mathcal{N}\,[\,\overline{\alpha}m = \overline{\beta}m \to R(\beta, n)\,]$$

This might sound improbable at first, but recall how we view elements of Baire space. Every infinite sequence of natural numbers may be thought of as being constructed step by step. If you state

$$\forall \alpha{\in}\mathcal{N}\,\exists n{\in}\mathbb{N}\,[\,R(\alpha, n)\,],$$

you claim that if someone (let us call that person your opponent) gives you an element α of $\mathcal{N}$ by giving you one by one its values $\alpha(0)$, $\alpha(1)$, $\alpha(2), \ldots$, at some point you are able to come up with a natural number n such that $R(\alpha, n)$. Suppose you do so after your opponent has given the first 100 values of α. Then for any β that agrees on the first 100 positions with α (i.e. $\overline{\alpha}100 = \overline{\beta}100$), also $R(\beta, n)$, as you do not yet know anything about the future values of α.

In Section 2.1 we have constructed a sequence α for which the statement $\alpha = \underline{0} \vee \alpha \# \underline{0}$ is reckless. Using the Continuity Principle one can obtain a contradiction from the assumption $\forall \alpha, \beta \in \mathcal{N} [\alpha = \beta \vee \alpha \# \beta]$. First note that we can code a sequence of elements $\alpha_0, \alpha_1, \ldots$ of $\mathcal{N}$ by just one element α of $\mathcal{N}$. To do so we use the map $\langle\,\rangle$ defined at (2.2) on page 12.

Definition 2.9. Let $\alpha \in \mathcal{N}$.
For each $n \in \mathbb{N}$, we define $\alpha^n \in \mathcal{N}$ by, for all $m \in \mathbb{N}$,

$$\alpha^n(m) \underset{\text{D}}{=} \alpha(\langle n, m \rangle).$$

Now suppose

$$\forall \alpha, \beta \in \mathcal{N} [\alpha = \beta \vee \alpha \# \beta]. \tag{2.3}$$

Define $R \subseteq \mathcal{N} \times \mathbb{N}$ by

$$R(\alpha, 0) \quad \leftrightarrows \quad \alpha^0 = \alpha^1 \qquad\qquad R(\alpha, 1) \quad \leftrightarrows \quad \alpha^0 \# \alpha^1.$$

By (2.3), in particular $\forall \alpha \in \mathcal{N} [\alpha^0 = \alpha^1 \vee \alpha^0 \# \alpha^1]$, so

$$\forall \alpha \in \mathcal{N} \exists n \in \mathbb{N} [R(\alpha, n)].$$

Applying the Continuity Principle yields:

$$\forall \alpha \in \mathcal{N} \exists m \in \mathbb{N} \exists n \in \mathbb{N} \forall \beta \in \mathcal{N} [\overline{\alpha} m = \overline{\beta} m \rightarrow R(\beta, n)]. \tag{2.4}$$

Note that $\underline{0}^0 = \underline{0} = \underline{0}^1$, so $R(\underline{0}, 0)$ and not $R(\underline{0}, 1)$.
Hence, by applying (2.4) for $\alpha = \underline{0}$, we can determine $m \in \mathbb{N}$ such that:

$$\forall \beta \in \mathcal{N} [\overline{\underline{0}} m = \overline{\beta} m \rightarrow R(\beta, 0)]. \tag{2.5}$$

Define $\beta \in \mathcal{N}$ by

$$\beta(n) \underset{\text{D}}{=} 0 \qquad\qquad \text{if } n \neq \langle 1, m \rangle,$$
$$\underset{\text{D}}{=} 1 \qquad\qquad \text{if } n = \langle 1, m \rangle.$$

Then, as $\langle 1, m \rangle > m$, $\overline{\beta} m = \overline{\underline{0}} m$. Hence, by (2.5), $R(\beta, 0)$.
But $\beta^0 = \underline{0} \neq \beta^1$. Contradiction.
So, using the Continuity Principle, we have proven

$$\neg \forall \alpha, \beta \in \mathcal{N} [\alpha = \beta \vee \alpha \# \beta].$$

In Section 2.3 we have introduced spreads. The argument given to justify the Continuity Principle for $\mathcal{N}$ can be applied to any spread.

Definition 2.10. *Brouwer's Generalized Continuity Principle* (GCP).
Let $\sigma \subseteq \mathcal{N}$ be a spread and $R \subseteq \sigma \times \mathbb{N}$.
If, for all α in σ, there exists a natural number n satisfying $R(\alpha, n)$, then

$$\forall \alpha \in \sigma \exists m \in \mathbb{N} \exists n \in \mathbb{N} \forall \beta \in \sigma [\overline{\alpha} m = \overline{\beta} m \rightarrow R(\beta, n)].$$

The (Generalized) Continuity Principle is a powerful tool. Using GCP one can for example prove that every real function is continuous (the definition of the real numbers will be given in Section 2.7). The Continuity Principle will play an important role in many proofs in this thesis.

2.5 Axioms of Choice

In this thesis we will investigate what becomes of the various forms and equivalents of the Axiom of Choice if we try to interpret them intuitionistically. Some forms of the Axiom of Choice are acceptable for the intuitionistic mathematician, as these follow from the way we think about elements of Baire space. The first of these is the First Axiom of Countable Choice

Definition 2.11. *First Axiom of Countable Choice* ($AC_{0,0}$)
Let $P \subseteq \mathbb{N} \times \mathbb{N}$ such that, for each natural number n, we can find at least one m in $\mathbb{N}$ with $P(n, m)$. Then there exists $\alpha \in \mathcal{N}$ such that $\forall n \in \mathbb{N}\, [\, P(n, \alpha(n))\,]$.

In classical set theory $AC_{0,0}$ is a theorem (i.e. a statement one can prove from ZF, the usual axiom system for set theory without the Axiom of Choice), as one may define a function $f : \mathbb{N} \to \mathbb{N}$, by

$$f(n) \underset{\mathrm{D}}{=} \mu m\, [\, P(n, m)\,].$$

Intuitionistically, this formula does not define an element of $\mathcal{N}$, as we may not be able to decide, for a given n, which $m \in \mathbb{N}$ is the smallest natural number satisfying $P(n, m)$.
However, $AC_{0,0}$ is accepted as an axiom, because it follows from the way we view $\mathcal{N}$. The elements of $\mathcal{N}$ are defined step by step. Hence for any subset P of $\mathbb{N} \times \mathbb{N}$ with

$$\forall m \in \mathbb{N}\, \exists n \in \mathbb{N}\, [\, P(m, n)\,],$$

we can define an element α as follows: in step n find $m \in \mathbb{N}$ with $P(n, m)$ and define $\alpha(n) \underset{\mathrm{D}}{=} m$. This α satisfies the requirement.

The First Axiom of Countable Choice is about relations on $\mathbb{N}$. The next choice axiom we consider is about relations between $\mathcal{N}$ and $\mathbb{N}$. Before we can formulate this axiom we need to be clear on what we mean by 'a function from $\mathcal{N}$ to $\mathbb{N}$'. We immediately define a more general notion, namely the one of 'a function from a spread to the natural numbers'.

Definition 2.12. Let $\gamma, \alpha \in \mathcal{N}$. We define

$$\begin{aligned} \gamma : \alpha \mapsto n \;&\underset{\mathrm{D}}{=}\; \gamma \text{ assigns to } \alpha \text{ the natural number } n, \\ &\underset{\mathrm{D}}{=}\; \exists k \in \mathbb{N}\, [\, \gamma(\overline{\alpha}k) = n + 1 \wedge \forall m \in \mathbb{N}\, [\, m \neq k \to \gamma(\overline{\alpha}m) = 0\,]\,]. \end{aligned}$$

For any spread σ and any $\gamma \in \mathcal{N}$ we define

$$\begin{aligned} \gamma : \sigma \to \mathbb{N} \;&\underset{\mathrm{D}}{=}\; \gamma \text{ defines a function from } \sigma \text{ to } \mathbb{N}, \\ &\underset{\mathrm{D}}{=}\; \forall \alpha \in \sigma\, \exists n \in \mathbb{N}\, [\, \gamma : \alpha \mapsto n\,]. \end{aligned}$$

For all $\alpha \in \sigma$, we write $\gamma(\alpha)$ for the unique natural number n such that $\gamma : \alpha \mapsto n$.

Why does it make sense to define functions from a spread σ to $\mathbb{N}$ like this? Well, such a function sends each element of σ to a natural number. If someone is building an element of the spread by giving its values step by step, at some point the function should be able to supply the image of this sequence. This is exactly what happens with the γ defined above. For any $\alpha \in \mathcal{N}$, for all $m \in \mathbb{N}$, as long as the initial segment $\overline{\alpha}m$ of α does not give enough information to determine the image of α, $\gamma(\overline{\alpha}m) = 0$. However, at some point we know what the image of α is, say after knowing the first k values of α we know its image is n. Then $\gamma(\overline{\alpha}k) = n + 1$.

Now we are ready to define the next choice axiom, the First Axiom of Continuous Choice.

Definition 2.13. *First Axiom of Continuous Choice* ($GAC_{1,0}$)
Let σ be a spread and $R \subseteq \sigma \times \mathbb{N}$ such that, for all $\alpha \in \sigma$, there exists $n \in \mathbb{N}$ with $R(\alpha, n)$. Then there exists a function $\gamma : \sigma \rightarrow \mathbb{N}$, such that, for all $\alpha \in \mathcal{N}$, $R(\alpha, \gamma(\alpha))$.

This axiom may be defended as follows. Let σ be a spread and $R \subseteq \sigma \times \mathbb{N}$ such that $\forall \alpha \in \sigma \, \exists n \in \mathbb{N} \, [\, R(\alpha, n) \,]$. We define $\gamma \in \mathcal{N}$ step by step. For each $a \in \mathbb{N}$ decide whether a gives enough information to find $n \in \mathbb{N}$ with $R(\alpha, n)$ for a sequence α starting with (the finite sequence coded by) a. If this is the case and there is no $b \in \mathbb{N}$ with $b \sqsubset a$ and $\gamma(b) \neq 0$, we define $\gamma(a) \underset{\mathrm{D}}{=} n + 1$. Otherwise, $\gamma(a) \underset{\mathrm{D}}{=} 0$. One may convince oneself that γ is a function from σ to $\mathbb{N}$ and, for all $\alpha \in \sigma$, $R(\alpha, \gamma(\alpha))$.

The last choice axiom we introduce is about relations on $\mathcal{N}$. We first define what we mean by 'a function from a spread to $\mathcal{N}$'.

Definition 2.14. Let $\alpha, \beta, \gamma \in \mathcal{N}$. We define:

$$\begin{aligned} \gamma : \alpha \mapsto \beta \; &\underset{\mathrm{D}}{=} \; \gamma \text{ assigns to } \alpha \text{ the sequence } \beta, \\ &\underset{\mathrm{D}}{=} \; \forall n \in \mathbb{N} \, [\, \gamma^n : \alpha \mapsto \beta(n) \,]. \end{aligned}$$

For any spread σ and $\gamma \in \mathcal{N}$ we define:

$$\begin{aligned} \gamma : \sigma \rightarrow \mathcal{N} \; &\underset{\mathrm{D}}{=} \; \gamma \text{ defines a function from } \sigma \text{ to } \mathcal{N}, \\ &\underset{\mathrm{D}}{=} \; \forall \alpha \in \sigma \, \exists \beta \in \mathcal{N} \, [\, \gamma : \alpha \mapsto \beta \,]. \end{aligned}$$

For all $\alpha \in \sigma$, we write $\gamma|\alpha$ for the unique sequence β such that $\gamma : \alpha \mapsto \beta$.

Definition 2.15. *Second Axiom of Continuous Choice* ($GAC_{1,1}$)
Let σ be a spread and $R \subseteq \sigma \times \mathcal{N}$ such that, for all $\alpha \in \sigma$, there exists $\beta \in \mathcal{N}$ with $R(\alpha, \beta)$. Then there exists a function $\gamma : \sigma \rightarrow \mathcal{N}$ such that, for all $\alpha \in \mathcal{N}$, $R(\alpha, \gamma|\alpha)$.

It takes a bit more effort to defend this axiom. For the justification of this axiom and a thorough explanation of all choice axioms introduced in this section the reader is referred to [4].
$GAC_{1,1}$ is the strongest axiom we have introduced in this section. Both $AC_{0,0}$ and $GAC_{1,0}$ are a consequence of $GAC_{1,1}$.

2.6 Fans

We have seen two examples of spreads up to now: $\mathcal{N}$ and $\mathcal{C}$. There is an important difference between these two spreads. When constructing an element of $\mathcal{N}$ you have infinitely many possible choices at each step (you can pick any natural number). On the other hand, you only have two options (zero and one) for each position when you are constructing an element of $\mathcal{C}$. A spread in which you only have finitely many possible choices in each step is called a fan. A precise definition of the notion of a fan is the following.

Definition 2.16. Let $\sigma : \mathbb{N}^* \to \{0, 1\}$.
We say σ is a *fan law* if and only if σ is a spread law, i.e. σ satisfies:

(L1) $\sigma(\langle\,\rangle) = 0$,

(L2) for all $s \in \mathbb{N}^*$, $\sigma(s) = 0 \leftrightarrows \exists n\,[\,\sigma(s * \langle\, n\,\rangle) = 0\,]$,

and in addition,

(L3) for all $s \in \mathbb{N}^*$, $\sigma(s) = 0 \to \exists k\in\mathbb{N}\,\forall n \in \mathbb{N}\,[\,\sigma(s * \langle n\rangle) = 0 \to n < k\,]$.

For any fan law σ the set

$$\{\alpha \in \mathcal{N} \mid \forall n\in\mathbb{N}\,[\,\sigma(\overline{\alpha}n) = 0\,]\},$$

i.e. the set of all infinite sequences that are admitted by σ, is called a *fan*.

One may view a fan as a tree, where in each node there are finitely many (but never zero) branches going up. The elements of a fan are constructed by walking up the tree and in each node picking one of the finitely many possible branches to a following node.

2.6.1 Fan Theorem

The Fan Theorem is a powerful tool used in intuitionistic mathematics. Brouwer used this theorem to prove that every real-valued (continuous) function defined on a closed interval is uniformly continuous. First we introduce the notion of a 'bar'.

Definition 2.17. Let σ be a fan and B a subset of $\mathbb{N}^*$.
We call B a *bar* in σ if and only if every α in σ has an initial segment $\overline{\alpha}n$ belonging to B.

The Fan Theorem says

every decidable bar has a finite subbar.

What do we mean by this? Let σ be a fan and B a decidable subset of $\mathbb{N}^*$ that is a bar in σ. Then the Fan Theorem states we can find a finite subset B' of B, such that B' is a bar in σ. One may formulate the Fan Theorem in the following equivalent way as well.

Definition 2.18. *Fan Theorem*
For every fan σ and for every decidable subset B of $\mathbb{N}^*$ that is a bar in σ, there exists a natural number M such that, for all $\alpha \in \sigma$, there exists a natural number $n \leq M$ with $\overline{\alpha}n \in B$.

Classically, the Fan Theorem is equivalent to its contraposition, which is known as König's Lemma:

every infinite finitely branching tree has an infinite path.

There are some expressions in the statement above that perhaps have to be explained. So let us start with that.

Definition 2.19. Let $T \subseteq \mathbb{N}^*$.
T is a *tree* if and only if for all $s \in \mathbb{N}^*$, for all $n \in \mathbb{N}$, if $s * \langle n \rangle$ belongs to T, then s belongs to T.
A tree T is *finitely-branching* if and only if for all elements s of T, there are only finitely many natural numbers n with $s * \langle n \rangle \in T$.
A tree T is *infinite* if and only if for all $n \in \mathbb{N}$, there exists $s \in T$ with $lg(s) = n$.
An element α of $\mathcal{N}$ is an *infinite path* of T if and only if for all $n \in \mathbb{N}$, $\overline{\alpha}n \in T$.

The classical mathematican may prove König's Lemma as follows. Let T be an infinite finitely-branching tree. The classical mathematician now constructs an infinite path α of T step by step. In the first step he notes that T is finitely-branching, hence there are only finitely many $n \in \mathbb{N}$ with $\langle n \rangle \in T$, say $n_0, \ldots, n_k$. As any element of T starts with either n_0 or n_1 or $\ldots$ or n_k and T is infinite, at least for one of the n_i the set

$$A_i \underset{\mathrm{D}}{=} \{ s \in \mathbb{N}^* \mid s(0) = n_i \}$$

is infinite. $\alpha(0)$ is defined as the smallest such n_i. Continuing this way, the classical mathematician defines, for each $n \in \mathbb{N}$,

$$\alpha(n) \underset{\mathrm{D}}{=} \mu k \, [\, \{ s \in T \mid \overline{\alpha}n * \langle k \rangle \sqsubseteq s \} \text{ is infinite} \,]$$

He proves by induction that such k exists for each n, hence α is a well-defined infinite path in T. So classically König's Lemma holds and thereby also the Fan Theorem holds.
This proof of König's Lemma is not acceptable for the intuitionistic mathematician. Have another look at how $\alpha(0)$ is defined. Although the union of the sets $A_0, \ldots, A_k$ is infinite, we may not be able to decide which of the sets

is infinite. We only know that they are not all finite. The same problem arises when defining $\alpha(1), \alpha(2), \ldots$.
Having seen this, we need not be suprised that König's Lemma does not hold intuitionistically. One could define a tree T by

$$\begin{aligned} s \in T \quad &\leftrightarrows \quad \exists n \in \mathbb{N}\,[\, s = \underline{\overline{0}}n \,\wedge\, (n < k_{99} \,\vee\, (n \geq k_{99} \,\wedge\, k_{99} \text{ is even})\,] \,\vee \\ &\qquad\; \exists n \in \mathbb{N}\,[\, s = \underline{\overline{1}}n \,\wedge\, (n < k_{99} \,\vee\, (n \geq k_{99} \,\wedge\, k_{99} \text{ is odd})\,]. \end{aligned}$$

Then T is an infinite finitely-branching tree, but it is reckless to say T has an infinite path. For suppose T has an infinite path, say α. Then

$$\begin{aligned} \alpha(0) = 0 \quad &\leftrightarrows \quad \forall n \in \mathbb{N}\,[\, n = k_{99} \,\rightarrow\, n \text{ is even}\,], \\ \alpha(0) = 1 \quad &\leftrightarrows \quad \forall n \in \mathbb{N}\,[\, n = k_{99} \,\rightarrow\, n \text{ is odd}\,]. \end{aligned}$$

As we are able to decide whether $\alpha(0) = 0$ or $\alpha(0) = 1$, it follows from the equivalencies above that we can decide

$$\forall n \in \mathbb{N}\,[\, n = k_{99} \,\rightarrow\, n \text{ is even}\,] \,\vee\, \forall n \in \mathbb{N}\,[\, n = k_{99} \,\rightarrow\, n \text{ is odd}\,].$$

But this is reckless.

Brouwer proved the Fan Theorem using his Principle of Bar Induction. For a full explanation of his argument the reader could consult [3] or [4].

2.6.2 Unrestricted Fan Theorem

Up to now we have only spoken about decidable bars. Leaving out the adjective 'decidable' in the formulation of the Fan Theorem gives us a more general form of this theorem. Note that this does not change anything for the classical mathematician. He assumes the law of excluded middle by which every bar is automatically a decidable bar.

Definition 2.20. *Unrestricted Fan Theorem*
Let σ be a fan and let $B \subseteq \mathbb{N}^*$ be a bar in σ. Then we can find $N \in \mathbb{N}$ such that $\forall \alpha \in \sigma\, \exists n \in \mathbb{N}\,[\, n \leq N \,\wedge\, (\alpha, n) \in B\,]$.

Theorem 2.21. *Assuming $GAC_{1,0}$, the Unrestricted Fan Theorem can be derived from the Fan Theorem.*

Proof. Let σ be a fan and $B \subseteq \mathbb{N}^*$ a bar in σ.
We will show that B has a finite subbar B'.
As B is a bar in σ,

$$\forall \alpha \in \sigma\, \exists n \in \mathbb{N}\,[\, \overline{\alpha}n \in B\,].$$

So, by $GAC_{1,0}$, there exists a function $\gamma : \sigma \to \mathbb{N}$ such that:

$$\forall \alpha \in \sigma\,[\, \overline{\alpha}(\gamma(\alpha)) \in B\,].$$

Define of subset C of $\mathbb{N}^*$ by

$$C \underset{\text{D}}{=} \{ s \in \mathbb{N}^* \mid \gamma(s) \neq 0 \}.$$

First note that C is a bar in σ.

Let $\alpha \in \sigma$.
$\gamma : \sigma \to \mathbb{N}$, so by definition there exists $n \in \mathbb{N}$ with $\gamma(\overline{\alpha}n) \neq 0$.
So $\forall \alpha \in \sigma\, \exists n \in \mathbb{N}\, [\, \overline{\alpha}n \in C\,]$.

C is a decidable subset of $\mathbb{N}^*$. Applying the Fan Theorem yields that C has a finite subbar C', say $C' = \{s_0, \ldots, s_n\}$.

The idea for the definition of B' is the following:

For $0 \leq i \leq n$:
Consider $\gamma(s_i)$ and distinguish two cases:

1. $\gamma(s_i) - 1 \;\leq\; lg(s_i)$.
   Make sure $\overline{s_i}(\gamma(s_i) - 1)$ is in B'.
2. $\gamma(s_i) - 1 \;>\; lg(s_i)$.
   Make sure all finite sequences of length $\gamma(s_i) - 1$ that start with s_i and are in σ are in B'.

To accomplish this we define $B' \subseteq \mathbb{N}^*$ by:

$$\begin{aligned} s \in B' \quad \leftrightarrows \quad & \exists i \leq n\, [\, \overline{s_i}(\gamma(s_i) - 1) = s\,] \vee \\ & \exists i \leq n\, [\, s_i \sqsubseteq s \;\wedge\; lg(s) = \gamma(s_i) - 1 \;\wedge\; \sigma(s) = 0\,]. \end{aligned}$$

Note that B' is a decidable subset of $\mathbb{N}^*$.
We will prove that B' is a finite subbar of B.

- $B' \subseteq B$
  Let $s \in B'$.
  It follows from the definition of B' that there are two possibilities:

  1. $\exists i \leq n\, [\, \overline{s_i}(\gamma(s_i) - 1) = s\,]$.
     By definition of γ this yields $s \in B$.
  2. $\exists i \leq n\, [\, s_i \sqsubseteq s \;\wedge\; lg(s) = \gamma(s_i) - 1 \;\wedge\; \sigma(s) = 0\,]$.
     Define $\alpha \in \mathcal{N}$ by

     $$\begin{aligned} \alpha(n) &\underset{\mathrm{D}}{=} s(n) && \text{if } n < lg(s), \\ &\underset{\mathrm{D}}{=} \mu i\, [\, \sigma(\overline{\alpha}n * \langle i \rangle) = 0\,] && \text{if } n \geq lg(s). \end{aligned}$$

     $\alpha \in \sigma$, so

     $$\overline{\alpha}(\gamma(\alpha)) \in B. \tag{2.6}$$

     As $s_i \sqsubseteq s = \overline{\alpha}(lg(s))$ and $\gamma(s_i) \neq 0$,

     $$\gamma(\alpha) = \gamma(s_i) - 1. \tag{2.7}$$

Combining (2.6) and (2.7) we conclude,

$$\overline{\alpha}(\gamma(s_i) - 1) \in B.$$

As $\gamma(s_i) - 1 = lg(s)$,

$$s = \overline{\alpha}(lg(s)) = \overline{\alpha}(\gamma(s_i) - 1) \in B.$$

So $B' \subseteq B$.

- B' is a bar in σ.
  Let $\alpha \in \sigma$.
  Determine $i \in \mathbb{N}$ such that $\overline{\alpha}(lg(s_i)) = s_i$ (C' is a bar in σ).
  Then $\overline{\alpha}(\gamma(s_i) - 1) \in B'$.

- B' is finite.
  For every $0 \leq i \leq n$ there are two possibilities:

  1. $\gamma(s_i) - 1 \ \leq \ lg(s_i)$
     s_i causes the addition of one extra element to B'.

  2. $\gamma(s_i) - 1 \ > \ lg(s_i)$
     σ is a fan, so for each $t \in \mathbb{N}^*$ with $\sigma(t) = 0$ there are only finitely many natural numbers k such that $\sigma(t * \langle k \rangle) = 0$.
     So there are only finitely many $t \in \sigma$ of length $\gamma(s_i) - 1$ that start with s_i.

  Any element in B' is in there because of either reason 1 or 2. So B' only has finitely many elements.

Hence B' is a finite subbar of B, as required. □

2.6.3 Extended Fan Theorem

We often use an extension of the Fan Theorem which states: if, for every element in a certain fan, we can find a natural number with a desired property, then we can find $N \in \mathbb{N}$ such that, for each element of the fan, we can find a natural number with the desired property below N.

Definition 2.22. *Extended Fan Theorem*
Let σ be a fan and $R \subseteq \sigma \times \mathbb{N}$ such that for each $\alpha \in \sigma$ there exists $n \in \mathbb{N}$ such that $R(\alpha, n)$. Then we can find $N \in \mathbb{N}$ such that $\forall \alpha \in \sigma \, \exists n \leq N \, [\, R(\alpha, n) \,]$.

Theorem 2.23. *Assuming $GAC_{1,0}$, the Extended Fan Theorem can be derived from the Fan Theorem.*

Proof. Let σ be a fan and $R \subseteq \sigma \times \mathbb{N}$ with the property

$$\forall \alpha \in \sigma \, \exists n \in \mathbb{N} \, [\, R(\alpha, n) \,].$$

By $GAC_{1,0}$, there exists a function $\gamma : \sigma \to \mathbb{N}$ satisfying

$$\forall \alpha \in \sigma \, [\, R(\alpha, \gamma(\alpha)) \,].$$

Define a subset B of $\mathbb{N}^*$ by, for all $s \in \mathbb{N}^*$,

$$s \in B \quad \leftrightarrows \quad \gamma(s) \neq 0.$$

B is decidable and, as γ is a function on σ, B is a bar in σ. So we can apply the Fan Theorem and find $M \in \mathbb{N}$ such that

$$\forall \alpha \in \sigma \, \exists n \leq M \, [\, \gamma(\overline{\alpha} n) \neq 0 \,].$$

Define $N \underset{\mathrm{D}}{=} max(\{\, \gamma(s) \mid lg(s) \leq M \,\})$. Then

$$\forall \alpha \in \sigma \, \exists n \leq N \, [\, R(\alpha, n) \,],$$

as required. □

The Extended Fan Theorem is much stronger than the Fan Theorem and, in contrast to the Unrestricted Fan Theorem, it does not hold classically. One could define for example $R \subseteq \mathcal{C} \times \mathbb{N}$ by, for all $\alpha \in \mathcal{C}$, for all $n \in \mathbb{N}$,

$$R(\alpha, n) \;\leftrightarrows\; (\alpha = \underline{0} \wedge n = 0) \vee \alpha(n) \neq 0.$$

The classical mathematician would say this relation satisfies:

$$\forall \alpha \in \mathcal{C} \, \exists n \in \mathbb{N} \, [\, R(\alpha, n) \,]. \tag{2.8}$$

To him, any element α of $\mathcal{C}$ is either equal to $\underline{0}$ (in which case $R(\alpha, 0)$) or is not equal to $\underline{0}$ (in that case he is convinced there exists a position n with $\alpha(n) \neq 0$). Clearly there is no natural number N, such that for all α in $\mathcal{C}$ we can find $n \leq N$ with $R(\alpha, n)$. Hence the Extended Fan Theorem does not hold classically.
The intuitionistic mathematician does not agree with (2.8), however. Hence the relation R does not satisfy the requirements of the Extended Fan Theorem.

2.7 The set of the real numbers

Starting from the two basic structures we have introduced, $\mathbb{N}$ and $\mathcal{N}$, we want to construct more complex structures. To begin with, we define the integers by $\mathbb{Z} \underset{\mathrm{D}}{=} (\mathbb{N} \times \mathbb{N}) / \sim_0$, where $\sim_0$ is the equivalence relation:

$$\langle m, n \rangle \sim_0 \langle p, q \rangle \quad \leftrightarrows \quad m + q = n + p.$$

Addition, multiplication and order on $\mathbb{Z}$ are defined as usual.
The rational numbers are defined by $\mathbb{Q} \underset{\mathrm{D}}{=} (\mathbb{Z} \times \mathbb{N}_{>0}) / (\sim_1)$, where $\sim_1$ is the equivalence relation:

$$\langle m, n \rangle \sim_1 \langle p, q \rangle \quad \leftrightarrows \quad m \cdot q = n \cdot p.$$

Addition, multiplication and order on $\mathbb{Q}$ are defined as usual.
Note that the equality on both $\mathbb{Z}$ and $\mathbb{Q}$ is decidable.

To define the real numbers we first introduce the set S of all (code numbers of) rational intervals. Let ρ be a bijective map from $\mathbb{N}$ to $\mathbb{Q}$. We say $m \in \mathbb{N}$ codes a rational interval if and only if there exist $m_0, m_1 \in \mathbb{N}$ such that

1. $m = \langle m_0, m_1 \rangle$,
2. $\rho(m_0) \leq \rho(m_1)$.

For each $m = \langle m_0, m_1 \rangle$ in S we define

$$m' \underset{\text{D}}{=} \rho(m_0) \quad \text{and} \quad m'' \underset{\text{D}}{=} \rho(m_1),$$
$$lg(m) \underset{\text{D}}{=} m'' - m'.$$

So m' and m'' are the left and the right endpoint of the interval coded by m and $lg(m)$ is the length of the interval coded by m.
For rational numbers p, q, we will usually just write $[p, q]$ for the rational interval $\langle \rho^{-1}(p), \rho^{-1}(q) \rangle$.

For all $a, b \in S$ we define

$$a \sqsubseteq b \underset{\text{D}}{=} b' \leq a' \leq a'' \leq b'' \qquad (a \text{ is contained in } b),$$
$$a \approx b \underset{\text{D}}{=} a' \leq b'' \wedge b' \leq a'' \qquad (a \text{ touches } b).$$

Notice we also defined a function lg and a relation $\sqsubseteq$ on the collection of finite sequences of natural numbers. By stating which collection we are working with it will be clear which notion we mean.

Definition 2.24. Let $x \in \mathcal{N}$.
We say x is a *real number*, i.e. $x \in \mathbb{R}$, if and only if x is a shrinking and shriveling sequence of rational intervals. That is,

(i). for all $n \in \mathbb{N}$, $x(n) \in S$,

(ii). for all $n \in \mathbb{N}$, $x(n+1) \sqsubseteq x(n)$ (shrinking)

(iii). for all $n \in \mathbb{N}$, there exists m in $\mathbb{N}$ such that $lg(x(n)) \leq 2^{-m}$ (shriveling)

For all $x, y \in \mathbb{R}$, we define

$$x \equiv y \underset{\text{D}}{=} \forall n \in \mathbb{N} [\, x(n) \approx y(n) \,],$$
$$x \# y \underset{\text{D}}{=} \exists n \in \mathbb{N} [\neg (\, x(n) \approx y(n))\,],$$

$$x < y \underset{\text{D}}{=} \exists n \in \mathbb{N} [\, x(n)'' < y(n)',],$$
$$x \leq y \underset{\text{D}}{=} \forall n \in \mathbb{N} [\, x(n)' \leq y(n)'' \,].$$

Note that equality on the reals is not decidable. We could, for example, define a real number ψ by

$$\begin{aligned} \psi(n) \underset{\mathrm{D}}{=} & [-\tfrac{1}{n}, \tfrac{1}{n}] && \text{if } n \leq k_{99}, \\ \underset{\mathrm{D}}{=} & [\tfrac{1}{k}, \tfrac{1}{k}] && \text{if } k = k_{99} \text{ and } k \leq n. \end{aligned}$$

As one can show

$$\begin{aligned} \psi \equiv 0 \quad & \leftrightarrows \quad k_{99} \text{ exists}, \\ \psi \,\#\, 0 \quad & \leftrightarrows \quad k_{99} \text{ does not exist}, \end{aligned}$$

we cannot decide $\psi \equiv 0 \vee \psi \,\#\, 0$.

Far more can be said about the construction of the real numbers, but as this is not of particular importance for this thesis we do not go into this subject any further.

Chapter 3

Discrete spreads and fans

In the previous chapter we have seen that equality on $\mathcal{N}$ is not decidable. One could, however, study subsets of $\mathcal{N}$ that do have a decidable equality, i.e. subsets A of $\mathcal{N}$ with

$$\forall \alpha, \beta \in A\,[\,\alpha = \beta \vee \alpha \# \beta\,]. \tag{3.1}$$

A subset A of $\mathcal{N}$ satisfying (3.1) is called *discrete*.

In this chapter we study discrete spreads and fans. We will see that these can be characterized in a surprising and nice way. The characterization of discrete fans leads us to an equivalent of the Fan Theorem, which we prove in the last section of this chapter.

3.1 Discrete spreads

In section 2.3 we have introduced spreads. When working with spreads we have a powerful tool, the (Generalized) Continuity Principle (Definition 2.10). Using GCP we will show that a spread is discrete if and only if it is enumerable.

Definition 3.1. Let $A \subseteq \mathcal{N}$.
A is *enumerable* iff there exist $\alpha_0, \alpha_1, \ldots$ in $\mathcal{N}$ such that $A = \{\alpha_0, \alpha_1, \ldots\}$.

We start with a lemma and show: for all α and β in a discrete spread, we can already decide whether $\alpha = \beta$ or $\alpha \# \beta$ after knowing finitely many values of α and β.

Lemma 3.2. *Let σ be a discrete spread. Then*

$$\forall \alpha \in \sigma\, \exists m \in \mathbb{N}\, \forall \beta \in \sigma\, [\,\overline{\alpha}m = \overline{\beta}m \rightarrow \alpha = \beta\,].$$

Proof. Let σ be a discrete spread.
We define for each $\alpha \in \mathcal{N}$

$$\alpha_{odd} \underset{\mathrm{D}}{=} (\alpha(1), \alpha(3), \alpha(5), \ldots) \qquad \alpha_{even} \underset{\mathrm{D}}{=} (\alpha(0), \alpha(2), \alpha(4), \ldots).$$

Similarly, s_{odd} and s_{even} are defined for $s \in \mathbb{N}^*$.

Define a spread law τ by, for all $s \in \mathbb{N}^*$,

$$\tau(s) = 0 \;\leftrightarrows\; \sigma(s_{odd}) = \sigma(s_{even}) = 0.$$

As, for all $\gamma \in \tau$, both γ_{odd} and γ_{even} are in the discrete spread σ,

$$\forall \gamma \in \tau\, [\, \gamma_{odd} = \gamma_{even} \,\vee\, \gamma_{odd} \# \gamma_{even}\,]. \tag{3.2}$$

Define a relation $R \subseteq \tau \times \mathbb{N}$ by

$$\gamma\, R\, 0 \;\leftrightarrows\; \gamma_{odd} = \gamma_{even}, \qquad \gamma\, R\, 1 \;\leftrightarrows\; \gamma_{odd} \# \gamma_{even}.$$

Then, by (3.2), for all γ in τ, $\gamma\, R\, 0$ or $\gamma\, R\, 1$. So using GCP,

$$\forall \gamma \in \tau\, \exists n, m \in \mathbb{N}\, \forall \delta \in \tau\, [\, \overline{\gamma} m = \overline{\delta} m \,\rightarrow\, \delta\, R\, n\,]. \tag{3.3}$$

Let $\alpha \in \sigma$.
Define $\gamma \in \tau$ such that

$$\gamma_{odd} \underset{\mathrm{D}}{=} \alpha \quad \text{and} \quad \gamma_{even} \underset{\mathrm{D}}{=} \alpha.$$

Then: $\gamma\, R\, 0$ and $\neg\, (\gamma\, R\, 1)$ (as $\gamma_{odd} = \alpha = \gamma_{even}$).
Determine $m \in \mathbb{N}$ such that $\forall \delta \in \tau\, [\, \overline{\gamma} m = \overline{\delta} m \,\rightarrow\, \delta\, R\, 0\,]$ (see (3.3)). Then

$$\forall \delta \in \tau\, [\, \overline{\gamma} m = \overline{\delta} m \,\rightarrow\, \delta_{odd} = \delta_{even}\,]. \tag{3.4}$$

Now let $\beta \in \tau$ with $\overline{\alpha} m = \overline{\beta} m$.
Define $\delta \in \tau$ by

$$\delta_{odd} \underset{\mathrm{D}}{=} \alpha \quad \text{and} \quad \delta_{even} \underset{\mathrm{D}}{=} \beta.$$

Then $\overline{\gamma} m = \overline{\delta} m$, so by (3.4),

$$\alpha = \delta_{odd} = \delta_{even} = \beta.$$

And we conclude,

$$\forall \beta \in \sigma\, [\, \overline{\alpha} m = \overline{\beta} m \,\rightarrow\, \alpha = \beta\,].$$

□

What does this lemma tell us? Let us view a spread as a tree again. Then the lemma states: if you are walking up a tree representing a discrete spread, from some point onwards there is only one way up (in each node there is only one direction to choose).
Using this lemma, we prove that every discrete spread is enumerable. To define such an enumeration we first enumerate the set of all finite sequences of natural numbers. For each admitted finite sequence (i.e. each node in the tree), we construct an element of the spread by starting at the given finite sequence

and choosing, at each step, the leftmost branch. We add the resulting infinite sequence to our list.
For every element α of the discrete spread, we can find a natural number m such that any sequence in the spread starting with $\overline{\alpha}m$ is equal to α. Hence, if you start in the node $\overline{\alpha}m$ and take, in each step, the 'leftmost branch up', you construct α (as α is the only path in the spread starting with $\overline{\alpha}m$), so α is in our enumeration. We make this idea precise in the proof of the following theorem.

Theorem 3.3. *For every spread σ, σ is discrete if and only if σ is enumerable.*

Proof. Let σ be a spread.
$\Rightarrow$) Suppose σ is discrete.
Let $s_0, s_1, \ldots$ be an enumeration of $\mathbb{N}^*$.
First define α by, for all $n \in \mathbb{N}$,

$$\alpha(n) \underset{\text{D}}{=} \mu k\, [\, \sigma(\overline{\alpha}n * \langle k \rangle) = 0\,].$$

α is well-defined, as σ is a spread and $\alpha \in \sigma$.
We define a sequence $\alpha_0, \alpha_1, \ldots$ as follows:

> Let $p \in \mathbb{N}$ and consider s_p.
> We distinguish two cases:
>
> 1. $\sigma(s_p) = 1$.
>   Then s_p is not admitted by σ.
>   Define $\alpha_p \underset{\text{D}}{=} \alpha$.
> 2. $\sigma(s_p) = 0$.
>   Define α_p by:
>
> $$\begin{aligned} \alpha_p(n) &\underset{\text{D}}{=} s_p(n) && \text{if } n < lg(s_p), \\ &\underset{\text{D}}{=} \mu k\, [\, \sigma(\overline{\alpha_p}n * \langle k \rangle) = 0\,] && \text{if } n \geq lg(s_p). \end{aligned}$$

Note that this sequence is well-defined and that for each p, $\alpha_p \in \sigma$.
So $\{\alpha_0, \alpha_1, \ldots\} \subseteq \sigma$.

Let $\beta \in \sigma$.
Determine, using Lemma 3.2, $m \in \mathbb{N}$ such that

$$\forall \gamma \in \sigma\, [\, \overline{\beta}m = \overline{\gamma}m \;\rightarrow\; \beta = \gamma\,]. \tag{3.5}$$

Determine $p \in \mathbb{N}$ such that $\overline{\beta}m = s_p$.
It follows from the definition of α_p that

$$\overline{\alpha_p}m = s_p = \overline{\beta}m.$$

As $\alpha_p \in \sigma$, by (3.5), $\beta = \alpha_p \in \{\alpha_0, \alpha_1, \ldots\}$.
So $\sigma \subseteq \{\alpha_0, \alpha_1, \ldots\}$.

Hence $\sigma = \{\alpha_0, \alpha_1, \ldots\}$ and therefore, σ is enumerable.

$\Leftarrow$) Suppose σ is enumerable.
Let $\{\alpha_0, \alpha_1, \ldots\}$ be an enumeration of σ.
Then,

$$\forall \alpha \in \sigma \, \exists n \in \mathbb{N} \, [\, \alpha = \alpha_n \,].$$

Applying GCP yields:

$$\forall \alpha \in \sigma \, \exists m, n \in \mathbb{N} \, \forall \gamma \in \sigma \, [\, \overline{\alpha} m = \overline{\gamma} m \; \rightarrow \; \gamma = \alpha_n \,].$$

Let $\alpha, \beta \in \sigma$.
Determine m, n such that

$$\forall \gamma \in \sigma \, [\, \overline{\alpha} m = \overline{\gamma} m \; \rightarrow \; \gamma = \alpha_n \,].$$

In particular, as $\overline{\alpha} m = \overline{\alpha} m$, $\alpha = \alpha_n$. Hence

$$\forall \gamma \in \sigma \, [\, \overline{\alpha} m = \overline{\gamma} m \; \rightarrow \; \gamma = \alpha \,].$$

This means that, by considering $\overline{\alpha} m$ and $\overline{\beta} m$, we are able to decide

$$\alpha = \beta \; \vee \; \alpha \# \; \beta.$$

So σ is discrete. □

3.2 Discrete fans

Every fan is in particular a spread, so Theorem 3.3 also holds for discrete fans. However, using the Fan Theorem, we can prove an even stronger statement for fans, namely

every discrete fan is a finite set. (3.6)

In the context of elementary intuitionistic analysis, this statement is equivalent to the Fan Theorem.

Theorem 3.4. *The statement 'every discrete fan is a finite set' is equivalent to the Fan Theorem*

Proof.
$\Rightarrow$) Assume: every discrete fan is a finite set.
Let σ be a fan and $B \subseteq \mathbb{N}^*$ a decidable bar in σ.
We define a function $\sigma' : \mathbb{N}^* \rightarrow \{0, 1\}$ by:

$$\begin{aligned} \sigma'(s) = 0 \quad \leftrightarrows \quad & (\, s \in \sigma \; \wedge \; \forall n \leq lg(s) \, [\, \overline{s} n \notin B \,] \,) \; \vee \\ & (\, s \in \sigma \; \wedge \; \exists n \leq lg(s) \, [\, \overline{s} n \in B \; \wedge \; \forall m < n \, [\, \overline{s} m \notin B \,] \; \wedge \\ & \forall i \in \mathbb{N} \, [\, n < i < lg(s) \; \rightarrow \; s(i) = \mu j \, [\, \sigma(\overline{s} i * \langle j \rangle) = 0 \,] \,] \,] \,). \end{aligned}$$

The idea of σ' is that in the beginning, you put all elements of σ also in σ', but once you are in B, you only take the 'leftmost way up' in the fan σ.

σ' is a fan law, as:

L1) The statement on the right-hand side of the $\leftrightarrows$ - sign is decidable, so σ' is a well-defined function from $\mathbb{N}^*$ to $\{0,1\}$. Clearly $\sigma'(\langle\,\rangle) = 0$.

L2) For any $s \in \mathbb{N}^*$ with $\sigma'(s) = 0$,

$$\sigma'(s * \langle\, \mu j\, [\sigma(\, s * \langle j \rangle) = 0]\, \rangle) = 0.$$

So

$$\forall s \in \mathbb{N}^*\, [\, \sigma'(s) = 0 \;\rightarrow\; \exists n \in \mathbb{N}\, [\, \sigma'(s * \langle n \rangle) = 0\,].$$

The other direction ($\leftarrow$) follows immediately from the way σ' is defined.

L3) From the fact that $\sigma' \subseteq \sigma$ and σ is a fan, it follows that

$$\forall s \in \mathbb{N}^*\, [\, \sigma'(s) = 0 \;\rightarrow\; \exists k \in \mathbb{N}\, \forall n \in \mathbb{N}\, [\, \sigma'(s * \langle n \rangle) = 0 \rightarrow n \leq k\,].$$

Furthermore, σ' is a discrete fan. We prove this as follows.

Let $\alpha, \beta \in \sigma'$.
Determine $k \in \mathbb{N}$ such that $\overline{\alpha}k \in B$.
It follows from the definition of σ' that:

$$\forall \beta \in \sigma'\, [\, \overline{\beta}k = \overline{\alpha}k \;\rightarrow\; \beta = \alpha\,].$$

Hence, by considering $\overline{\beta}k$, we can decide:

$$\alpha = \beta \,\vee\, \alpha \# \beta.$$

As σ' is a discrete fan, we can apply the assumption and conclude that σ' is finite. Say $\sigma' = \{\alpha_0, \ldots, \alpha_p\}$.

Define, for $0 \leq i \leq n$,

$$n_i \underset{\mathrm{D}}{=} \mu j\, [\, \overline{\alpha_i} j \in B\,].$$

And define:

$$B' \underset{\mathrm{D}}{=} \{\overline{\alpha_0}n_0, \ldots, \overline{\alpha_0}n_p\}.$$

Then clearly $B' \subseteq B$.

The only thing left to show is that B' is a bar in σ.

Let $\alpha \in \sigma$.
B is a decidable bar in σ, so we can define $n \underset{\mathrm{D}}{=} \mu i\,[\,\overline{\alpha}i \in B\,]$.
Define $\beta \in \mathcal{N}$ by:

$$\begin{aligned} \beta(k) &\underset{\mathrm{D}}{=} \alpha(k) && \text{if } k \leq n, \\ &\underset{\mathrm{D}}{=} \mu j\,[\,\overline{\beta}k * \langle j \rangle \in \sigma\,] && \text{if } k > n. \end{aligned}$$

Then $\beta \in \sigma'$, so we can determine $i \in \mathbb{N}$ such that $\beta = \alpha_i$.
As $\overline{\alpha_i}n = \overline{\beta}n = \overline{\alpha}n$, it follows from the definition of n that $n = n_i$.
And we conclude

$$\overline{\alpha}n = \overline{\alpha_i}n = \overline{\alpha_i}n_i \in B'.$$

So, for all α in σ, there exists a natural number n such that $\overline{\alpha}n$ is in B'. Hence B' is a finite subbar of B.

$\Leftarrow$) Assume the Fan Theorem.
Let σ be a discrete fan.
According to Lemma 3.2,

$$\forall \alpha \in \sigma\, \exists m \in \mathbb{N}\, \forall \beta \in \sigma\, [\,\overline{\alpha}m = \overline{\beta}m \;\rightarrow\; \alpha = \beta\,].$$

Using the Unrestricted Fan Theorem we can find $M \in \mathbb{N}$ such that

$$\forall \alpha \in \sigma\, \exists m \leq M\, \forall \beta \in \sigma\, [\,\overline{\alpha}m = \overline{\beta}m \;\rightarrow\; \alpha = \beta\,].$$

So

$$\forall \alpha, \beta \in \sigma\, [\,\overline{\alpha}M = \overline{\beta}M \;\rightarrow\; \alpha = \beta\,]. \tag{3.7}$$

As σ is a fan, there are only finitely many $s \in \mathbb{N}^*$ with $lg(s) = M$ and $\sigma(s) = 0$. Combining this with (3.7) gives that σ is finite, as required. □

Chapter 4

On open, closed and enumerable subsets of $\mathcal{N}$

In Chapter 5 we will examine whether there exist choice functions on various collections of subsets of $\mathcal{N}$. The collection of the closed subsets of $\mathcal{N}$ and the collection of the open subsets of $\mathcal{N}$ are two of these collections. Classically, there are a number of equivalent ways to define the notions of 'closed' and 'open'. Intuitionistically, not all those definitions are equivalent. Before we can go into the problem of the existence of choice functions we have to be clear on what collections we are talking about. In this chapter we study the different definitions and their dependencies.

4.1 All kinds of closed and open

In classical mathematics the most commonly used definition of 'open' is the one that states: a subset A of $\mathcal{N}$ is open iff every element of A has a neighbourhood that is contained in A. Considering $\mathcal{N}$ with the metric given by

$$d(\alpha, \beta) \underset{\mathrm{D}}{=} \sum_{n=0}^{\infty} sg(\alpha(n) - \beta(n)) \cdot 2^{-n},$$

where, for every natural number m,

$$sg(m) = 0 \leftrightarrows m = 0 \quad \text{and} \quad sg(m) = 1 \leftrightarrows m \neq 0,$$

this notion of 'open' corresponds with the intuitionistic notion of 'weakly open'. One could also define an open subset of $\mathcal{N}$ as a union of basic open subsets of $\mathcal{N}$, where a subset A of $\mathcal{N}$ is basic open iff there exists $s \in \mathbb{N}^*$ with

$$A = \{\alpha \in \mathcal{N} \mid \overline{\alpha}(lg(s)) = s\}.$$

This definition corresponds with the intuitionistic notion of 'effectively open', provided we start from a decidable collection of basic open sets, i.e. $A \subseteq \mathcal{N}$ is effectively open iff there exists a decidable subset B of $\mathbb{N}^*$ such that

$$A = \bigcup_{s \in B} \{\alpha \in \mathcal{N} \mid \overline{\alpha}(lg(s)) = s\}.$$

A precise definition of these two notions of open is the following.

Definition 4.1. Let $G \subseteq \mathcal{N}$.
We say G is *weakly open* if and only if for every $\alpha \in G$, there exist $n \in \mathbb{N}$, such that every $\gamma \in \mathcal{N}$ passing through $\overline{\alpha}n$ belongs to G.
We say G is *effectively open* if and only if there exists $\beta \in \mathcal{N}$ such that, for all $\alpha \in \mathcal{N}$, α belongs to G if and only if there exists $n \in \mathbb{N}$ with $\overline{\alpha}n \in D_\beta$ (i.e. $\beta(\overline{\alpha}n) = 1$).

In the definition of effectively open we could just as well replace D_β by E_β. To see this, define for each $\beta \in \mathcal{N}$ an element γ_β of $\mathcal{N}$ by:

$$\begin{aligned} \gamma_\beta(n) &\underset{\text{D}}{=} 1 && \text{if } \exists k \leq n\,[\,\beta(k) \neq 0 \wedge \beta(k) - 1 \sqsubseteq n\,], \\ &\underset{\text{D}}{=} 0 && \text{otherwise.} \end{aligned}$$

Then:

$$\begin{aligned} \{\alpha \in \mathcal{N} \mid \exists n\,[\,\overline{\alpha}n \in E_\beta\,]\} &= \{\alpha \in \mathcal{N} \mid \exists n, k\,[\,\beta(k) = \overline{\alpha}n + 1\,]\}, \\ &= \{\alpha \in \mathcal{N} \mid \exists m\,[\,\gamma_\beta(\overline{\alpha}m) = 1\,]\}, \\ &= \{\alpha \in \mathcal{N} \mid \exists m\,[\,\overline{\alpha}m \in D_{\gamma_\beta}\,]\}. \end{aligned}$$

One can easily show that every effectively open subset of $\mathcal{N}$ is also weakly open. The converse is not true intuitionistically. We will show this using the Second Axiom of Continuous Choice ($GAC_{1,1}$).

Theorem 4.2. *Not: every weakly open subset of $\mathcal{N}$ is effectively open.*

Proof. Suppose every weakly open subset of $\mathcal{N}$ is effectively open.
Define, for every $\alpha \in \mathcal{N}$, a subset G_α of $\mathcal{N}$ by

$$G_\alpha \underset{\text{D}}{=} \{\gamma \in \mathcal{N} \mid \forall n\,[\,\alpha(n) = 0\,]\}.$$

G_α is weakly open for each $\alpha \in \mathcal{N}$ (just pick $n = 1$ for every $\gamma \in G_\alpha$). So from the assumption it follows that, for each α in $\mathcal{N}$, G_α is effectively open, i.e.:

$$\forall \alpha {\in} \mathcal{N}\, \exists \beta {\in} \mathcal{N}\, \forall \gamma {\in} \mathcal{N}\,[\,\gamma \in G_\alpha \leftrightarrows \exists n\,[\,\beta(\overline{\gamma}n) = 1\,]\,].$$

According to $GAC_{1,1}$, there exists a function $\delta : \mathcal{N} \to \mathcal{N}$ such that

$$\forall \alpha {\in} \mathcal{N}\, \forall \gamma {\in} \mathcal{N}\,[\,\gamma \in G_\alpha \leftrightarrows \exists n\,[\,(\delta|\alpha)(\overline{\gamma}n) = 1\,]\,].$$

Consider $\alpha = \underline{0}$.
$G_{\underline{0}} = \mathcal{N}$, so in particular $\underline{1} \in G_{\underline{0}}$. Hence,

$$\exists n\,[\,(\delta|\underline{0})(\overline{\underline{1}}n) = 1\,].$$

Determine n such that $(\delta|\underline{0})(\overline{\underline{1}}n) = 1$ and define $m \underset{\text{D}}{=} \overline{\underline{1}}n$.
Determine k such that $\delta^m(\overline{\underline{0}}k) \neq 0$.

By definition, $(\delta|\underline{0})(m) = \delta^m(\underline{0})$.
So, for all $\alpha \in \mathcal{N}$ with $\overline{\alpha}k = \overline{\underline{0}}k$, $\delta^m(\alpha) = \delta^m(\underline{0}) = 1$. Hence,

$$\forall \alpha \in \mathcal{N} [\, \overline{\alpha}k = \overline{\underline{0}}k \;\rightarrow\; \underline{1} \in G_\alpha \,]. \tag{4.1}$$

Define $\alpha \in \mathcal{N}$ by:

$$\begin{aligned} \alpha(n) &\underset{\text{D}}{=} 0 && \text{if } n \leq k, \\ &\underset{\text{D}}{=} 1 && \text{if } n > k. \end{aligned}$$

Then $\overline{\alpha}k = \overline{\underline{0}}k$ and by (4.1), $\underline{1} \in G_\alpha$.
But $\alpha(k+1) = 1$, so $\neg\forall n\,[\,\alpha(n) = 0\,]$ and $G_\alpha = \emptyset$. Contradiction.
We conclude: not every weakly open subset of $\mathcal{N}$ is effectively open. □

Let us move on to the closed subsets of $\mathcal{N}$. To begin with, we give a definition of the notion of the 'complement of a subset of $\mathcal{N}$'.

Definition 4.3. Let $A \subseteq \mathcal{N}$.
The *complement of* A, notation $A^\neg$, is defined by

$$A^\neg \underset{\text{D}}{=} \{\alpha \in \mathcal{N} \mid \alpha \notin A\}.$$

So $A^\neg$ is the set of all α in $\mathcal{N}$ for which the assumption 'α belongs to A' leads to a contradiction. Note that not for every subset A of $\mathcal{N}$, $(A^\neg)^\neg = A$.

Having two different notions of open, the idea: 'a set is closed if and only if it is the complement of an open set', also leads to two notions of 'closed subset of $\mathcal{N}$'. Note that we say 'it is the complement of an open set' and not 'its complement is open'. Classically these two statement are equivalent, but intuitionistically they are not.
We also introduce a third notion of closed, namely 'sequentially closed'.

Definition 4.4. Let $F \subseteq \mathcal{N}$.
The set F is *weakly closed* if and only if there exists a weakly open subset G of $\mathcal{N}$ such that F is the complement of G.
We say F is *effectively closed* if and only if F is the complement of an effectively open set. Or, equivalently, F is effectively closed if and only if there exists $\beta \in \mathcal{N}$ such that, for all $\alpha \in \mathcal{N}$, α belongs to F iff for all $n \in \mathbb{N}$, $\overline{\alpha}n \in D_\beta$.
We call F *sequentially closed* if and only if, for every $\gamma \in \mathcal{N}$, if, for every $n \in \mathbb{N}$, there exists $\alpha \in F$ passing through $\overline{\gamma}n$, then γ itself belongs to F.

We will now investigate how those different notions of being 'closed' are related.

Theorem 4.5. *Every weakly closed subset of $\mathcal{N}$ is sequentially closed.*

Proof. Let F be a weakly closed subset of $\mathcal{N}$.
Suppose $\gamma \in \mathcal{N}$ and $\forall n \in \mathbb{N}\, \exists \alpha \in F\, [\, \overline{\gamma}n = \overline{\alpha}n\,]$. We have to prove: $\gamma \in F$.
As F is weakly closed, we can find a weakly open set G such that $F = G^{\neg}$. Note that:

$$\begin{aligned} \gamma \in F &\leftrightarrows \gamma \notin G, \\ &\leftrightarrows \neg\,(\, \exists n \forall \beta\, [\, \overline{\gamma}n = \overline{\beta}n \,\rightarrow \beta \in G\,]\,), \\ &\leftrightarrows \forall n \neg\,(\, \forall \beta\, [\, \overline{\gamma}n = \overline{\beta}n \rightarrow \beta \in G\,]\,). \end{aligned}$$

Let $n \in \mathbb{N}$.
By assumption, there exists $\alpha \in F$, such that $\overline{\gamma}n = \overline{\alpha}n$.
As $\alpha \in F$, $\neg\,(\, \forall \beta\, [\, \overline{\alpha}n = \overline{\beta}n \rightarrow \beta \in G\,]\,)$. Hence, using $\overline{\gamma}n = \overline{\alpha}n$,

$$\neg\,(\, \forall \beta\, [\, \overline{\gamma}n = \overline{\beta}n \rightarrow \beta \in G\,]\,).$$

This holds for all $n \in \mathbb{N}$, so $\gamma \in F$ and F is sequentially closed. □

The converse is not true constructively: not every sequentially closed subset is weakly closed. Let P be an unsolved mathematical problem and define

$$F \underset{\mathrm{D}}{=} \{\, \alpha \in \mathcal{N} \mid P \,\vee\, \neg P \,\}.$$

F is sequentially closed, but not weakly closed.
For suppose F is weakly closed. Then we can find a weakly open set G such that $F = G^{\neg}$ and, for all $\alpha \in \mathcal{N}$,

$$\neg\neg(\alpha \in F) \;\rightarrow\; \neg\neg(\alpha \notin G).$$

As $\neg\neg(\alpha \notin G)$ is equivalent to $\alpha \notin G$, for all $\alpha \in \mathcal{N}$,

$$\neg\neg(\alpha \in F) \;\rightarrow\; \alpha \in G^{\neg} = F.$$

So if $\neg\neg(P \,\vee\, \neg P)$, then $P \,\vee\, \neg P$.
But $\neg\neg(P \,\vee\, \neg P)$ is true intuitionistically and $P \,\vee\, \neg P$ may be reckless [5].

It is clear that every effectively closed subset of $\mathcal{N}$ is also weakly closed. However, as in the case of the open subsets of $\mathcal{N}$, the converse does not hold. The proof of this theorem resembles the proof of Theorem 4.2.

Theorem 4.6. *Not: every weakly closed subset of $\mathcal{N}$ is effectively closed.*

Proof. Just as in the proof of Theorem 4.2, define, for every $\alpha \in \mathcal{N}$,

$$G_\alpha \underset{\mathrm{D}}{=} \{\gamma \in \mathcal{N} \mid \forall n\, [\, \alpha(n) = 0\,]\}.$$

G_α is weakly open, and therefore, $F_\alpha \underset{\mathrm{D}}{=} (G_\alpha)^{\neg}$ is weakly closed.
Suppose F_α is effectively closed for each α, then:

$$\forall \alpha\, \exists \beta\, \forall \gamma\, [\, \gamma \in F_\alpha \leftrightarrows \forall n\, [\, \beta(\overline{\gamma}n) = 1\,]\,].$$

By $GAC_{1,1}$, there exists a function $\delta : \mathcal{N} \to \mathcal{N}$ such that

$$\forall\alpha\forall\gamma\,[\,\gamma \in F_\alpha \leftrightarrows \forall n\,[\,(\delta|\alpha)(\overline{\gamma}n) = 1\,]\,].$$

Consider $\underline{0}$.
$G_{\underline{0}} = \mathcal{N}$, so $F_{\underline{0}} = \emptyset$ and in particular

$$\underline{1} \notin F_{\underline{0}}. \tag{4.2}$$

Suppose $\exists n\,[\,(\delta|\underline{0})(\overline{\underline{1}}n) \neq 1\,]$.

Find such n_0 and find m such that

$$\forall\alpha\,[\,\overline{\underline{0}}m = \overline{\alpha}m \to (\delta|\alpha)(\overline{\underline{1}}n_0) \neq 1\,]. \tag{4.3}$$

Define $\alpha \in \mathcal{N}$ by

$$\begin{aligned} \alpha(n) &\underset{\text{D}}{=} 0 && \text{if } n \leq m,\\ &\underset{\text{D}}{=} 1 && \text{if } n > m. \end{aligned}$$

Then $\overline{\alpha}m = \overline{\underline{0}}m$ so, by (4.3), $(\delta|\alpha)(\overline{\underline{1}}n_0) \neq 1$ and $\underline{1} \notin F_\alpha$.
But $\alpha(m+1) = 1$, so $\neg\forall n\,[\,\alpha(n) = 0\,]$ and $F_\alpha = \mathcal{N}$. Contradiction.

Hence $\neg\exists n\,[\,(\delta|\underline{0})(\overline{\underline{1}}n) \neq 1\,]$, which means

$$\forall n\,[\,(\delta|\underline{0})(\overline{\underline{1}}n) = 1\,].$$

By definition of δ, $\underline{1} \in F_{\underline{0}}$, contradicting (4.2).
We conclude: not every weakly closed subset of $\mathcal{N}$ is effectively closed. □

We summarize the results on closed subsets of $\mathcal{N}$ as follows:
For every subset A of $\mathcal{N}$,

A is effectively closed$\to$ A is weakly closed $\to$ A is sequentially closed.

In Chapter 2 we have introduced spreads (Definition 2.8). Every spread is sequentially closed. In addition, spreads are located subsets of $\mathcal{N}$. A set A of $\mathcal{N}$ is called *located* if and only if we can decide, for all $s \in \mathbb{N}^*$, whether there exists α in A passing through s or not.

Example 4.7. Consider the subset A of $\mathcal{N}$ defined by:

$$\alpha \in A \leftrightarrows \alpha = \underline{0} \vee (\,\alpha = \underline{1} \wedge k_{99} \text{ does not exist}\,).$$

A is an effectively closed subset of $\mathcal{N}$. For we can define $\beta \in \mathcal{N}$ by:

$$\beta(n) = 1 \leftrightarrows \exists m\,[\,n = \overline{\underline{0}}m \vee (n = \overline{\underline{1}}m \wedge m < k_{99})\,].$$

Then,

$$\alpha \in A \leftrightarrows \forall n\,[\,\beta(\overline{\alpha}n) = 1\,].$$

So in particular A is sequentially closed. However, A is not a spread, because A is not located: we cannot decide whether or not there exists α in A passing through $\langle\,1\,\rangle$.

4.2 Sequentially closed subsets of $\mathcal{N}$

In the previous section we have introduced three notions of 'closed'. In this section we study one of them, namely the notion: 'sequentially closed'. Our aim is to find a way to characterize finitely enumerable and enumerable subsets of $\mathcal{N}$ that are sequentially closed.

Definition 4.8. Let $A \subseteq \mathcal{N}$.
We define the *closure of* A, notation $\overline{A}$, by

$$\overline{A} =_{\mathrm{D}} \{\gamma \in \mathcal{N} \mid \forall n \in \mathbb{N}\, \exists \alpha \in A\, [\overline{\gamma}n = \overline{\alpha}n\}\,].$$

Note that a subset A of $\mathcal{N}$ is sequentially closed if and only if $A = \overline{A}$.

Furthermore, for any subset A of $\mathbb{N}$, $\overline{A}$ is sequentially closed. For let $A \subseteq \mathcal{N}$. It is clear that $\overline{A} \subseteq \overline{\overline{A}}$. Now suppose $\gamma \in \overline{\overline{A}}$.
To prove $\gamma \in \overline{A}$, we have to find, for all $n \in \mathbb{N}$, $\alpha \in A$ with $\overline{\gamma}n = \overline{\alpha}n$.

Let $n \in \mathbb{N}$.
As $\gamma \in \overline{\overline{A}}$, there exists $\beta \in \overline{A}$, with

$$\overline{\gamma}n = \overline{\beta}n. \tag{4.4}$$

As $\beta \in \overline{A}$, there exists $\alpha \in A$, with

$$\overline{\beta}n = \overline{\alpha}n. \tag{4.5}$$

Combining (4.4) and (4.5) yields,

$$\overline{\gamma}n = \overline{\alpha}n.$$

Hence, for all $n \in \mathbb{N}$, there exists $\alpha \in A$ with $\overline{\gamma}n = \overline{\alpha}n$, and therefore, $\gamma \in \overline{A}$.
So $\overline{A} = \overline{\overline{A}}$.

4.2.1 Finitely enumerable subsets of $\mathcal{N}$

First we consider the finitely enumerable subsets of $\mathcal{N}$. A subset A of $\mathcal{N}$ is called *finitely enumerable* iff A is of the form $\{\alpha_0, \ldots, \alpha_k\}$, for certain $k \in \mathbb{N}$ and $\alpha_i \in \mathcal{N}$. Not every finitely enumerable subset of $\mathcal{N}$ is sequentially closed. Define for example $\alpha_0, \alpha_1 \in \mathcal{N}$ by:

$$\alpha_0 =_{\mathrm{D}} \underline{0}.$$

$$\begin{aligned} \alpha_1(n) &=_{\mathrm{D}} 0 && \text{if } n \leq k_{99}, \\ &=_{\mathrm{D}} 1 && \text{if } n > k_{99}. \end{aligned}$$

Clearly $\{\alpha_0, \alpha_1\}$ is finitely enumerable. But this set is not sequentially closed. For we may define $\gamma \in \mathcal{N}$ by

$$\gamma(n) \underset{\text{D}}{=} 0 \qquad \text{if } n \leq k_{99} \vee (n > k_{99} \wedge k_{99} \text{ is odd}),$$
$$\underset{\text{D}}{=} 1 \qquad \text{if } n > k_{99} \wedge k_{99} \text{ is even}.$$

Then, for all $n \in \mathbb{N}$, $\overline{\gamma}n = \overline{\alpha_0}n \vee \overline{\gamma}n = \overline{\alpha_1}n$, so $\gamma \in \overline{\{\alpha_0, \alpha_1\}}$.
Note that:

$$\gamma = \alpha_0 \quad \leftrightarrows \quad \forall n \in \mathbb{N}\,[\,n = k_{99} \rightarrow n \text{ is even}\,],$$
$$\gamma = \alpha_1 \quad \leftrightarrows \quad \forall n \in \mathbb{N}\,[\,n = k_{99} \rightarrow n \text{ is odd}\,].$$

Therefore, we cannot decide $\gamma = \alpha_0 \vee \gamma = \alpha_1$. Hence we have no reason to affirm $\gamma \in \{\alpha_0, \alpha_1\}$ and $\{\alpha_0, \alpha_1\}$ is not sequentially closed.

To characterize the finitely enumerable sequentially closed subsets of $\mathcal{N}$ we first show that these are spreads. This allows us to use the Continuity Principle to prove any further properties of these subsets of $\mathcal{N}$.

Lemma 4.9. *For all $k \in \mathbb{N}$, for all $\alpha_0, \ldots, \alpha_k \in \mathcal{N}$,*

$\overline{\{\alpha_0, \ldots, \alpha_k\}}$ *is a spread.*

Proof. Let $k \in \mathbb{N}$ and $\alpha_0, \ldots, \alpha_k \in \mathcal{N}$.
Define $\sigma : \mathbb{N}^* \rightarrow \{0, 1\}$ by, for all $s \in \mathbb{N}^*$,

$$\sigma(s) = 0 \leftrightarrows \exists i \leq k\,[\,s = \overline{\alpha_i}(lg(s))\,].$$

The statement on the right-hand side is decidable so σ is a well-defined function. It is clear that σ is a spread law (defintion 2.8).
Note that:

$$\alpha \in \sigma \leftrightarrows \forall n\,[\,\sigma(\overline{\alpha}n) = 0\,],$$
$$\leftrightarrows \forall n\,\exists i[\,\overline{\alpha}n = \overline{\alpha_i}n\,],$$
$$\leftrightarrows \alpha \in \overline{\{\alpha_0, \ldots, \alpha_k\}}.$$

So $\sigma = \overline{\{\alpha_0, \ldots, \alpha_k\}}$ and $\overline{\{\alpha_0, \ldots, \alpha_k\}}$ is a spread. □

A subset A of $\mathcal{N}$ is called *finite* iff there exist $k \in \mathbb{N}$ and $\alpha_0, \ldots, \alpha_k \in \mathcal{N}$ such that $A = \{\alpha_0, \ldots, \alpha_k\}$ and in addition, for all $i < j \leq k$, $\alpha_i \mathrel{\#} \alpha_j$. In the next theorem we prove that a finitely enumerable subset of $\mathcal{N}$ is sequentially closed if and only if it is finite.

Theorem 4.10. *For all $k \in \mathbb{N}$, for all $\alpha_0, \ldots, \alpha_k \in \mathcal{N}$,*

$$\overline{\{\alpha_0, \ldots, \alpha_k\}} = \{\alpha_0, \ldots, \alpha_k\} \Leftrightarrow \forall i, j \leq k\,[\,\alpha_i = \alpha_j \vee \alpha_i \# \alpha_j\,].$$

Proof. Let $k \in \mathbb{N}$.
$\Leftarrow$) Suppose $\forall i, j \leq k\,[\,\alpha_i = \alpha_j \vee \alpha_i \# \alpha_j\,]$.
It is clear that $\{\alpha_0, \ldots, \alpha_k\} \subseteq \overline{\{\alpha_0, \ldots, \alpha_k\}}$.

Let $\gamma \in \overline{\{\alpha_0, \ldots, \alpha_k\}}$.
We can assume, for all $i < j \leq k$, $\alpha_i \# \alpha_j$.
Define

$$M \underset{\mathrm{D}}{=} max(\,\{\,\mu p\,[\,\overline{\alpha_i}p \neq \overline{\alpha_j}p\,]\,|\,i < j \leq k\,\}\,).$$

Then

$$\forall i, j \leq k\,[\,i \neq j \rightarrow \overline{\alpha_i}M \neq \overline{\alpha_j}M\,].$$

As $\gamma \in \overline{\{\alpha_0, \ldots, \alpha_k\}}$, by definition

$$\forall n \exists i \leq k\,[\,\overline{\gamma}n = \overline{\alpha_i}n\,].$$

Find i such that $\overline{\gamma}M = \overline{\alpha_i}M$.
Then $\gamma = \alpha_i \in \{\alpha_0, \ldots, \alpha_k\}$.
So $\overline{\{\alpha_0, \ldots, \alpha_k\}} \subseteq \{\alpha_0, \ldots, \alpha_k\}$.

$\Rightarrow$) Suppose $\overline{\{\alpha_0, \ldots, \alpha_k\}} = \{\alpha_0, \ldots, \alpha_k\}$.
Then

$$\forall \gamma \in \overline{\{\alpha_0, \ldots, \alpha_k\}}\, \exists i \leq k\,[\,\gamma = \alpha_i\,].$$

As $\overline{\{\alpha_0, \ldots, \alpha_k\}}$ is a spread (Lemma 4.9), we can apply the Continuity Principle.
This yields

$$\forall \gamma \in \overline{\{\alpha_0, \ldots, \alpha_k\}}\, \exists n, i\, \forall \beta \in \overline{\{\alpha_0, \ldots, \alpha_k\}}\,[\,\overline{\gamma}n = \overline{\beta}n \rightarrow \beta = \alpha_i\,]. \tag{4.6}$$

Now let $i, j \leq k$.
We have to show: $\alpha_i = \alpha_j \vee \alpha_i \# \alpha_j$.
As $\alpha_i \in \overline{\{\alpha_0, \ldots, \alpha_k\}}$, we can find, by applying (4.6), $n \in \mathbb{N}$ such that

$$\forall \beta \in \overline{\{\alpha_0, \ldots, \alpha_k\}}\,[\,\overline{\alpha_i}n = \overline{\beta}n \rightarrow \beta = \alpha_i\,].$$

We distinguish two cases:

1. $\overline{\alpha_i}n \neq \overline{\alpha_j}n$.
   Then $\alpha_i \# \alpha_j$.

2. $\overline{\alpha_i}n = \overline{\alpha_j}n$.
   Note that $\alpha_j \in \overline{\{\alpha_0, \ldots, \alpha_k\}}$, and therefore, $\alpha_i = \alpha_j$.

So we can decide: $\alpha_i = \alpha_j \vee \alpha_i \# \alpha_j$, as required. □

4.2.2 Enumerable subsets of $\mathcal{N}$

In the previous sections we have found a way to characterize finitely enumerable subsets A of $\mathcal{N}$ with the property $\overline{A} = A$. This makes us wonder whether we can find such a characterization for enumerable subsets A of $\mathcal{N}$ as well. The Continuity Principle played a crucial role in the proof of the finitely enumerable case. We were allowed to use this principle, because $\overline{A}$ is a spread for every finitely enumerable subset A of $\mathcal{N}$. Unfortunately, this is not in general the case for enumerable subsets of $\mathcal{N}$.

Example 4.11. Consider the set $\{\alpha_0, \alpha_1, \ldots\}$, defined by

$$\alpha_n \underset{\mathrm{D}}{=} \underline{0} \qquad \text{if } n < k_{99},$$
$$\underset{\mathrm{D}}{=} \underline{1} \qquad \text{if } n \geq k_{99}.$$

Suppose $\overline{\{\alpha_0, \alpha_1, \ldots\}}$ is a spread, say σ.

Then we are able to decide

$$\sigma(\langle 1 \rangle) = 0 \ \vee \ \sigma(\langle 1 \rangle) \neq 0.$$

Note that

$$\sigma(\langle 1 \rangle) = 0 \ \leftrightarrows \ k_{99} \text{ exists.}$$

Hence we can decide

$$k_{99} \text{ exists } \vee \ k_{99} \text{ does not exist.}$$

But this statement is reckless.

This is a Brouwerian counterexample. Using the Continuity Principle, one can obtain a contradiction from the statement: for every $\alpha \in \mathcal{N}$, the set $\overline{\{\alpha^0,\, \alpha^1,\, \ldots\}}$ is a spread.
In the example above $\overline{\{\alpha_0, \alpha_1, \ldots\}} = \{\alpha_0, \alpha_1, \ldots\}$. For suppose $\gamma \in \overline{\{\alpha_0, \alpha_1, \ldots\}}$. Then we can determine $i \in \mathbb{N}$ such that $\overline{\gamma}1 = \overline{\alpha_i}1$, whence $\gamma = \alpha_i \in \{\alpha_0, \alpha_1, \ldots\}$.

Definition 4.12. Let $X \subseteq \mathcal{N}$.
X is called *strictly analytic* iff there exists a continuous function $f : \mathcal{N} \to \mathcal{N}$ such that:

$$X = \emptyset \ \vee \ X = Ran(f).$$

We will see that the set $\overline{\{\alpha^0,\, \alpha^1, \ldots\}}$ is strictly analytic for each $\alpha \in \mathcal{N}$. This allows us to use the Continuity Principle nevertheless when proving properties of the enumerable sequentially closed subsets of $\mathcal{N}$.

Lemma 4.13. *Let $\alpha \in \mathcal{N}$ and define $A \underset{D}{=} \{\alpha^0, \alpha^1, \ldots\}$.*
If A is sequentially closed, i.e. $A = \overline{A}$, then, for all $\gamma_0 \in \overline{A}$, there exists $n \in \mathbb{N}$, such that, for all $\gamma_1 \in \overline{A}$ with $\overline{\gamma_0}n = \overline{\gamma_1}n$, $\gamma_0 = \gamma_1$.

Proof. First we show $\overline{A}$ is strictly analytic by defining a continuous function $f : \mathcal{N} \to \mathcal{N}$ with $\overline{A} = Ran(f)$. Start from the observation that for every $\gamma \in \mathcal{N}$,

$$\gamma \in \overline{A} \leftrightarrows \forall n \, \exists m \, [\, \overline{\gamma}n = \overline{\alpha^m}n \,].$$

Hence, by applying the Minimal Axiom of Choice [1].

$$\gamma \in \overline{A} \leftrightarrows \exists \beta \in \mathcal{N} \, \forall n \, [\, \overline{\gamma}n = \overline{\alpha^{\beta(n)}}n \,].$$

We define the function f as follows:

$$\begin{aligned} f(\beta)(n) &\underset{D}{=} \alpha^{\beta(n)}(n) && \text{if } \forall k \leq n \, [\, \overline{f(\beta)}k = \overline{\alpha^{\beta(k)}}k \,], \\ &\underset{D}{=} \alpha^{\beta(m)}(n) && \text{if } \exists k \leq n \, [\, \overline{f(\beta)}k \neq \overline{\alpha^{\beta(k)}}k \,] \\ & && \text{and } m = \mu i \, [\, \overline{f(\beta)}(i+1) \neq \overline{\alpha^{\beta(i+1)}}(i+1) \,]. \end{aligned}$$

We claim: $\overline{A} = Ran(f)$. One proves this as follows.

$\subseteq$) Let $\gamma \in \overline{A}$.
Then, by definition,

$$\forall n \, \exists m \, [\, \overline{\gamma}n = \overline{\alpha^m}n \,].$$

According to the Minimal Axiom of Choice there exists $\beta \in \mathcal{N}$ such that

$$\forall n \, [\, \overline{\gamma}n = \overline{\alpha^{\beta(n)}}n \,].$$

It follows from the definition of f that $\gamma = f(\beta) \in Ran(f)$.

$\supseteq$) Let $\gamma \in Ran(f)$.
Find β such that $\gamma = f(\beta)$.
One can prove that $\forall n \, \exists m \, [\, \overline{f(\beta)}n = \overline{\alpha^m}n \,]$.
Hence $\gamma \in \overline{A}$.

It is clear that f is continuous as

$$\forall \beta_0, \beta_1 \in \mathcal{N} \, [\, \overline{\beta_0}n = \overline{\beta_1}n \, \to \, \overline{f(\beta_0)}n = \overline{f(\beta_1)}n \,]. \tag{4.7}$$

Now we are ready to prove the lemma.
Assume $A = \overline{A}$ and let f be as defined above.
As $Ran(f) = \overline{A} = A$,

$$\forall \alpha \in \mathcal{N} \, \exists i \in \mathbb{N} \, [\, f(\alpha) = \alpha_i \,].$$

[1] The Minimal Axiom of Choice is $AC_{0,0}$ as defined on page 15, but with the extra condition that $P \subseteq \mathbb{N} \times \mathbb{N}$ be decidable. In that case, one may define $\alpha \in \mathcal{N}$ by $\alpha(n) \underset{D}{=} \mu m \, [\, P(n, m) \,]$.

Applying CP yields

$$\forall\alpha\in\mathcal{N}\,\exists m, i\in\mathbb{N}\,\forall\beta\in\mathcal{N}\,[\,\overline{\alpha}m = \overline{\beta}m \;\rightarrow\; f(\beta) = \alpha_i\,].$$

Let $\gamma_0\in\overline{A}$ and determine $\alpha\in\mathcal{N}$ such that $\gamma_0 = f(\alpha)$.
Determine natural numbers m and i such that

$$\forall\beta\in\mathcal{N}\,[\,\overline{\alpha}m = \overline{\beta}m \;\rightarrow\; f(\beta) = \alpha_i\,].$$

Then in particular $\gamma_0 = f(\alpha) = \alpha_i$, and therefore,

$$\forall\beta\in\mathcal{N}\,[\,\overline{\alpha}m = \overline{\beta}m \;\rightarrow\; f(\beta) = \gamma_0\,]. \tag{4.8}$$

Let $\gamma_1 \in \overline{A}$ with $\overline{\gamma_0}m = \overline{\gamma_1}m$.
We have to show: $\gamma_0 = \gamma_1$

Determine $\beta \in \mathcal{N}$ such that $\gamma_1 = f(\beta)$ and define $\delta \in \mathcal{N}$ by

$$\begin{aligned} \delta(k) &\underset{\text{D}}{=} \alpha(k) && \text{if}\, k < m,\\ &\underset{\text{D}}{=} \beta(k) && \text{if}\, k \geq m. \end{aligned}$$

$\overline{\delta}m = \overline{\alpha}m$, so using (4.7)

$$\overline{f(\delta)}m = \overline{f(\alpha)}m = \overline{\gamma_0}m = \overline{\gamma_1}m = \overline{f(\beta)}m.$$

From place m onwards δ and β are the same. So by the way f is defined we see

$$f(\delta) = f(\beta) = \gamma_1. \tag{4.9}$$

On the other hand $\overline{\delta}m = \overline{\alpha}m$, so by (4.8)

$$f(\delta) = f(\alpha) = \gamma_0. \tag{4.10}$$

Combining (4.9) and (4.10), we conclude $\gamma_0 = \gamma_1$. $\square$

For every subset A of $\mathcal{N}$, an element α of A is called an *isolated point of* A iff there exists a natural number n such that all elements of A that start with $\overline{\alpha}n$ are equal to α. Lemma 4.13 shows that intuitionistically all elements of a sequentially closed subset of $\mathcal{N}$ are isolated. This is not the case classically. Consider for example $A \subseteq \mathcal{N}$ defined by

$$A \underset{\text{D}}{=} \{\,\underline{0},\ \langle 1\rangle * \underline{0},\ \langle 0,1\rangle * \underline{0},\ \langle 0,0,1\rangle * \underline{0},\ \ldots\,\}.$$

From a classical point of view this set is sequentially closed, but $\underline{0}$ is not isolated in A as there is no natural number n such that every element of A that starts with $\overline{0}n$ is equal to $\underline{0}$.
Note that constructively A is not sequentially closed. We could define $\gamma \in \mathcal{N}$ by, for all $n \in \mathbb{N}$,

$$\begin{aligned} \gamma(n) &\underset{\text{D}}{=} 0 && \text{if } n \neq k_{99},\\ &\underset{\text{D}}{=} 1 && \text{if } n = k_{99}. \end{aligned}$$

Then $\gamma \in \overline{A}$, but it is reckless to say $\gamma \in A$.

Corollary 4.14. *For all $\alpha \in \mathcal{N}$,*

$$\overline{\{\alpha^0, \alpha^1, \ldots\}} = \{\alpha^0, \alpha^1, \ldots\} \Rightarrow \forall i, j\, [\, \alpha^i = \alpha^j \vee \alpha^i \# \alpha^j \,].$$

Unfortunately, the converse of Corollary 4.14 does not hold. Define for example $\alpha \in \mathcal{N}$ such that, for each $n \in \mathbb{N}$,

$$\begin{aligned} \alpha^n(k) &= 1 && \text{if } k = n, \\ &= 0 && \text{if } k \neq n. \end{aligned}$$

Then

$$A = \{\alpha^0, \alpha^1, \ldots\} = \{\langle 1 \rangle * \underline{0},\ \langle 0, 1 \rangle * \underline{0},\ \langle 0, 0, 1 \rangle * \underline{0},\ \ldots\}.$$

Clearly $\forall i, j\, [\, \alpha^i = \alpha^j \vee \alpha^i \# \alpha^j \,]$.
But $\overline{A} \neq A$, as $\underline{0} \in \overline{A}$ and $\underline{0} \notin A$.

So we need to add some extra conditions to characterize $\alpha \in \mathcal{N}$ with the property $\overline{\{\alpha^0, \alpha^1, \ldots\}} = \{\alpha^0, \alpha^1, \ldots\}$. We start with some definitions.

Definition 4.15.
We define the *successor function* $S : \mathbb{N} \to \mathbb{N}$ by,

$$S(n) \underset{\text{D}}{=} n + 1 \qquad \text{for all } n \in \mathbb{N}.$$

For all $\alpha, \beta \in \mathcal{N}$, we define the *composition of* α *and* β, notation $\alpha \circ \beta$, by

$$\alpha \circ \beta(n) \underset{\text{D}}{=} \alpha(\beta(n)) \qquad \text{for all } n \in \mathbb{N}.$$

For all $m \in \mathbb{N}$, we write $S^{(m)}$ for $\underbrace{S \circ \ldots \circ S}_{m\, times}$ and note

$$S^{(m)}(n) = n + m \qquad \text{for all } n \in \mathbb{N}.$$

Combining all these definitions we see that, for every $\alpha \in \mathcal{N}$, for every $m \in \mathbb{N}$, the element $\alpha \circ S^m$ of $\mathcal{N}$ is given by

$$(\alpha \circ S^m)(n) = \alpha(n + m) \qquad \text{for all } n \in \mathbb{N}.$$

So we obtain the infinite sequence $\alpha \circ S^m$ by deleting the first m values of the infinite sequence α.

Definition 4.16. Let $s, t \in \mathbb{N}$.
We say s and t are *incompatible*, notation $s \perp t$, if and only if

$$\neg(s \sqsubseteq t) \quad \text{and} \quad \neg(t \sqsubseteq s).$$

Note that any infinite sequence of natural numbers passing through t does not pass through s and the other way around.

Theorem 4.17. *For every* $\alpha \in \mathcal{N}$,

$\overline{\{\alpha^0, \alpha^1, \ldots\}} = \{\alpha^0, \alpha^1, \ldots\} \quad \Leftrightarrow \quad$ *There exist* $s_0, s_1, \ldots \in \mathbb{N}^*$ *and* $\beta_0, \beta_1, \ldots \in \mathcal{N}$ *such that*

(i) $\{\alpha^0, \alpha^1, \ldots\} = \{s_0 * \beta_0, s_1 * \beta_1, \ldots\}$,

(ii) $\forall i, j \in \mathbb{N}\,[\, i \neq j \rightarrow s_i \perp s_j \,]$,

(iii) $\forall \gamma \in \overline{\{\alpha^0, \alpha^1, \ldots\}}\, \exists i \in \mathbb{N}\,[\, \overline{\gamma}(lg(s_i)) = s_i \,]$.

Proof. Let $\alpha \in \mathcal{N}$ and define $A \underset{\text{D}}{=} \{\alpha^0, \alpha^1, \ldots\}$.
$\Rightarrow$) Assume $\overline{A} = A$.
We define $s_0, s_1, \ldots \in \mathbb{N}^*$ and $\beta_0, \beta_1, \ldots \in \mathcal{N}$ step by step. While defining these sequences we also define a third sequence $m_0, m_1, \ldots \in \mathbb{N}$, which we use later on to show that the first two sequences have the required properties.

Step 1 According to Lemma 4.13 we can determine $m_0 \in \mathbb{N}$ such that

$$\forall \gamma \in \overline{A}\,[\, \overline{\gamma} m_0 = \overline{\alpha^0} m_0 \;\rightarrow\; \gamma = \alpha^0 \,].$$

Define $s_0 \underset{\text{D}}{=} \overline{\alpha^0} m_0$ and $\beta_0 \underset{\text{D}}{=} \alpha^0 \circ S^{m_0}$.

Step n+1 Suppose we have already defined $s_0, \ldots s_k$ and $\beta_0, \ldots \beta_k$.
Consider α^{n+1}.
By Corollary 4.14, $\forall i\,[\, \alpha^i = \alpha^{n+1} \vee \alpha^i \# \alpha^{n+1} \,]$.
Distinguish two cases:

1. $\exists i < n+1\,[\, \alpha^i = \alpha^{n+1} \,]$.
   Define $m_{n+1} \underset{\text{D}}{=} 0$ and continue with step $n+2$.
2. $\forall i < n+1\,[\, \alpha^i \# \alpha^{n+1} \,]$.
   Determine m_{n+1} such that

$$\forall \gamma \in \overline{A}\,[\, \overline{\gamma} m_{n+1} = \overline{\alpha^{n+1}} m_{n+1} \;\rightarrow\; \gamma = \alpha^{n+1} \,].$$

   Define $s_{k+1} \underset{\text{D}}{=} \overline{\alpha^{n+1}} m_{n+1}$ and $\beta_{k+1} \underset{\text{D}}{=} \alpha^{n+1} \circ S^{m_{n+1}}$.

Now we have to show $s_0, s_1 \ldots$ and $\beta_0, \beta_1 \ldots$ have properties (i), (ii) and (iii).

(i). $\{\alpha^0, \alpha^1, \ldots\} = \{s_0 * \beta_0, s_1 * \beta_1, \ldots\}$.

$\subseteq$) We will prove this with the principle of course-of-values-induction.
Let $n \in \mathbb{N}$ and assume for all $i < n$, $\alpha^i \in \{s_0 * \beta_0, s_1 * \beta_1, \ldots\}$.

Case 1: $\exists i < n\,[\, \alpha^i = \alpha^n \,]$.
Determine such i.
As $i < n$ we can apply the induction hypothesis and find $k \in \mathbb{N}$ with $\alpha^i = s_k * \beta_k$.
So $\alpha^n = \alpha^i = s_k * \beta_k \in \{s_0 * \beta_0, s_1 * \beta_1, \ldots\}$.

Case 2: $\forall i < n\,[\,\alpha^i \# \alpha^n\,]$.

This means that in the construction of $s_0, s_1 \ldots$ and $\beta_0, \beta_1 \ldots$ new s_k and β_k are added to the list in step n.
Determine k such that s_k, β_k are defined in step n.
Then $s_k = \overline{\alpha^n} m_n$ and $\beta_k = \alpha^n \circ S^{m_n}$.
Hence $\alpha_n = \beta_k * s_k \in \{s_0 * \beta_0, s_1 * \beta_1, \ldots\}$.

$\supseteq$) Let $k \in \mathbb{N}$.
We can determine $n \in \mathbb{N}$ such that $s_k = \overline{\alpha^n} m_n$ and $\beta_k = \alpha^n \circ S^{m_n}$.
So $s_k * \beta_k = \overline{\alpha^n} m_n * (\alpha^n \circ S^{m_n}) = \alpha^n \in \{\alpha^0, \alpha^1, \ldots\}$.

(ii). $\forall i, j\,[\,i \neq j \;\rightarrow\; s_i \perp s_j\,]$.
Let $i, j \in \mathbb{N}$ with $i \neq j$. We will prove $s_i \perp s_j$.
Determine $p, q \in \mathbb{N}$ such that s_i is constructed in step p and s_j is constructed in step q. It follows from the way these are constructed that

$$\alpha^p \# \alpha^q. \tag{4.11}$$

Suppose $s_i \sqsubseteq s_j$.

Then $m_p \leq m_q$ and therefore, as $\overline{\alpha^q} m_q = s_j$,

$$\overline{\alpha^q} m_p = s_i = \overline{\alpha^p} m_p.$$

By definition of m_p,

$$\forall \gamma \in \overline{A}\,[\,\overline{\gamma} m_p = \overline{\alpha^p} m_p \;\rightarrow\; \gamma = \alpha^p\,].$$

So in particular, as $\alpha^q \in \overline{A}$ and $\overline{\alpha^q} m_p = s_i = \overline{\alpha^p} m_p$,

$$\alpha^q = \alpha^p.$$

Contradicting (4.11).

Hence $s_i \not\sqsubseteq s_j$.
In a similar way we see that $s_j \not\sqsubseteq s_i$ and we conclude $s_i \perp s_j$.

(iii). $\forall \gamma \in \overline{\{\alpha^0, \alpha^1, \ldots\}}\, \exists i\,[\,\overline{\gamma}(lg(s_i)) = s_i\,]$.
Let $\gamma \in \overline{\{\alpha^0, \alpha^1, \ldots\}}$.
We have to find $i \in \mathbb{N}$ with $\overline{\gamma}(lg(s_i)) = s_i$.
As

$$\overline{\{\alpha^0, \alpha^1, \ldots\}} = \{\alpha^0, \alpha^1, \ldots\} = \{s_0 * \beta_0, s_1 * \beta_1, \ldots\},$$

we see that

$$\gamma \in \{s_0 * \beta_0, s_1 * \beta_1, \ldots\}.$$

Determine i such that $\gamma = s_i * \beta_i$.
Then $\overline{\gamma}(lg(s_i)) = s_i$.

$\Leftarrow$) Suppose there exist $s_0, s_1, \ldots \in \mathbb{N}^*$ and $\beta_0, \beta_1, \ldots \in \mathcal{N}$ satisfying requirements (i), (ii) and (iii).

It is clear that $A \subseteq \overline{A}$.

Let $\gamma \in \overline{A}$.
By property (iii) we can determine $i \in \mathbb{N}$ with $\overline{\gamma}(lg(s_i)) = s_i$.
Using property (ii) we see that, for all $j \in \mathbb{N}$ with $i \neq j$,

$$\overline{\gamma}(lg(s_i)) \neq \overline{s_j * \beta_j}(lg(s_i)). \tag{4.12}$$

We have assumed $\gamma \in \overline{A} = \overline{\{s_0 * \beta_0, s_1 * \beta_1, \ldots\}}$, and therefore,

$$\forall n \in \mathbb{N}\, \exists j \in \mathbb{N}\, [\, \overline{\gamma}n = \overline{s_j * \beta_j}n\,]. \tag{4.13}$$

Combining (4.12) and (4.13) yields

$$\forall n \in \mathbb{N}\, [\, \overline{\gamma}n = \overline{s_i * \beta_i}n\,].$$

So $\gamma = s_i * \beta_i \in \{s_0 * \beta_0, s_1 * \beta_1, \ldots\} = \{\alpha^0, \alpha^1, \ldots\}$. $\square$

Chapter 5

On the existence and non-existence of certain choice functions

In this chapter we study the following version of the Axiom of Choice, formulated by Zermelo in 1908:

> for every family $\mathcal{F}$ of non-empty sets, there exists a function $f : \mathcal{F} \to \bigcup \mathcal{F}$ such that for each $S \in \mathcal{F} : f(S) \in S$.

Given a family $\mathcal{F}$, a function satisfying the above condition is called *a choice function on $\mathcal{F}$*. We will consider various families of non-empty sets and investigate whether one can define, in an effective way, a choice function on the family.

5.1 Choice functions on collections of subsets of $\mathbb{N}$

The Axiom of Choice deals with collections of non-empty sets. Intuitionistically a subset A of $\mathbb{N}$ is called *not empty* iff $\forall n\,[\,n \notin A\,]$. Note that this is equivalent to $\neg\neg\exists n\,[\,n \in A\,]$. We want to define a stronger notion.

Definition 5.1. Let $A \subseteq \mathbb{N}$.
A is *inhabited* if and only if there exists a natural number n with $n \in A$.

A subset A of $\mathbb{N}$ may be not empty while we are unable to prove A is inhabited. For example, let P be a statement we can, at the moment, not prove nor reject (for example the Riemann Hypothesis) and define:

$$A \underset{\mathrm{D}}{=} \{n \in \mathbb{N} \mid (n = 0 \wedge P) \vee (n = 1 \wedge \neg P)\}.$$

As we can prove $\neg\neg(P \vee \neg P)$, we have $\neg\neg(0 \in A \vee 1 \in A)$, which means A is not empty. But we cannot prove $0 \in A$, nor $1 \in A$, so A is not inhabited.

In the special case of decidable subsets of $\mathbb{N}$, for any $\alpha \in \mathcal{N}$, stating D_α is inhabited and D_α is not empty mean, respectively,

$$\exists n\,[\,\alpha(n) = 1\,] \quad \text{and} \quad \neg\neg\exists n \in \mathbb{N}\,[\,\alpha(n) = 1\,].$$

Markov's Principle says these two statements are equally strong. This principle is accepted in some branches of constructive mathematics. However, using the Brouwer-Kripke-axiom one can construct a sequence $\alpha(0)$, $\alpha(1)$, ... such that the second statement is intuitionistically true, but the first one is reckless [4]. We will only consider collections of inhabited subsets of $\mathbb{N}$. In the remainder of the thesis we will use both the term 'non-empty' and the term 'inhabited' for the notion defined in 5.1.

Theorem 5.2. *Define* $A \underset{\text{D}}{=} \{\alpha \in \mathcal{N} \mid D_\alpha$ *is inhabited*$\}$.
There exists a function $f : A \to \mathbb{N}$ *such that*

$$\forall \alpha \in A\,[\,\alpha(f(\alpha)) = 1\,].$$

Proof. Let $\alpha \in A$ and define:

$$f(\alpha) \underset{\text{D}}{=} \mu i\,[\,\alpha(i) = 1\,].$$

As $\alpha \in A$ we know D_α is inhabited, so f is well-defined.
Clearly $\alpha(f(\alpha)) = 1$. □

Apparently, we can define a choice function on the collection of inhabited decidable subsets of $\mathbb{N}$. Now let us consider the collection of inhabitedly enumerable subsets of $\mathbb{N}$. These are defined as follows.

Definition 5.3. For any $\alpha \in \mathcal{N}$ we define

$$IE_\alpha \underset{\text{D}}{=} \{\alpha(n) \mid n \in \mathbb{N}\}.$$

Let $X \subseteq \mathbb{N}$.
We say that α *inhabitedly enumerates* X iff X coincides with IE_α.
X is an *inhabitedly enumerable subset* of $\mathbb{N}$ iff some element α of $\mathcal{N}$ inhabitedly enumerates X.

Two different elements of $\mathcal{N}$ may inhabitedly enumerate the same subset of $\mathbb{N}$. Consider, for example, $\alpha \underset{\text{D}}{=} (1, 2, 2, 2, 2, \ldots)$ and $\beta \underset{\text{D}}{=} (2, 1, 1, 1, 1, \ldots)$.

$$IE_\alpha = \{1, 2\} = IE_\beta,$$

but $\alpha \neq \beta$.
When defining a choice function on the collection of inhabitedly enumerable subsets of $\mathcal{N}$ using this parametrization, we of course want the choice function to assign the same value to the above defined α and β. And in general, we want, for all α, β in $\mathcal{N}$ with $IE_\alpha = IE_\beta$, $f(\alpha) = f(\beta)$. Unfortunately, this is impossible, as the following theorem points out.

Theorem 5.4. *There is no function $f : \mathcal{N} \to \mathbb{N}$ such that*

(i). $\forall \alpha \in \mathcal{N} \, \exists n \in \mathbb{N} \, [\, f(\alpha) = \alpha(n) \,]$,

(ii). $\forall \alpha, \beta \in \mathcal{N} \, [\, IE_\alpha = IE_\beta \; \to \; f(\alpha) = f(\beta) \,]$.

Proof. Suppose $f : \mathcal{N} \to \mathbb{N}$ satisfies these requirements.
Applying the Continuity Principle yields

$$\forall \alpha \in \mathcal{N} \, \exists m \, \forall \beta \in \mathcal{N} \, [\, \overline{\alpha} m = \overline{\beta} m \; \to \; f(\alpha) = f(\beta) \,].$$

Note that necessarily $f(\underline{0}) = 0$ and $f(\underline{1}) = 1$.
Find m_0, m_1 such that

$$\forall \beta \in \mathcal{N} \, [\, \overline{\underline{0}} m_0 = \overline{\beta} m_0 \; \to \; f(\beta) = 0 \,]. \tag{5.1}$$

$$\forall \beta \in \mathcal{N} \, [\, \overline{\underline{1}} m_1 = \overline{\beta} m_1 \; \to \; f(\beta) = 1 \,]. \tag{5.2}$$

Define $\beta_0, \beta_1 \in \mathcal{N}$ by

$$\begin{aligned} \beta_0(n) &\underset{\mathrm{D}}{=} 0 && \text{if } n \le m_0, \\ &\underset{\mathrm{D}}{=} 1 && \text{if } n > m_0. \end{aligned}$$

$$\begin{aligned} \beta_1(n) &\underset{\mathrm{D}}{=} 1 && \text{if } n \le m_1, \\ &\underset{\mathrm{D}}{=} 0 && \text{if } n > m_1. \end{aligned}$$

Then

$$IE_{\beta_0} = IE_{\beta_1}.$$

But by (5.1) and (5.2)

$$f(\beta_0) = f(\underline{0}) = 0 \quad \text{and} \quad f(\beta_1) = f(\underline{1}) = 1.$$

Contradiction. So there is no such function. □

Definition 5.5. Let $\alpha \in \mathcal{N}$ and $k \in \mathbb{N}$. Define:

$$\begin{aligned} \gamma_\alpha^k(n) &\underset{\mathrm{D}}{=} \alpha(0) && \text{if } \exists n_0, \ldots, n_{k-1} < n \, \forall i, j < k \, [\, i \ne j \; \to \; \alpha(n_i) \ne \alpha(n_j) \,], \\ &\underset{\mathrm{D}}{=} \alpha(n) && \text{otherwise.} \end{aligned}$$

For each $\alpha \in \mathcal{N}$, we define $IE_\alpha^k \underset{\mathrm{D}}{=} IE_{\gamma_\alpha^k}$.
The set $\{ IE_\alpha^k \mid \alpha \in \mathcal{N} \}$ is the collection of all inhabitedly enumerable subsets of the natural numbers with at most k elements.

Note that we cannot even define a choice function on the collection of inhabitedly enumerable subsets of $\mathbb{N}$ with at most two elements. We can just copy the proof of Theorem 5.4 replacing IE_α by IE_α^2.

5.2 Choice functions on collections of subsets of $\mathcal{C}$

We move from families of subsets of $\mathbb{N}$ to families of subsets of $\mathcal{C}$. To be able to define a choice function on a collection $\mathcal{F}$ of non-empty subsets of $\mathcal{C}$ we want a way to parametrize $\mathcal{F}$.

Definition 5.6. Let X be a set and $\mathcal{F}$ a collection of subsets of X.
A *$\mathcal{C}$-parametrization* of $\mathcal{F}$ is a surjective map $V : \mathcal{C} \to \mathcal{F}$, that is, a map V such that $\mathcal{F} = \{V(\alpha) \mid \alpha \in \mathcal{C}\}$.
The notion of a '$\mathcal{N}$-parametrization' is defined in a similar way.

Note that in the previous section we have used $\mathcal{N}$-parametrizations of collections of subsets of $\mathbb{N}$. In this section we mainly consider $\mathcal{C}$-parametrizations. Therefore, we will focus on these now and leave out the prefix '$\mathcal{C}$' if there is no confusion possible.

Let X be a set. When we are able to $\mathcal{C}$-parametrize a collection $\mathcal{F}$ of non-empty subsets of X, we can define a choice function on this collection as a function $f : \mathcal{C} \to X$.
As we do not require the parametrization V to be injective, it may happen that two distinct elements α and β of $\mathcal{C}$ are mapped by V to the same element of $\mathcal{F}$ (compare this with the case of enumerable subsets of $\mathbb{N}$, Section 5.1). We want a choice function on $\mathcal{F}$ to assign the same image to α and β. This leads to the following definition.

Definition 5.7. Let X be a set, $\mathcal{F}$ a collection of non-empty subsets of X.
Suppose $\mathcal{F}$ is parametrized by the map $V : \mathcal{C} \to \mathcal{F}$.
A function $f : \mathcal{C} \to X$ is a *choice function on $\mathcal{F}$ corresponding to V* iff

(i). $\forall \alpha \in \mathcal{C}\, [\, f(\alpha) \in V(\alpha)\,]$,

(ii). $\forall \alpha, \beta \in \mathcal{C}\, [\, V(\alpha) = V(\beta) \to f(\alpha) = f(\beta)\,]$.

This raises the question:

does the existence of a choice function depend on the chosen parametrization?

Luckily, the answer is 'no'. For let $\mathcal{F}$ be a collection of non-empty subsets of a set X. Now suppose V, W are $\mathcal{C}$-parametrizations of $\mathcal{F}$. That is, V and W are surjective functions from $\mathcal{C}$ to $\mathcal{F}$ such that

$$\{V(\alpha) \mid \alpha \in \mathcal{C}\} \;=\; \mathcal{F} \;=\; \{W(\alpha) \mid \alpha \in \mathcal{C}\} \tag{5.3}$$

Suppose there exists a choice function f on $\mathcal{F}$ corresponding to V. We will prove that in this case there is also a choice function on $\mathcal{F}$ corresponding to W. It follows from (5.3) that

$$\forall \alpha \in \mathcal{C}\, \exists \beta \in \mathcal{C}\, [\, W(\alpha) = V(\beta)\,].$$

According to $GAC_{1,1}$, there exists a function $\gamma : \mathcal{C} \to \mathcal{C}$ such that

$$\forall \alpha \in \mathcal{C}\, [\, W(\alpha) = V(\gamma|\alpha)\,].$$

Define a function $g : \mathcal{C} \to \mathcal{C}$ by, for all $\alpha \in \mathcal{C}$,

$$g(\alpha) \underset{\text{D}}{=} f(\gamma|\alpha).$$

We claim that g is a choice function on $\mathcal{F}$ corresponding to W. We show that g satisfies the two requirements of Definition 5.7.

(i). Let $\alpha \in \mathcal{C}$.
$g(\alpha) = f(\gamma|\alpha) \in V(\gamma|\alpha) = W(\alpha)$.

(ii). Let $\alpha, \beta \in \mathcal{C}$ and suppose $W(\alpha) = W(\beta)$.
Then $V(\gamma|\alpha) = W(\alpha) = W(\beta) = V(\gamma|\beta)$ and therefore, $f(\gamma|\alpha) = f(\gamma|\beta)$.
Hence $g(\alpha) = g(\beta)$.

So whether or not there exists a choice function on $\mathcal{F}$ does not depend on the chosen $\mathcal{C}$-parametrization. Note that the same holds for $\mathcal{N}$-parametrizations. From now on we will usually leave out the phrase 'corresponding to V' when talking about choice functions.

In the following sections we study collections $\mathcal{F}$ of non-empty subsets of $\mathcal{C}$ and ask ourselves:

Can we parametrize $\mathcal{F}$?

And for each collection the main question will be:

Can we define a choice function on $\mathcal{F}$?

We will also make some comments about a number of collections of subsets of $\mathcal{N}$ as some of the proofs given in the next sections also hold if one replaces $\mathcal{C}$ by $\mathcal{N}$.

5.2.1 Finitely enumerable subsets of $\mathcal{C}$

Clearly there exists a choice function on the collection of all subsets of $\mathcal{C}$ with exactly one element. Intuitively we could say: just send each set to its unique member. To make this precise, we define a parametrization $V : \alpha \mapsto \{\alpha\}$. Then the identity map on $\mathcal{C}$ is a choice function on this collection.

Definition 2.9 gives us a way to parametrize, for each $k \in \mathbb{N}$, the collection of all subsets of $\mathcal{C}$ with at most k elements. For we can define:

$$V_k^{fe} : \alpha \mapsto \{\alpha^0, \ldots, \alpha^{k-1}\}.$$

Unfortunately, even for subsets of $\mathcal{C}$ with at most two elements we cannot define a choice function. The proof of this fact resembles the proof of the non-existence of a choice function on the collection of inhabitedly enumerable subsets of the natural numbers (Theorem 5.4).

Theorem 5.8. *There is no choice function f on the collection of subsets of $\mathcal{C}$ with at most 2 elements.*

Proof. Suppose such a function f exists. Then $f : \mathcal{C} \to \mathcal{C}$ satisfies

(i). $\forall \alpha \in \mathcal{C}\, [\, f(\alpha) = \alpha^0 \vee f(\alpha) = \alpha^1\,]$,

(ii). $\forall \alpha, \beta \in \mathcal{C}\, [\, \{\alpha^0, \alpha^1\} = \{\beta^0, \beta^1\} \to f(\alpha) = f(\beta)\,]$.

Then

$$\forall \alpha \in \mathcal{C}\, \exists i \in \{0, 1\}\, [\, f(\alpha) = \alpha^i\,].$$

Applying the Continuity Principle yields

$$\forall \alpha \in \mathcal{C}\, \exists n, i\, \forall \beta \in \mathcal{C}\, [\overline{\alpha} n = \overline{\beta} n \to f(\beta) = \beta^i\,].$$

In particular we can find $n, i \in \mathbb{N}$ such that

$$\forall \beta \in \mathcal{C}\, [\, \overline{\underline{0}} n = \overline{\beta} n \to f(\beta) = \beta^i\,]. \tag{5.4}$$

Let us assume $i = 0$ (the case $i = 1$ is proven in a similar way).
Determine $k \in \mathbb{N}$ such that

$$\langle 0, k \rangle > n \quad \text{and} \quad \langle 1, k \rangle > n.$$

Define $\beta, \gamma \in \mathcal{C}$ by

$$\beta(\langle p, q \rangle) =_{\text{D}} 1 \qquad \text{if } \langle p, q \rangle = \langle 0, k \rangle,$$
$$=_{\text{D}} 0 \qquad \text{otherwise.}$$

$$\gamma(\langle p, q \rangle) =_{\text{D}} 1 \qquad \text{if } \langle p, q \rangle = \langle 1, k \rangle,$$
$$=_{\text{D}} 0 \qquad \text{otherwise.}$$

Then $\{\beta^0, \beta^1\} = \{\gamma^0, \gamma^1\}$. So by property (ii) of f:

$$f(\beta) = f(\gamma).$$

On the other hand, $\overline{\beta} n = \overline{\gamma} n = \overline{\underline{0}} n$. Hence, using (5.4),

$$f(\beta) = \beta^0 \quad \text{and} \quad f(\gamma) = \gamma^0.$$

But $\beta^0 = \underline{0} \neq \gamma^0$. Contradiction! ☐

5.2.2 Sequentially closed subsets of $\mathcal{C}$

The previous section seems to destroy all hopes of finding choice functions for certain subcollections of $\mathcal{C}$: even for the collection of all subsets of $\mathcal{C}$ with at most two elements we are unable to construct a choice function. Fortunately there are some further results. In Chapter 4 we have met the sequentially closed subsets of $\mathcal{N}$. We can parametrize the collection of all sequentially closed subsets of $\mathcal{C}$ generated by two elements as follows,

$$V_2^{sc} : \alpha \mapsto \overline{\{\, \alpha^0,\, \alpha^1 \,\}}.$$

We are able to define a choice function on this collection.

Theorem 5.9. *There exists a choice function on the collection of sequentially closed subsets of $\mathcal{C}$ generated by two elements.*

Proof. Let $\alpha \in \mathcal{C}$.
We define $f(\alpha)$ step by step.

$$\begin{aligned} f(\alpha)(n) &\underset{\text{D}}{=} \alpha^1(n) \qquad && \text{if } \exists k \leq n\,[\, \alpha^0(k) \neq \alpha^1(k)\,] \text{ and } \alpha^1(m) < \alpha^0(m) \\ & && \text{where } m = \mu k[\, \alpha^0(k) \neq \alpha^1(k)\,], \\ &\underset{\text{D}}{=} \alpha^0(n) && \text{otherwise.} \end{aligned}$$

So f sends α to the element of $\overline{\{\alpha^0,\, \alpha^1\}}$ that is lexicografically the first.
It is clear that f satifies the two requirements of Definition 5.7. ☐

Note that for the function f defined in this theorem, not for all $\alpha \in \mathcal{C}$:

$$f(\alpha) = \alpha^0 \;\vee\; f(\alpha) = \alpha^1.$$

Therefore, f is not a choice function as required in Theorem 5.8.

The proof of the theorem above works just as well for $\mathcal{N}$ in stead of $\mathcal{C}$. Furthermore it can be generalized to the collection of sequentially closed subsets of $\mathcal{N}$ generated by k elements, where k is any natural number. Hence, for each k we can define a choice function on the collection of sequentially closed subsets of $\mathcal{N}$ generated by k elements.
Taking the union of all those collections yields the collection of closures of finitely enumerable subsets of $\mathcal{C}$ (or $\mathcal{N}$). This collection can be parametrized by

$$V_{fe}^{sc} : \alpha \;\mapsto\; \{\alpha^1, \ldots, \alpha^{\alpha(0)+1}\}.$$

Also in this case, a choice function may be defined by taking the element that is lexicografically the first.

The collection of closures of enumerable subsets of $\mathcal{C}$ may be parametrized as follows:

$$V_{\omega}^{sc} : \alpha \;\mapsto\; \overline{\{\alpha^0, \alpha^1, \ldots\}}.$$

There is no choice function on this collection.

Theorem 5.10. *There is no choice function on the collection of closures of enumerable subsets of* $\mathcal{C}$.

Proof. Suppose $f : \mathcal{C} \to \mathcal{C}$ is a choice function on the collection of closures of enumerable subsets of $\mathcal{C}$.
Note that $V^{sc}_{\omega}(\underline{0}) = \{\underline{0}\}$ and $V^{sc}_{\omega}(\underline{1}) = \{\underline{1}\}$. Hence $f(\underline{0}) = \underline{0}$ and $f(\underline{1}) = \underline{1}$.
Using the Continuity Principle, find $k_0, k_1 \in \mathbb{N}$ such that

$$\forall \alpha \in \mathcal{C}\, [\, \overline{\alpha}k_0 = \overline{\underline{0}}k_0 \rightarrow f(\alpha)(0) = f(\underline{0})(0)\,], \tag{5.5}$$

and

$$\forall \alpha \in \mathcal{C}\, [\, \overline{\alpha}k_1 = \overline{\underline{1}}k_1 \rightarrow f(\alpha)(0) = f(\underline{1})(0)\,]. \tag{5.6}$$

Define $\gamma_0, \gamma_1 \in \mathcal{C}$ such that, for all $k \in \mathbb{N}$,

$$\begin{aligned} \gamma_0^k &= \underline{0} \quad \text{if } k \le k_0, & \gamma_1^k &= \underline{1} \quad \text{if } k \le k_1, \\ &= \underline{1} \quad \text{if } k > k_0. & &= \underline{0} \quad \text{if } k > k_1. \end{aligned}$$

Then certainly $\overline{\gamma_0}k_0 = \overline{\underline{0}}k_0$, so by (5.5),

$$f(\gamma_0)(0) = 0.$$

And $\overline{\gamma_1}k_1 = \overline{\underline{1}}k_1$, so by (5.6),

$$f(\gamma_1)(0) = 1.$$

Hence $f(\gamma_0) \neq f(\gamma_1)$.
But $V^{sc}_{\omega}(\gamma_0) = \{\underline{0}, \underline{1}\} = V^{sc}_{\omega}(\gamma_1)$. Contradiction! □

Of course, the proof given above also holds for the collection of closures of enumerable subsets of $\mathcal{N}$. Hence there is no choice function on this collection either.
A subcollection of the sequentially closed subsets of $\mathcal{N}$ is the collection of spreads. Every spread is uniquely represented by its spread law. The collection $\mathcal{S} =_{D} \{\sigma \in \mathcal{N} \mid \sigma \text{ is a spread law}\}$ is strictly analytic, as we can define a surjective function $g : \mathcal{N} \to \mathcal{S}$ by, for all $\alpha \in \mathcal{N}$, for all $a \in \mathbb{N}^*$,

$$\begin{aligned} g(\alpha)(a) &=_{D} 0 \quad \text{if } \alpha^0(a) = 0 \vee a = \langle\rangle \vee (\, lg(a) \ge 1 \wedge \\ &\qquad\qquad g(\alpha)(\overline{a}(lg(a)-1)) = 0 \wedge a(lg(a)-1) = \alpha^1(\overline{a}(lg(a)-1)\,)), \\ &=_{D} 1 \quad \text{otherwise.} \end{aligned}$$

And therefore, the collection of spreads can be parametrization by

$$V_{spr} : \alpha \mapsto \{\beta \in \mathcal{N} \mid \beta \text{ is admitted by the spread law } g(\alpha)\}$$

There exists a choice function on the collection of spreads. The idea of this function is that it assigns to every spread its 'leftmost path'. A precise construction of a choice function on this collection is given as follows.

Theorem 5.11. *There exists a choice function on the collection of spreads.*

Proof. Let $\alpha \in \mathcal{N}$.
We define $f(\alpha)$ by

$$f(\alpha)(n) \underset{\mathrm{D}}{=} \mu i\,[\,g(\alpha)(\langle f(\alpha)(0), \ldots, f(\alpha(n-1)), i\rangle) = 0\,] \qquad \text{for all } n \in \mathbb{N}.$$

As $g(\alpha)$ is a spread law this function is well-defined.
Notice that for all $\alpha \in \mathcal{N}$ and all $n \in \mathbb{N}$,

$$g(\alpha)(\overline{f(\alpha)}n) = 0.$$

Hence, for all $\alpha \in \mathcal{N}$, $f(\alpha) \in V_{spr}(\alpha)$.
No two different spread laws represent the same spread. So, for all $\alpha, \beta \in \mathcal{N}$ with $V_{spr}(\alpha) = V_{spr}(\beta)$, $g(\alpha) = g(\beta)$. It follows immediately from the definition of f that in that case $f(\alpha) = f(\beta)$.
So f is a choice function on the collection of spreads. □

On the collection of the sequentially closed subsets of $\mathcal{N}$ we cannot define a functions that sends every set to its 'leftmost' element, as, in contrast to spreads, these sets are not necessarily located.

5.2.3 Effectively open subsets of $\mathcal{C}$

In Chapter 4 we have introduced two notions of 'open'. The notion of 'weakly open' is, as the name suggest, quite weak. That makes it hard to parametrize the collection of all non-empty weakly open subsets of $\mathcal{C}$. Therefore, we will only consider the effectively open subsets of $\mathcal{C}$.
Recall that a subset A of $\mathcal{C}$ is effectively open if and only if, there exists $\beta \in \mathcal{N}$ such that $A = \{\alpha \in \mathcal{C} \mid \exists n \in \mathbb{N}\,[\beta(\overline{\alpha}n) = 1]\}$. This gives us a natural way to $\mathcal{N}$-parametrize the collection of effectively open subsets of $\mathcal{C}$, namely

$$V_{eo} : \beta \mapsto \{\alpha \in \mathcal{C} \mid \exists n \in \mathbb{N}\,[\,\beta(\overline{\alpha}n) = 1\,]\}.$$

But not every effectively open subset of $\mathcal{C}$ is non-empty. Notice that, for all $\beta \in \mathcal{N}$, V_{eo} is non-empty if and only if there exists $n \in \{0,1\}^*$ with $\beta(n) = 1$.
To define a parametrization V_{ieo} of the collection of non-empty effectively open subsets of $\mathcal{C}$ we first define, for every $\beta \in \mathcal{N}$, an element γ_β of $\mathcal{N}$ by

$$\begin{aligned} \gamma_\beta(n) &\underset{\mathrm{D}}{=} 1 && \text{if } n = \beta(0), \\ &\underset{\mathrm{D}}{=} \beta(n) && \text{if } n \neq \beta(0). \end{aligned}$$

Then we define for all $\beta \in \mathcal{N}$,

$$V_{ieo} : \beta \mapsto \{\alpha \in \mathcal{C} \mid \exists n \in \mathbb{N}\,[\gamma_\beta(\overline{\alpha}n) = 1\,]\}.$$

There is no choice function on this collection.

Theorem 5.12. *There is no choice function on the collection of non-empty effectively open subsets of $\mathcal{C}$.*

Proof. Suppose $f : \mathcal{C} \to \mathcal{C}$ is a choice function on the collection of non-empty effectively open subsets of $\mathcal{C}$.
Using the Contuinity Principle, it follows that

$$\forall \beta \in \mathcal{N}\, \exists n \in \mathbb{N}\, \forall \gamma \in \mathcal{N}\, [\, \overline{\beta}n = \overline{\gamma}n \to f(\beta)(0) = f(\gamma)(0)\,]. \tag{5.7}$$

Define $\beta_0, \beta_1 \in \mathcal{N}$ by

$$\begin{aligned} \beta_0(n) &\underset{\text{D}}{=} \langle 0 \rangle && \text{if } n = 0, \\ &\underset{\text{D}}{=} 0 && \text{otherwise.} \end{aligned}$$

$$\begin{aligned} \beta_1(n) &\underset{\text{D}}{=} \langle 1 \rangle && \text{if } n = 0, \\ &\underset{\text{D}}{=} 0 && \text{otherwise.} \end{aligned}$$

Notice that $V_{ieo}(\beta_0)$ consists of all elements of $\mathcal{C}$ starting with 0 and $V_{ieo}(\beta_1)$ consists of all elements of $\mathcal{C}$ starting with 1.
Hence $f(\beta_0)(0) = 0$ and $f(\beta_1)(0) = 1$.
By (5.7) we can determine $n_0, n_1 \in \mathbb{N}$ such that, for $i = 0, 1$,

$$\forall \gamma \in \mathcal{N}\, [\, \overline{\beta_i}n_i = \overline{\gamma}n_i \to f(\gamma)(0) = i\,]. \tag{5.8}$$

Define $k \underset{\text{D}}{=} max(n_0, n_1)$.
Define $\gamma_0, \gamma_1 \in \mathcal{N}$ by

$$\begin{aligned} \gamma_0(n) &\underset{\text{D}}{=} \langle 0 \rangle && \text{if } n = 0, \\ &\underset{\text{D}}{=} 1 && \text{if } n > k, \\ &\underset{\text{D}}{=} 0 && \text{otherwise.} \end{aligned}$$

$$\begin{aligned} \gamma_1(n) &\underset{\text{D}}{=} \langle 1 \rangle && \text{if } n = 0, \\ &\underset{\text{D}}{=} 1 && \text{if } n > k, \\ &\underset{\text{D}}{=} 0 && \text{otherwise.} \end{aligned}$$

Then $\overline{\gamma_0}n_0 = \overline{\beta_0}n_0$, so by (5.8), $f(\gamma_0)(0) = 0$.
In a similar way we see that $f(\gamma_1)(0) = 1$. So

$$f(\gamma_0) \neq f(\gamma_1).$$

But $V_{ieo}(\gamma_0) = \mathcal{N} = V_{ieo}(\gamma_1)$. Contradiction! □

Chapter 6

Sets of representatives for equivalence relations

In 1902 Henri Lebesgue published his dissertation “Intégral, longueur, aire” in which he formulates his integration theory. He introduces a new notion of measure and of integration in order to obtain more measurable subsets of $\mathbb{R}$ and more measurable and integrable functions from $\mathbb{R}$ to $\mathbb{R}$ than Riemann, with his notion, had been able to define.
This raises the question whether perhaps *every* bounded subset of the reals is Lebesgue measurable. In 1905 Giuseppe Vitali showed that a classical mathematician accepting the Axiom of Choice must answer this question negatively.

Definition 6.1. Let X be a set and $R \subseteq X^2$ an equivalence relation on X. We say A is a *set of representatives* for R iff

(i). $\forall a, a' \in A\, [\, a\, R\, a' \;\rightarrow\; a = a'\,]$,

(ii). $\forall x \in X \exists a \in A\, [\, x\, R\, a\,]$.

Vitali considered the equivalence relation $\sim_V$ on $[0, 1]$ defined by:

$$x \sim_V y \quad \leftrightarrows \quad x - y \in \mathbb{Q}. \tag{6.1}$$

One of the classical equivalents of the Axiom of Choice states:

every equivalence relation has a set of representatives.

Vitali showed that a set of representatives for the relation defined in (6.1) cannot be Lebesgue measurable.
Lebesgue himself did not accept the Axiom of Choice and for him it remained possible that every bounded subset of the reals is Lebesgue measurable [6].

In this chapter we study several equivalence relations and see what we can say about the existence of a set of representatives. We begin by considering some classes of relations on the set $\mathbb{N}$ of the natural numbers and after that we examine various relations on Cantor space $\mathcal{C}$. There is a connection between the existence of sets of representatives and the existence of choice functions discussed in the previous chapter. This connection is the topic of the last section of this chapter.

6.1 Equivalence relations on $\mathbb{N}$

In Section 2.2 we introduced decidable, enumerable and co-enumerable subsets of the natural numbers. We now extend these notions to relations on $\mathbb{N}$.

Definition 6.2. Let $R \subseteq \mathbb{N} \times \mathbb{N}$.
R is *decidable relation on* $\mathbb{N}$ if and only if the set $\{ \langle n, m\rangle \mid n\, R\, m \}$ is a decidable subset of $\mathbb{N}$. For any $\alpha \in \mathcal{N}$, we say α *decides* R if and only if D_α coincides with this set.
The notions of enumerable and co-enumerable relation on $\mathbb{N}$ are defined analogously.

We first consider decidable equivalence relations on $\mathbb{N}$. These have decidable sets of representatives as you can just pick the smallest element from each equivalence class. This idea is made precise as follows.

Theorem 6.3. *Any decidable equivalence relation R on $\mathbb{N}$ has a decidable set of representatives.*

Proof. Define

$$A \underset{\mathrm{D}}{=} \{n \in \mathbb{N} \mid \neg \exists m \leq n\, [\, n\, R\, m\,]\}.$$

As R is a decidable subset of $\mathbb{N} \times \mathbb{N}$, A is a decidable subset of $\mathbb{N}$.
It is clear that A satifies the two requirements of Definition 6.1. □

Now suppose R is an equivalence relation on $\mathbb{N}$ with a system of representatives A. Let n, m be natural numbers. By property (ii) of A, we can find $a, b \in A$ such that $n\, R\, a$ and $m\, R\, b$.
If $a = b$ then, by the transitivity of R, $n\, R\, m$. On the other hand, if $a \neq b$, by property (i) of A, $\neg(n\, R\, m)$. As the equality on the natural numbers is decidable, this means we can decide $n\, R\, m \vee \neg(n\, R\, m)$. In other words: any relation on the natural numbers that has a system of representatives as defined above is decidable.
We do not want to restrict ourselves to decidable relations and choose to weaken the notion of a set of representatives.

Definition 6.4. Let X be a set and $R \subseteq X^2$ an equivalence relation on X.
A is a *weak set of representatives* for R iff

(i). $\forall a, a' \in A\, [\, a\, R\, a' \rightarrow a = a'\,]$,

(ii). $\forall x \in X \neg\neg\exists a \in A\, [\, x\, R\, a\,]$.

Note that property (ii) is equivalent to:

(ii)'. $\forall x \in X \neg\forall a \in A\, [\, \neg(a\, R\, x)\,]$

We will now consider enumerable and co-enumerable relations on R. First note that not every enumerable equivalence relation on $\mathbb{N}$ has a decidable weak set of representatives. For we could define $R \subseteq \mathbb{N} \times \mathbb{N}$ by

$$n\, R\, m \leftrightarrows n = m \vee (((n = 0 \wedge m = 1) \vee (n = 1 \wedge m = 0)) \wedge k_{99} \text{ exists}).$$

R is an enumerable relation on the natural numbers.
Suppose R has a decidable weak set of representatives A.
As we can decide whether both 0 and 1 are in A or not, we are also able to decide whether k_{99} exists or not. But that is a reckless statement.

Using the same idea as in Theorem 6.3 (i.e. picking the smallest element from each equivalence class), we first prove that every enumerable equivalence relation on $\mathbb{N}$ has a co-enumerable weak set of representatives.

Theorem 6.5. *Every enumerable equivalence relation on* $\mathbb{N}$ *has a co-enumerable weak set of representatives.*

Proof. Let R be an enumerable equivalence relation on $\mathbb{N}$.
Find $\alpha \in \mathcal{N}$ enumerating R.
On page 12 we explained how every natural number n codes a finite sequence of natural numbers. For every $i \leq lg(n)$, $n(i)$ denotes the $(i+1)$-th term of the sequence coded by n. Using this, we define $\beta \in \mathcal{N}$ by:

$$\begin{aligned} \beta(n) &\underset{\text{D}}{=} n(0)+1 && \text{if } \exists j < n(0)\, \exists i \leq n(1)\, [\, \alpha(i) = \langle\, n(0), j\,\rangle\,], \\ &\underset{\text{D}}{=} 0 && \text{otherwise.} \end{aligned}$$

Note that

$$n \in E_\beta \quad \leftrightarrows \quad \exists j < n\, [\, n\, R\, j\,]. \tag{6.2}$$

We will show that CE_β is a weak set of representatives for R.

(i). $\forall a, a' \in CE_\beta\, [\, a\, R\, a' \,\rightarrow\, a = a'\,]$.
We prove this as follows.
Let $a, a' \in CE_\beta$ with $a\, R\, a'$.
As α enumerates R, we can find k such that $\alpha(k) = \langle\, a,\, a'\,\rangle$.
Suppose $a' < a$.

It follows from (6.2) that $a \in E_\beta$.
But we have assumed $a \in CE_\beta = N \setminus E_\beta$. Contradiction.

So $a \leq a'$.
In a similar way we see: $a' \leq a$.
Hence $a = a'$.

(ii). $\forall n \in \mathbb{N}\, \neg\forall a \in CE_\beta\, [\, \neg(a\, R\, n)\,]$.
We prove this by course-of-values induction.
Let $n \in \mathbb{N}$ and assume, for all $j < n$,

$$\neg\forall a \in CE_\beta\, [\, \neg(a\, R\, j)\,].$$

Suppose: $\forall a \in CE_\beta [\neg(a\, R\, n)]$.

Then it follows from the induction hypothesis and the fact that R is an equivalence relation that $\forall j < n [\neg(j\, R\, n)]$. Hence

$$\neg \exists j < n [j\, R\, n].$$

According to (6.2), $n \notin E_\beta$, so $n \in CE_\beta$.
As R is an equivalence relation, $n\, R\, n$.
Hence there is an element of CE_β that is equivalent to n (namely n itself). Contradiction.

So $\neg \forall a \in CE_\beta [\neg(a\, R\, n)]$.

□

Using the idea of 'picking the smallest element of each equivalence class' has resulted in finding a way to define, for every enumerable equivalence relation on $\mathbb{N}$, a co-enumerable weak set of representatives. This raises the question whether we can also define, for every enumerable equivalence relation on $\mathbb{N}$, an enumerable weak set of representatives. The answer to this question is 'no'.

Theorem 6.6. *Not: every enumerable equivalence relation on* $\mathbb{N}$ *has an enumerable weak set of representatives.*

Proof. Suppose every enumerable equivalence relation on $\mathbb{N}$ has an enumerable weak set of representatives.
For each $\alpha \in \mathcal{N}$, we define a relation R_α on $\mathbb{N}$ by, for all $n, m \in \mathbb{N}$,

$$n\, R_\alpha\, m \leftrightarrows n = m \vee \Big(\big((n = 0 \wedge m = 1) \vee (n = 1 \wedge m = 0)\big) \wedge \exists k \in \mathbb{N} [\alpha(k) \neq 0] \Big).$$

For all $\alpha \in \mathcal{N}$, $0\, R_\alpha\, 1$ iff there exists a natural number k with $\alpha(k) \neq 1$.
Note that, for all $\alpha \in \mathcal{N}$, R_α is an enumerable equivalence relation on $\mathbb{N}$. Hence it follows from the assumption that:

$$\forall \alpha \in \mathcal{N}\, \exists \beta \in \mathcal{N} [E_\beta \text{ is a weak set of representatives for } R_\alpha].$$

According to $GAC_{1,1}$, there exists a function $\gamma : \mathcal{N} \to \mathcal{N}$ such that

$$\forall \alpha \in \mathcal{N} [E_{\gamma|\alpha} \text{ is a weak set of representatives for } R_\alpha].$$

Let us consider $\underline{0}$.
Suppose: $0 \in E_{\gamma|\underline{0}} \wedge 1 \in E_{\gamma|\underline{0}}$

Determine $p_0, p_1 \in \mathbb{N}$ such that

$$0 = (\gamma|\underline{0})(p_0) - 1 \quad \text{and} \quad 1 = (\gamma|\underline{0})(p_1) - 1.$$

Determine $k \in \mathbb{N}$ such that

$$\forall \delta \in \mathcal{N} [\overline{\delta}k = \overline{\underline{0}}k \to ((\gamma|\delta)(p_0) = (\gamma|\underline{0})(p_0) \wedge (\gamma|\delta)(p_1) = (\gamma|\underline{0})(p_1))].$$

Define $\delta \in \mathcal{N}$ by,

$$\delta(n) \underset{\mathrm{D}}{=} 0 \qquad \text{if } n \leq k,$$
$$\underset{\mathrm{D}}{=} 1 \qquad \text{if } n > k.$$

Then $\overline{\delta}k = \overline{\underline{0}}k$, so

$$(\gamma|\delta)(p_0) = (\gamma|\underline{0})(p_0) = 0{+}1 \quad \text{and} \quad (\gamma|\delta)(p_1) = (\gamma|\underline{0})(p_1) = 1{+}1.$$

So 0 and 1 are both in $E_{\gamma|\delta}$.
But as $\delta(k+1) = 1$, $0\, R_\delta\, 1$. Contradiction.

We conclude:

$$\neg(\, 0 \in E_{\gamma|\underline{0}} \,\wedge\, 1 \in E_{\gamma|\underline{0}}\,). \tag{6.3}$$

On the other hand, $E_{\gamma|\underline{0}}$ is a weak system of representatives for $R_{\underline{0}}$, so by applying property (ii) of Definition 6.4 for $n = 0$ and $n = 1$:

$$\neg\neg\, \exists m_0 \in E_{\gamma|\underline{0}}\, [\, m_0\, R_{\underline{0}}\, 0\,] \;\wedge\; \neg\neg\, \exists m_1 \in E_{\gamma|\underline{0}}\, [\, m_1\, R_{\underline{0}}\, 1\,].$$

As $R_{\underline{0}}$ is the trivial relation, it follows that

$$\neg\neg(0 \in E_{\gamma|\underline{0}}) \,\wedge\, \neg\neg(1 \in E_{\gamma|\underline{0}}\,). \tag{6.4}$$

In general, for all propositions A and B,

$$\text{if } \neg\neg A \,\wedge\, \neg\neg B, \text{ then } \neg\neg(A \,\wedge\, B).$$

By defining:

$$A \underset{\mathrm{D}}{=} 0 \in E_{\gamma|\underline{0}} \quad \text{and} \quad B \underset{\mathrm{D}}{=} 1 \in E_{\gamma|\underline{0}}$$

we see that (6.3) and (6.4) contradict each other.

So not every enumerable equivalence relation on the natural numbers has an enumerable weak set of representatives. □

We will now see that every co-enumerable equivalence relation on $\mathbb{N}$ has an enumerable weak set of representatives. The proof of this fact is very similar to the proof of Theorem 6.5.

Theorem 6.7. *Every co-enumerable equivalence relation on* $\mathbb{N}$ *has an enumarable weak set of representatives.*

Proof. Let R be a co-enumerable equivalence relation on $\mathbb{N}$.
Find $\alpha \in \mathcal{N}$ co-enumerating R and define $\beta \in \mathcal{N}$ by:

$$\beta(n) \underset{\mathrm{D}}{=} n(0) + 1 \qquad \text{if } \forall j < n(0)\, \exists k \leq n(1)\, [\, \alpha(k) = \langle\, n(0),\, j\,\rangle\,],$$
$$\underset{\mathrm{D}}{=} 0 \qquad \text{otherwise.}$$

We claim that E_β is a weak systeem of representatives for R. In a similar way as in the proof of Theorem 6.5 one can show that E_β satisfies requirement (i) of Definition 6.4.

To prove that E_β satisfies requirement (ii) as well, we first show, for all $n \in \mathbb{N}$,

$$n \in E_\beta \quad \Leftrightarrow \quad \forall j < n \, \exists k \in \mathbb{N} \, [\, \alpha(k) = \langle \, n, \, j \, \rangle \,]. \tag{6.5}$$

Let $n \in \mathbb{N}$.

$\Rightarrow$) Suppose $n \in E_\beta$.
Determine p such that $n = \beta(p) - 1$.
Then, by definition of β,

$$\forall j < n \, \exists k \leq p(1) \, [\, \alpha(k) = \langle \, n, \, j \, \rangle \,].$$

$\Leftarrow$) Suppose $\forall j < n \, \exists k \in \mathbb{N} \, [\, \alpha(k) = \langle \, n, \, j \, \rangle \,]$.
Find, for each $j < n$, a natural number k_j with $\alpha(k_j) = \langle \, n, \, j \, \rangle$.
Define $p \underset{\text{D}}{=} \max\limits_{j<n} k_j$.
Then: $\beta(\langle \, n, \, p \, \rangle) = n$.
So $n \in E_\beta$.

We now prove by course-of-values induction that E_β has property (ii)'.
Let $n \in \mathbb{N}$ and assume:

$$\forall j < n \, \neg \forall a \in E_\beta \, [\, \neg (j \, R \, a) \,].$$

Suppose $\forall a \in E_\beta \, [\, \neg (n \, R \, a) \,]$.

As $n \, R \, n$, this means in particular $n \notin E_\beta$.
Using (6.5) we conclude

$$\neg \forall j < n \, \exists k \in \mathbb{N} \, [\, \alpha(k) = \langle \, n, \, j \, \rangle \,]. \tag{6.6}$$

From the induction hypothesis together with the fact that R is an equivalence relation it follows that

$$\forall j < n \, [\, \neg (n \, R \, j) \,].$$

As R is co-enumerated by α this means:

$$\forall j < n \, \neg \neg \exists k \in \mathbb{N} \, [\, \alpha(k) = \langle \, n, \, j \, \rangle \,]. \tag{6.7}$$

For each $n \in \mathbb{N}$, for all propositions $A_0, \ldots, A_{n-1}$,

$$\text{if } \bigwedge_{j<n} \neg \neg \, A_j, \text{ then } \neg \neg \bigwedge_{j<n} A_j.$$

By defining, for all $j < n$,

$$A_j \underset{\text{D}}{=} \exists k \in \mathbb{N} \, [\, \alpha(k) = \langle \, n, \, j \, \rangle \,]$$

we see that (6.6) and (6.7) contradict each other.

So $\neg \forall a \in E_\beta \, [\, \neg (n \, R \, a) \,]$.
Hence E_β is a weak set of representatives for R. □

One can also prove, in a way comparable to the proof of Theorem 6.6, that not every co-enumerable equivalence relation on $\mathbb{N}$ has a co-enumerable weak set of representatives. We omit that proof here.

6.2 The prisoners and their hats

As promised, we return to the problem of the prisoners formulated in the introduction. Recall, the situation is as follows. Countably infinitely many inmates, numbered 0, 1, 2, ... are placed in line facing the positive direction (i.e. each prisoner can see infinitely many hats). The guard randomly assigns either a black or a white hat to each prisoner. Then he asks them one by one to guess the colour of their hats, without the other prisoners being able to hear the answers. If the prisoner guesses correctly, then he is released. If not, he has to stay in prison for the rest of his life. After being given the rules of the game and their number, the prisoners get one hour to discuss and come up with a strategy. What is the best they can do?[1]

As said, the classical mathematician accepting the Axiom of Choice, claims they can ensure that at most finitely many guess wrong. His plan is as follows. Consider the equivalence relation on $\mathcal{C}$ defined by, for all $\alpha, \beta \in \mathcal{C}$,

$$\alpha \sim_a \beta \leftrightarrows \exists n \forall m > n \, [\, \alpha(m) = \beta(m) \,]. \tag{6.8}$$

In other words: α and β are equivalent if and only if they are almost everywhere the same.
How does this relation help the prisoners? Each distribution of hats can be seen as a sequence of zeros and ones (saying each black hat is a one and each white hat a zero). Before they line up, the prisoners together select, using the Axiom of Choice(!), one representative from each equivalence class. During the game there are only finitely many hats a prisoner cannot see, so he can decide in which equivalence class the actual hat distribution is. He guesses the hat colour he would have if the chosen representative of that class were the actual situation. As the sequence resulting from the guesses of the prisoners and the actual sequence are in the same equivalence class, they differ only in finitely many positions. This means at most finitely many prisoners guess wrong [7].

6.2.1 Equivalence relation on $\mathcal{C}$

Let us investigate the equivalence relation that helped out the prisoners (see (6.8)). What can we say intuitionistically about the existence of a set of representatives for this relation? First of all it has no enumerable set of representatives as any enumerable subset of $\mathcal{C}$ is positively incomplete.

Definition 6.8. Let R be an equivalence relation on $\mathcal{C}$ and $X \subseteq \mathcal{C}$.
X is called *positively R-incomplete* if and only if there exists $\beta \in \mathcal{C}$ such that, for all $\alpha \in X$, $\neg(\alpha \, R \, \beta)$.

[1] A nice puzzle: change the situation to 100 prisoners, each of them being able to hear the answers of the inmates behind him. How can they ensure (without cheating or using the Axiom of Choice) at most one of them guesses wrong?

Theorem 6.9. *Any enumerable subset of $\mathcal{C}$ is positively $\sim_a$-incomplete.*

Proof. Let $\alpha_0, \alpha_1, \ldots \in \mathcal{C}$.
We will define $\beta \in \mathcal{C}$ such that $\forall i \in \mathbb{N}\,[\,\neg(\beta \sim_a \alpha_i)\,]$.
To achieve this we define, for all $n \in \mathbb{N}$,

$$\beta(n) \underset{\text{D}}{=} 1 - \alpha_i(n) \qquad \text{where } i \underset{\text{D}}{=} \mu j\,[\,2^{j+1} \nmid n\,].$$

Let $i \in \mathbb{N}$.
For all natural numbers k,

$$\beta(2^i \cdot (2k+1)) = 1 - \alpha_i(2^i \cdot (2k+1)) \neq \alpha_i(2^i \cdot (2k+1)).$$

Hence

$$\forall m\, \exists n\,[\,n > m \,\wedge\, \beta(n) \neq \alpha_i(n)\,],$$

and therefore, $\neg\,(\beta \sim_a \alpha_i)$. □

Definition 6.10. Let R be an equivalence relation on $\mathcal{C}$ and $X \subseteq \mathcal{C}$.
X is called *R-independent* if and only if for all $\alpha, \beta \in \mathcal{C}$,

if $\alpha \,\#\, \beta$, then $\neg(\alpha\, R\, \beta)$.

There exists an $\sim_a$-independent subset of $\mathcal{C}$ that is 'as big as $\mathcal{C}$'. We may draw this conclusion from the following result.

Theorem 6.11. *There exists a function $f : \mathcal{C} \to \mathcal{C}$ such that:*

$$\forall \alpha, \beta \in \mathcal{C}\,[\,\alpha \,\#\, \beta \,\to\, \neg\,(f(\alpha) \sim_a f(\beta))\,]. \tag{6.9}$$

Proof. Define $f : \mathcal{C} \to \mathcal{C}$ by,

$$f(\alpha)(n) \underset{\text{D}}{=} \alpha(n(0)).$$

Now suppose α, $\beta \in \mathcal{C}$ and $\alpha \,\#\, \beta$.
Then we can find $p \in \mathbb{N}$ such that $\alpha(p) \neq \beta(p)$.
For all $m \in \mathbb{N}$,

$$f(\alpha)(\langle p, m\rangle) = \alpha(p) \neq \beta(p) = f(\beta)(\langle p, m\rangle).$$

Hence $\neg\,(f(\alpha) \sim_a f(\beta))$. □

Ran(f), where f as defined in the proof above, is positively $\sim_a$-incomplete as we can define $\gamma \in \mathcal{C}$ by:

$$\gamma(n) \underset{\text{D}}{=} 0 \qquad \text{if } n = \langle 0, 2k\rangle \text{ for certain } k \in \mathbb{N},$$
$$\underset{\text{D}}{=} 1 \qquad \text{otherwise.}$$

For all $\alpha \in \mathcal{C}$ and all $m \in \mathbb{N}$,

$$f(\alpha)(\langle 0, m\rangle) = \alpha(0).$$

While for each $k \in \mathbb{N}$:

$$\gamma(\langle 0, 2k\rangle) = 0 \quad \text{and} \quad \gamma(\langle 0, 2k+1\rangle) = 1.$$

Hence

$$\forall \alpha {\in} \mathcal{C}\, \forall m\, \exists n > m\, [\, f(\alpha)(n) \neq \gamma(n)\,].$$

So $\forall \alpha {\in} \mathcal{C}\, [\, \neg\, (\gamma \sim_a f(\alpha))\,]$.

This makes us wonder whether every function $f : \mathcal{C} \to \mathcal{C}$ satisfying (6.9) is positively $\sim_a$-incomplete. Saying $\neg\, (\gamma \sim_a \delta)$ means:

$$\neg \exists n\, \forall m > n\, [\, \gamma(m) = \delta(m)\,].$$

This statement is quite weak. Therefore, we introduce a stronger notion.

Definition 6.12. Let $\gamma, \delta \in \mathcal{N}$.
We say γ is *strongly not* $\sim_a$ *related to* δ, notation $\gamma \not\#_a \delta$, if and only if

$$\forall m \exists n > m\, [\, \gamma(n) \neq \delta(n)\,].$$

Note that both Theorem 6.9 and Theorem 6.11 still hold if we replace $\neg\, (\alpha \sim_a \beta)$ by $\alpha \not\#_a \beta$. With this strong interpretation of being not $\sim_a$-equivalent we can prove that every function f satisfying (6.9) is positively $\sim_a$-incomplete (where we replace all occurrences of $\neg(\alpha \sim_a \beta)$ by $\alpha \not\#_a \beta$).

Theorem 6.13. *For any function* $f : \mathcal{C} \to \mathcal{C}$ *satisfying:*

$$\forall \alpha, \beta \in \mathcal{C}\, [\, \alpha \# \beta \;\to\; f(\alpha) \not\#_a f(\beta)\,]$$

there exists $\gamma \in \mathcal{C}$ *such that for all* α *in* $\mathcal{C}$, $\gamma \not\#_a f(\alpha)$.

Proof. Suppose $f : \mathcal{C} \to \mathcal{C}$ satisfies

$$\forall \alpha, \beta \in \mathcal{C}\, [\, \alpha \# \beta \to \forall m\, \exists n > m\, [\, f(\alpha)(n) \neq f(\beta)(n)\,]\,]. \tag{6.10}$$

We will define a strictly increasing sequence $k_{-1}, k_0, k_1, \ldots \in \mathbb{N}$ and a sequence $\gamma_0, \gamma_1, \ldots \in \mathcal{C}$ such that, for all $\alpha \in \mathcal{C}$, for all $m \in \mathbb{N}$,

$$\exists n {\in} \mathbb{N}\, [\, k_m \leq n < k_{m+1} \;\wedge\; f(\alpha)(n) \neq \gamma_{m+1}(n)\,].$$

Then we define $\gamma \in \mathcal{N}$ by

$$\gamma(n) \underset{\mathrm{D}}{=} \gamma_m(n) \qquad \text{where } m \underset{\mathrm{D}}{=} \mu i\, [\, k_{i-1} \leq n < k_i\,]$$

and show that γ has the required property.

The two sequences are defined as follows.

First define $k_{-1} \underset{\text{D}}{=} 0$ and do the following for each $m \in \mathbb{N}$:
Define $\gamma_m \in \mathcal{C}$ by:

$$\begin{aligned} \gamma_m &\underset{\text{D}}{=} 1 - f(\underline{0})(n) && \text{if } n \leq k_{m-1}, \\ &\underset{\text{D}}{=} f(\underline{0})(n) && \text{if } n > k_{m-1}. \end{aligned}$$

Determine, using the Continuity Principle, $p \in \mathbb{N}$ such that

$$\forall \alpha \in \mathcal{C} \, [\, \overline{\alpha}p = \overline{\underline{0}}p \; \rightarrow \; f(\alpha)(k_{m-1}) = f(\underline{0})(k_{m-1}) \,].$$

Note that

$$\forall \alpha \in \mathcal{C} \, [\, \overline{\alpha}p = \overline{\underline{0}}p \; \rightarrow \; f(\alpha)(k_{m-1}) \neq \gamma_m(k_{m-1}) \,]. \tag{6.11}$$

Define $\sigma \underset{\text{D}}{=} \{\alpha \in \mathcal{C} \mid \overline{\alpha}p \neq \overline{\underline{0}}p \,\}$.
All α in σ are apart from $\underline{0}$, so by applying (6.10) with $m = k_{m-1}$:

$$\forall \alpha \in \sigma \, \exists n > k_{m-1} \, [\, f(\alpha)(n) \neq f(\underline{0})(n) \,].$$

As σ is a fan, we can apply the Fan Theorem and find N such that

$$\forall \alpha \in \sigma \, \exists n \leq N \, [\, n > k_{m-1} \; \wedge \; f(\alpha)(n) \neq f(\underline{0})(n) \,].$$

Define $k_m \underset{\text{D}}{=} N + 1$.
For all $n > k_{m-1}$, $\gamma_m(n) = f(\underline{0})(n)$, and therefore,

$$\forall \alpha \in \sigma \, \exists n \, [\, k_{m-1} < n < k_m \; \wedge \; f(\alpha)(n) \neq \gamma_m(n) \,] \,]. \tag{6.12}$$

For each $\alpha \in \mathcal{C}$ we can decide whether $\overline{\alpha}p = \overline{\underline{0}}p$ or $\overline{\alpha}p \neq \overline{\underline{0}}p$, so combining (6.11) and (6.12) yields

$$\forall \alpha \in \mathcal{C} \, \exists n \, [\, k_{m-1} \leq n < k_m \; \wedge \; f(\alpha)(n) \neq \gamma_m(n) \,]. \tag{6.13}$$

Continue with step $m + 1$

As announced we define $\gamma \in \mathcal{N}$ by, for all $n \in \mathbb{N}$,

$$\gamma(n) \underset{\text{D}}{=} \gamma_m(n) \qquad \text{where } m \underset{\text{D}}{=} \mu i \, [\, k_{i-1} \leq n < k_i \,].$$

The sequence $k_{-1}, k_0, k_1, \ldots$ is strictly increasing. Hence it follows from (6.13) that, for all $\alpha \in \mathcal{C}$,

$$\forall m \, \exists n > m \, [\, \gamma(n) \neq f(\alpha)(n) \,]$$

and the constructed γ satisfies:

$$\forall \alpha \in \mathcal{C} \, [\, \gamma \not\#_a f(\alpha) \,].$$

□

6.2.2 Connection with the Vitali relation

In the previous section we studied the following equivalence relation on $\mathcal{C}$:

$$\alpha \sim_a \beta \leftrightarrows \exists m \forall n > m\, [\, \alpha(n) = \beta(n)\,].$$

This relation is connected to the relation defined by Vitali on $\mathbb{R}$, which we discussed briefly in the introduction of this chapter. Recall

$$x \sim_V y \leftrightarrows x - y \in \mathbb{Q}.$$

We will show that there exists a function $g : \mathcal{C} \to \mathbb{R}$ such that, for all $\alpha, \beta \in \mathcal{C}$,

$$\alpha \sim_a \beta \leftrightarrows g(\alpha) \sim_V g(\beta). \tag{6.14}$$

Hence, for all α and β in $\mathcal{C}$, if we know whether or not $g(\alpha)$ and $g(\beta)$ are $\sim_V$-equivalent, then we also know whether or not α and β themselves are $\sim_a$-equivalent. This means the relation $\sim_a$ is in some sense easier than $\sim_V$. If we fully understand the Vitali relation, then we also understand the relation $\sim_a$.

Theorem 6.14. *There exists a function $g : \mathcal{C} \to \mathbb{Q}$ such that, for all $\alpha, \beta \in \mathcal{C}$,*

$$\alpha \sim_a \beta \leftrightarrows g(\alpha) \sim_V g(\beta).$$

Proof. To define such a function g we use the infinite sequence of rational numbers $\frac{1}{0!}, \frac{1}{1!}, \frac{1}{2!}, \frac{1}{3!}, \ldots$. This sequence has the special property that the sum of every one of its infinite subsequences is positively irrational, that is, if $k_0, k_1, k_2, \ldots$ is a strictly increasing infinite sequence of natural numbers, then, for all $q \in \mathbb{Q}$,

$$\sum_{n=0}^{\infty} \frac{1}{k_n!} \# q$$

We define $g : \mathcal{C} \to \mathbb{R}$, by

$$g(\alpha) \underset{\text{D}}{=} \sum_{n=0}^{\infty} \frac{1}{n!} \cdot \alpha(n).$$

We have to show that g satisfies (6.14).
First suppose $\alpha \sim_a \beta$.
Then we can find $m \in \mathbb{N}$ such that for all $n > m$, $\alpha(n) = \beta(n)$, which yields,

$$\begin{aligned} g(\alpha) - g(\beta) &= \sum_{n=0}^{\infty} \frac{1}{n!} \cdot \alpha(n) - \sum_{n=0}^{\infty} \frac{1}{n!} \cdot \beta(n), \\ &= \sum_{n=0}^{\infty} \frac{1}{n!} \cdot (\alpha(n) - \beta(n)), \\ &= \sum_{n=0}^{m} \frac{1}{n!} \cdot (\alpha(n) - \beta(n)) \in \mathbb{Q}. \end{aligned}$$

So $g(\alpha) \sim_V g(\beta)$.

Now suppose $g(\alpha) \sim_V g(\beta)$.
Then we can find $a \in \mathbb{Z}$ and $b \in \mathbb{N}$ such that

$$\frac{a}{b} = \sum_{n=0}^{\infty} \frac{1}{n!} \cdot (\alpha(n) - \beta(n)). \tag{6.15}$$

Determine $N \in \mathbb{N}$ such that $b \,|\, N!$. Using (6.15),

$$\frac{a}{b} - \sum_{n=0}^{N} \frac{1}{n!} \cdot (\alpha(n) - \beta(n)) = \sum_{n=N+1}^{\infty} \frac{1}{n!} \cdot (\alpha(n) - \beta(n)).$$

As $N! \cdot (\frac{a}{b} - \sum_{n=0}^{N} \frac{1}{n!} \cdot (\alpha(n) - \beta(n))) \in \mathbb{Z}$, also

$$\sum_{n=N+1}^{\infty} \frac{1}{n!} \cdot (\alpha(n) - \beta(n)) \in \mathbb{Z}.$$

Note that

$$\begin{aligned} |\, N! \cdot \sum_{n=N+1}^{\infty} \frac{1}{n!} \cdot (\alpha(n) - \beta(n)) \,| &\leq N! \cdot \sum_{n=N+1}^{\infty} \frac{1}{n!}, \\ &\leq \frac{N!}{(N+1)!} \cdot \sum_{n=0}^{\infty} \frac{1}{n!}, \\ &\leq \frac{1}{N+1} \cdot e < 1. \end{aligned}$$

Hence,

$$\sum_{n=N+1}^{\infty} \frac{1}{n!} \cdot (\alpha(n) - \beta(n)) = 0. \tag{6.16}$$

As for each k,

$$|\sum_{n=k+1}^{\infty} \frac{1}{n!} \cdot (\alpha(n) - \beta(n)) \,| < \frac{1}{k!},$$

it follows from (6.16) that, for all $k \geq N + 1$, $\alpha(n) = \beta(n)$.
So $\alpha \sim_a \beta$. □

Note that the function g defined in the proof above also satisfies, for all $\alpha, \beta \in \mathcal{C}$,

$$\alpha \not\#_a \beta \rightarrow \forall q \in \mathbb{Q} [\, g(\alpha) - g(\beta) \# q \,].$$

Hence, by composing the function defined in Theorem 6.11 with g, we conclude, there exists a function $h : \mathcal{C} \rightarrow \mathbb{R}$, such that

$$\forall \alpha, \beta \in \mathcal{C} [\, \alpha \# \beta \rightarrow \forall q \in \mathbb{Q} [\, h(\alpha) - h(\beta) \# q \,]\,]. \tag{6.17}$$

A function satisfying (6.17) is called *Vitali independent*. This notion is quite strong. We also introduce a weaker notion.

Definition 6.15. Let $h : \mathcal{C} \to \mathbb{R}$.
h is *weakly Vitali independent* if and only if for all $\alpha, \beta \in \mathcal{C}$, for all $q \in \mathbb{Q}$ with $q \neq 0$, $h(\alpha) + q \# h(\beta)$.

Let us first show that this notion is indeed weaker. We start with a lemma.

Lemma 6.16. *For all x, y, $z \in \mathbb{R}$,*

$$\textit{if } x \# y, \textit{ then } z \# x \vee z \# y \tag{6.18}$$

Proof. Let x, y, $z \in \mathbb{R}$ and suppose $x \# y$.
We may assume, without loss of generality, $x < y$.
Recall, every real number is a shrinking and shriveling sequence of intervals with rational endpoints. As $x \# y$, we can find, by definition, $n \in \mathbb{N}$ with

$$\neg\,(\,x(n) \approx y(n)\,).$$

Find $m \in \mathbb{N}$, such that

$$z(m)'' - z(m)' < y(n)' - x(n)''.$$

Define $k \underset{\text{D}}{=} max(n, m)$, then

$$\neg\,(\,z(k) \approx x(k)\,) \;\vee\; \neg\,(\,z(k) \approx y(k)\,).$$

Hence $z \# x \vee z \# y$. □

The basic idea of the following proof will reoccur a number of times in the proofs of Section 7.3 and relies on Lemma 6.16.

Theorem 6.17. *Every Vitali independent function from $\mathcal{C}$ to $\mathbb{R}$ is weakly Vitali independent.*

Proof. Let $h : \mathcal{C} \to \mathbb{R}$ be Vitali independent.
Let $\alpha, \beta \in \mathcal{C}$ and $q \in \mathbb{Q}$ with $q \neq 0$. We have to show:

$$h(\alpha) \# h(\beta) + q.$$

As $q \neq 0$,

$$h(\beta) + q \# h(\beta).$$

Hence, by applying Lemma 6.16 with $x = h(\beta) + q,\ y = h(\beta)$ and $z = h(\alpha)$,

$$h(\alpha) \# h(\beta) + q \ \vee\ h(\alpha) \# h(\beta).$$

In the first case, we are immediately ready. In the second case, $\alpha \# \beta$ and we can apply the fact that h is Vitali independent, which yields

$$h(\alpha) \# h(\beta) + q.$$

So, h is weakly Vitali independent. □

Every weakly Vitali independent function is *positively Vitali incomplete.* That is, for every weakly Vitali independent function $h : \mathcal{C} \to \mathbb{R}$, there exists a real number x such that, for all $\alpha \in \mathcal{C}$, for all $q \in \mathbb{Q}$, $x \# h(\alpha) + q$. In the proof of this statement (Theorem 6.19) we use the following notion.

Definition 6.18. Let $x \in \mathbb{R}$ and $a, b \in \mathbb{Q}$. We say $[a, b]$ is *apart from* x, notation $[a, b] \# x$, if and only if $x < a$ or $x > b$.

Theorem 6.19. *Every weakly Vitali independent function from $\mathcal{C}$ to $\mathbb{R}$ is positively Vitali incomplete.*

Proof. Let $h : \mathcal{C} \to \mathbb{R}$ be weakly Vitali independent.
Then, for all $\alpha, \beta \in \mathcal{C}$, for all $q \in \mathbb{Q}$ with $q \neq 0$,

$$h(\alpha) \# h(\beta) + q. \tag{6.19}$$

Let $q_0, q_1, \ldots$ be an enumeration of the set of all rational numbers that are unequal to 0 (we denote this set by $\mathbb{Q}_{\neq 0}$).
We will define a shrinking and shriveling sequence of intervals with rational endpoints (i.e. a real number) inductively, such that in step n we ensure:

$$\forall \alpha \in \mathcal{C}\, [\, h(\alpha) + q_n \# [\, a_n, b_n \,]\,].$$

To start,

$$[\, a_{-1}, b_{-1} \,] \underset{\mathrm{D}}{=} [\, 1, 2 \,].$$

Now let $n \in \mathbb{N}$ and assume that $[\, a_{n-1}, b_{n-1} \,]$ has already been defined.
Determine $r \in \mathbb{Q}$ such that $r \neq q_n$ and

$$h(\underline{0}) + r \in [\, a_{n-1}, b_{n-1} \,].$$

It follows from (6.19) that

$$\forall \alpha {\in} \mathcal{C}\, [\, h(\alpha) \# h(\underline{0}) + (r - q_n)\,].$$

By applying the Fan Theorem, we can determine $N \in \mathbb{N}$ such that

$$\forall \alpha {\in} \mathcal{C}\, [\, |\, (h(\alpha) + q_n) - (h(\underline{0}) + r)\, | > \tfrac{1}{N}\,].$$

Determine $a_n, b_n \in \mathbb{Q}$[2] with

(i). $h(\underline{0}) + r \in [\, a_n, b_n \,] \subseteq [\, a_{n-1}, b_{n-1} \,]$,

(ii). $0 < b_n - a_n < min(\, \frac{1}{N}, \frac{1}{2}(b_{n-1} - a_{n-1})\,)$.

The so-defined sequence satisfies, for all $\alpha \in \mathcal{C}$, for all $n \in \mathbb{N}$,

$$h(\alpha) + q_n \# [\, a_n, b_n \,]. \tag{6.20}$$

Define $x = ([\, a_0, b_0 \,], [\, a_1, b_1 \,], \ldots)$.
As, for all $n \in \mathbb{N}$, $x \in [\, a_n, b_n \,]$, it follows from (6.20) that,

$$\forall \alpha {\in} \mathcal{C}\, \forall q {\in} \mathbb{Q}\, [\, x \# h(\alpha) + q \,]$$

and h is positively Vitali incomplete. □

In Section 7.3 we will see that there is also a connection between weakly Vitali independent functions and the existence of a Hamel basis for $\mathbb{R}$ over $\mathbb{Q}$.

[2] Note that finding such $a_n, b_n \in \mathbb{Q}$ does not require any form of the Axiom of Choice. One can explicitly give a general algorithm to define them.

6.3 Equivalence relations and choice functions

In the previous chapter we have discussed choice functions. There is a connection between choice functions and equivalence relations. We will explain this connection for equivalence relations on $\mathcal{N}$ and $\mathcal{N}$-parametrizations, but the same connection exists for equivalence relations on $\mathcal{C}$ and $\mathcal{C}$-parametrizations.

For any equivalence relation $\sim$ on $\mathcal{N}$, the collection $\mathcal{F}$ of equivalence classes of $\sim$ is a collection of non-empty subsets of $\mathcal{N}$. This collection can be parametrized by the map $V : \mathcal{N} \to \mathcal{F}$, defined by, for all $\alpha \in \mathcal{N}$,

$$V : \alpha \mapsto \{\beta \in \mathcal{N} \mid \alpha \sim \beta\}.$$

A choice function $f : \mathcal{N} \to \mathcal{N}$ on $\mathcal{F}$ corresponding to V satisfies

(i). $\forall \alpha \in \mathcal{N}\, [\, f(\alpha) \in V(\alpha)\,]$,

(ii). $\forall \alpha, \beta \in \mathcal{N}\, [\, V(\alpha) = V(\beta) \to f(\alpha) = f(\beta)\,]$.

Using this, one easily proves that,

(i). $\forall \alpha \in \mathcal{N}\, [\, f(\alpha) \sim \alpha\,]$,

(ii). $\forall \alpha, \beta \in \mathcal{N}\, [\, f(\alpha) \sim f(\beta) \to f(\alpha) = f(\beta)\,]$.

Hence, for any choice function f on the collection of equivalence classes of $\sim$, $Ran(f)$ is a set of representatives for $\sim$.

On the other hand, for every collection $\mathcal{F}$ of non-empty subsets of a set X parametrized by a function $V : \mathcal{N} \to \mathcal{F}$, one may define an equivalence relation $\sim$ on $\mathcal{N}$ by, for all $\alpha, \beta \in \mathcal{N}$,

$$\alpha \sim \beta \leftrightarrows V(\alpha) = V(\beta). \tag{6.21}$$

A choice function on $\mathcal{F}$ has to satisfy

$$\forall \alpha, \beta \in \mathcal{N}\, [\, V(\alpha) = V(\beta) \to f(\alpha) = f(\beta)\,],$$

which means a choice function has to respect the equivalence relation just defined in (6.21).

6.3.1 Equivalence relation given by the inhabitedly enumerable subsets of $\mathbb{N}$

In section 5.1 we studied the collection of inhabitedly enumerable subset of $\mathbb{N}$. This collection is $\mathcal{N}$-parametrized by the map

$$\alpha \mapsto IE_\alpha,$$

where $IE_\alpha = \{n \in \mathbb{N} \mid \exists k \in \mathbb{N}\, [\, \alpha(k) = n\,]\}$.
As explained above, this gives rise to the following equivalence relation on $\mathcal{N}$:

$$\alpha \sim \beta \leftrightarrows IE_\alpha = IE_\beta \qquad \text{for all } \alpha, \beta \in \mathcal{N}.$$

Does this equivalence relation have a weak set of representatives?
Let us first consider an easier case. We study this equivalence relation on $\mathcal{C}$. Then the answer is 'yes'.

Theorem 6.20. *The equivalence relation on $\mathcal{C}$ defined by*

$$\alpha \sim \beta \leftrightarrows IE_\alpha = IE_\beta$$

has a weak set of representatives.

Proof. Define $A \subseteq \mathcal{C}$ by

$$A \underset{\text{D}}{=} \{\underline{0}, \underline{1}, \langle 1 \rangle * \underline{0}\}.$$

We will show that A is a weak set of representatives for $\sim$.
It is clear that A is independent. So we only have to show:

$$\forall \alpha \in \mathcal{C} \neg\neg \exists \beta \in A [\, \alpha \sim \beta \,].$$

Let $\alpha \in \mathcal{C}$ and suppose $\neg \exists \beta \in A [\, \alpha \sim \beta \,]$. Then

$$\neg (\alpha \sim \underline{0}) \wedge \neg (\alpha \sim \underline{1}) \wedge \neg (\alpha \sim \langle 1 \rangle * \underline{0})$$

Note that

$$\begin{aligned} \neg (\alpha \sim \underline{0}) &\leftrightarrows \neg (\forall n \exists m [\, \alpha(n) = \underline{0}(m)] \wedge \forall m \exists n [\, \underline{0}(m) = \alpha(n)] \,) \\ &\leftrightarrows \neg (\forall n [\, \alpha(n) = 0 \,] \wedge \exists n [\, \alpha(n) = 0 \,] \,) \\ &\leftrightarrows \neg\neg \exists n [\, \alpha(n) = 1 \,] \end{aligned} \tag{6.22}$$

and

$$\neg (\alpha \sim \underline{1}) \leftrightarrows \neg\neg \exists n [\, \alpha(n) = 0 \,] \tag{6.23}$$

and

$$\begin{aligned} \neg (\alpha \sim \langle 1 \rangle * \underline{0}) &\leftrightarrows \neg (\forall n \exists m [\, \alpha(n) = (\langle 1 \rangle * \underline{0})(m) \,] \wedge \forall m \exists n [\, (\langle 1 \rangle * \underline{0})(m) = \alpha(n) \,]) \\ &\leftrightarrows \neg (\, \exists n [\, \alpha(n) = 0 \,] \wedge \exists n [\, \alpha(n) = 1 \,] \,). \end{aligned} \tag{6.24}$$

For any two propositions A and B,

if $\neg\neg A \wedge \neg\neg B$, then $\neg\neg(A \wedge B)$.

Hence by defining

$$A \underset{\text{D}}{=} \exists n [\, \alpha(n) = 0 \,] \quad \text{and} \quad B \underset{\text{D}}{=} \exists n [\, \alpha(n) = 1 \,],$$

we see that (6.22), (6.23) and (6.24) lead to a contradiction.
So $\neg\neg \exists \beta \in A [\, \alpha \sim \beta \,]$ and A is a weak set of representatives for $\sim$. □

We move on to $\mathcal{N}$. What can we say about the existence of a weak set of representatives for $\sim$ in this case? Well, we may define a subset A of $\mathcal{N}$ by, for all $\alpha \in \mathcal{N}$,

$$\alpha \in A \leftrightarrows \forall m \in \mathbb{N}\,[\,\alpha(m) \leq \alpha(m{+}1) \wedge (\alpha(m) = \alpha(m{+}1) \;\rightarrow\; \alpha(m{+}1) = \alpha(m{+}2))\,].$$

The set A consists of all increasing sequences of natural numbers, which remain constant as soon as a repetition occurs. Clearly A is independent, that is, for all $\alpha, \beta \in A$, if $IE_\alpha = IE_\beta$, then $\alpha = \beta$. But can we also prove:

$$\forall \alpha {\in} \mathcal{N}\, \neg\neg \exists \gamma {\in} A\,[\,IE_\gamma = IE_\alpha\,]? \tag{6.25}$$

Inspired by an idea of Solovay, Moschovakis has proven

$$\forall \alpha {\in} \mathcal{N}\, \neg\neg \exists \beta \in \mathcal{N}\,[\,IE_\alpha = D_\beta\,], \tag{6.26}$$

using both Markov's Principle and Brouwer's Principle of Bar Induction [8]. Under the assumption of Markov's Principle (6.25) and (6.26) are equivalent. We will prove this equivalence first, after which we present Moschovakis' proof of (6.26).

Definition 6.21. *Markov's Principle* (MP)
Markov's Principle states, for every decidable subset P of $\mathbb{N}$,

$$\text{if } \neg\neg\exists n {\in} \mathbb{N}\,[\,P(n)\,], \text{ then } \exists n {\in} \mathbb{N}\,[\,P(n)\,].$$

Brouwer does not accept this principle. Using the so-called *Brouwer-Kripke axiom*, he constructed an infinite sequence of natural numbers $\alpha(0), \alpha(1), \ldots$ such that $\neg\neg\exists n\,[\,\alpha(n) = 0\,]$, but $\exists n\,[\,\alpha(n) = 0\,]$ is reckless. For a full explanation of this argument the reader is referred to [4].
Markov's Principle does not contradict any of the (intuitionistic) axioms we assume in this thesis.

Theorem 6.22. *Under the assumption of Markov's Principle,*

$$\forall \alpha {\in} \mathcal{N}\, \neg\neg \exists \gamma {\in} A\,[\,IE_\gamma = IE_\alpha\,] \;\Leftrightarrow\; \forall \alpha {\in} \mathcal{N}\, \neg\neg \exists \beta \in \mathcal{N}\,[\,IE_\alpha = D_\beta\,].$$

Proof.
$\Rightarrow$) This follows immediately from the fact that, for all $\gamma \in A$, IE_γ is decidable.

$\Leftarrow$) Assume $\forall \alpha {\in} \mathcal{N}\, \neg\neg \exists \beta \in \mathcal{N}\,[\,IE_\alpha = D_\beta\,]$.
Let $\alpha \in \mathcal{N}$ and suppose

$$\neg \exists \gamma {\in} A\,[\,IE_\gamma = IE_\alpha\,]. \tag{6.27}$$

We will derive a contradiction.
Note that for any two statements X and Y,

$$\text{if } X \vdash \neg Y, \text{ then } \neg\neg X \vdash \neg Y.$$

Hence we can assume there exists $\beta \in \mathcal{N}$, such that

$$IE_\alpha = D_\beta.$$

Then it follows from (6.27) that $\neg\exists\gamma\in A\,[\,IE_\gamma = D_\beta\,]$, and therefore,

$$\forall\gamma\in A\,\neg\neg\exists n\in\mathbb{N}\,[\,\neg(n \in IE_\gamma \leftrightarrows n \in D_\beta)\,].$$

The statement between parentheses is a decidable property of the natural numbers, so we can apply Markov's Principle, and conclude

$$\forall\gamma\in A\,\exists n\in\mathbb{N}\,[\,\neg(n \in IE_\gamma \leftrightarrows n \in D_\beta)\,]. \tag{6.28}$$

As $\alpha(0) \in IE_\alpha = D_\beta$, D_β is inhabited and we can find the smallest natural number p with $p \in D_\beta$. Note that $\underline{p} \in A$, so using (6.28), we can determine $k \in \mathbb{N}$ with

$$\neg(k \in IE_{\underline{p}} \leftrightarrows k \in D_\beta).$$

As $p \in D_\beta$, we conclude

$$k \notin IE_{\underline{p}} \wedge k \in D_\beta.$$

Applying this idea repeatedly we define $\gamma \in \mathcal{N}$ by,

$$\begin{aligned} \gamma(0) &\underset{\mathrm{D}}{=} k, \\ \gamma(n+1) &\underset{\mathrm{D}}{=} \mu i\,[\,i \in D_\beta \wedge i \notin IE_{\overline{\gamma}n*\underline{\gamma(n)}}\,] \qquad \text{for all } n\in\mathbb{N}. \end{aligned}$$

Then $\gamma \in A$ and $IE_\gamma = D_\beta$. Contradiction.
Hence,

$$\neg\neg\exists\gamma\in A\,[\,IE_\gamma = IE_\alpha\,].$$

□

As said, Moschovakis also uses Brouwer's Principle of Bar Induction in her proof. This principle states the following.

Definition 6.23. *Brouwer's Principle of Bar Induction* (BI)
Brouwer's Principle of Bar Induction states:

if $B \subseteq \mathbb{N}^*$ is a decidable bar in $\mathcal{N}$ and A is a subset of $\mathbb{N}$ such that

(i). $B \subseteq A$,

(ii). for all $s \in \mathbb{N}^*$, if, for all $n \in \mathbb{N}$, $s * \langle n \rangle \in A$, then $s \in A$,

then $\langle\,\rangle \in A$.

We will not discuss the justification of this principle here, but again we refer the reader to [4].

Theorem 6.24. *Assuming Markov's Principle and Bar Induction,*

$$\forall \gamma \in \mathcal{N} \neg\neg \exists \alpha \in \mathcal{N} [D_\alpha = IE_\gamma].$$

Proof. Let $\gamma \in \mathcal{N}$.
Suppose

$$\neg \exists \alpha \in \mathcal{N} [D_\alpha = IE_\gamma]. \tag{6.29}$$

We will derive a contradiction.
From (6.29) it follows that,

$$\forall \alpha \in \mathcal{N} \neg \forall n \in \mathbb{N} [\alpha(n) \neq 0 \leftrightarrows \exists k [\gamma(k) = n]]. \tag{6.30}$$

Let $\alpha \in \mathcal{N}$.
Define, for all $n \in \mathbb{N}$,

$$R_\alpha(n) \underset{\text{D}}{=} \alpha(n) \neq 0 \wedge \neg \exists k [\gamma(k) = n],$$
$$S_\alpha(n) \underset{\text{D}}{=} \alpha(n) = 0 \wedge \exists k [\gamma(k) = n].$$

Claim 1: $\neg\neg \exists n [R_\alpha(n) \vee S_\alpha(n)]$.

Suppose $\neg \exists n [R_\alpha(n) \vee S_\alpha(n)]$.

Then, for all $n \in \mathbb{N}$,

$$\neg R_\alpha(n) \wedge \neg S_\alpha(n)$$

In that case,

$$\forall n \in \mathbb{N} [\alpha(n) \neq 0 \leftrightarrows \exists k [\gamma(k) = n]]. \tag{6.31}$$

For let $n \in \mathbb{N}$.

$\rightarrow$) Suppose $\alpha(n) \neq 0$.
As $\neg R_\alpha(n)$, $\neg\neg \exists k [\gamma(k) = n]$.
Applying MP yields $\exists k [\gamma(k) = n]$.

$\leftarrow$) Suppose $\exists k [\gamma(k) = n]$.
As $\neg S_\alpha(n)$, $\alpha(n) \neq 0$.

So we have proved (6.31), but this contradicts (6.30).

Hence $\neg\neg \exists n [R_\alpha(n) \vee S_\alpha(n)]$.

Define a subset B of $\mathbb{N}^*$ by: for all $s \in \mathbb{N}^*$, $s \in B$ iff for all $j < lg(s)$, either,

(i). $s(j) = 0 \wedge \exists k < lg(s) [\gamma(k) = j]$, or

(ii). $s(j) \neq 0 \wedge (\gamma(s(j) - 1) \neq j \vee \exists k < s(j) - 1 [\gamma(k) = j])$.

To show B is a bar in $\mathcal{N}$, we first prove the following.

Claim 2: $\exists n\,[\,R_\alpha(n)\;\vee\;S_\alpha(n)\,]\;\rightarrow\;\exists n\,[\,\overline{\alpha}n \in B\,]$.

Suppose: $\exists n\,[\,R_\alpha(n)\;\vee\;S_\alpha(n)\,]$.
Determine such n. There are two possibilities.

- $R_\alpha(n)$
  In this case, $\alpha(n) \neq 0\;\wedge\;\forall k\,[\,\gamma(k) \neq n\,]$.
  So in particular, $\gamma(\alpha(n)-1) \neq n$.
  Hence $\overline{\alpha}(n+1) \in B$, as $j = n$ satisfies (ii).

- $S_\alpha(n)$
  Then, by definition, $\alpha(n) = 0$ and $\exists k\,[\,\gamma(k) = n\,]$.
  Determine such k.
  Then $\overline{\alpha}(max(n,k)+1) \in B$, as $j = n$ satisfies (i).

Combining Claim 1 and Claim 2 yields

$$\forall \alpha \in \mathcal{N}\,\neg\neg\exists n\,[\,\overline{\alpha}n \in B\,].$$

As B is a decidable subset of $\mathbb{N}^*$, we can apply MP and we conclude

$$\forall \alpha \in \mathcal{N}\,\exists n\,[\,\overline{\alpha}n \in B\,]$$

and B is a bar in $\mathcal{N}$.

Define a subset A of $\mathbb{N}^*$ by, for all $s \in \mathbb{N}^*$,

$$\begin{aligned} s \in A \;\leftrightarrows\;& \exists j < lg(s)\,[\,(\,s(j) = 0 \wedge \exists k\,[\,\gamma(k) = j\,]\,) \vee \\ & (\,s(j) \neq 0\;\wedge\;[\gamma(s(j)-1) = j\;\rightarrow\;\exists y < s(j)-1\,[\,\gamma(y) = j\,]\,]\,)\,] \end{aligned}$$

Claim 3: $\forall s \in \mathbb{N}^*\,[\,\forall n\,[\,s * \langle n\rangle \in A\,]\;\rightarrow\;s \in A\,]$.

Let $s \in \mathbb{N}^*$ and assume, for all $n \in \mathbb{N}$, $s * \langle n\rangle \in A$.
Suppose $\neg\,(s \in A)$.

As $s * \langle 0\rangle \in A$, there exists $k \in \mathbb{N}$ with

$$\gamma(k) = lg(s). \tag{6.32}$$

For all n, $s * \langle n+1\rangle \in A$, and therefore, for all n,

$$\gamma(n) = lg(s)\;\rightarrow\;\exists y < n\,[\,\gamma(y) = lg(s)\,]. \tag{6.33}$$

Combining (6.32) and (6.33), we can define a strictly decreasing sequence of natural numbers. Contradiction.

So $\neg\neg(s \in A)$.

Notice that moving '$\exists k$' to the front in the definition of A does not make a difference. Therefore, $\neg\neg(s \in A)$ is a statement of the form

$$\neg\neg\exists k\,[\,\text{decidable condition}\,]$$

and we can apply MP.
Hence $s \in A$.

It is clear that $B \subseteq A$. So A satisfies all requirements of Definition 6.23 and we conclude $A(\langle\,\rangle)$. Contradiction!

We conclude: $\neg\neg\exists\alpha\in\mathcal{N}\,\forall n\in\mathbb{N}\,[\,\alpha(n) \neq 0 \leftrightarrows \exists k \in \mathbb{N}\,[\,\gamma(k) = n\,]\,]$. □

Chapter 7

Hamel basis

In this chapter we study an other equivalent of the Axiom of Choice. Before we can formulate this statement we need some definitions.
Let F be a field. A *vector space over* F is a set V with two operations, addition and scalar multiplication (multiplication with elements of F). These operations have to satisfy a number of criteria, which we do not mention here.

Definition 7.1. Let V be a vector space over a field F.
A subset B of V is a *(Hamel) basis* for V over F if and only if B satisfies the following conditions:

(i). B is *linearly independent*, that is, for all $n \in \mathbb{N}$, for any finite sequence $b_0, \ldots, b_n \in B$ with $\forall i < j\, [\, b_i \neq b_j\,]$, for all $t_0, \ldots, t_n \in F$,

$$\text{if } \neg \forall i\, [\, t_i = 0\,], \text{ then } t_0 \cdot b_0 + \ldots + t_n \cdot b_n \neq 0.$$

(ii). B is *complete*, that is, for all $x \in V$, there exist $n \in \mathbb{N}$, $t_0, \ldots, t_n \in F$ and $b_0, \ldots b_n \in B$, such that

$$x = t_0 \cdot b_0 + \ldots + t_n \cdot b_n.$$

Using the Axiom of Choice, the classical mathematician can prove,

$$\textit{every vector space has a basis.} \tag{7.1}$$

In 1984 Andreas Blass has proven that (7.1) is classically equivalent to the Axiom of Choice. In this chapter we study $\mathbb{R}$ as a vector space over $\mathbb{Q}$ and investigate the existence of a basis.

7.1 A Hamel independent function from $\mathcal{C}$ to $\mathbb{R}$

When trying to find a basis for $\mathbb{R}$ over $\mathbb{Q}$ one could just attempt to construct larger and larger independent subsets of $\mathbb{R}$. In this section we will show that there exists an independent subset of $\mathbb{R}$ that is 'as big as $\mathcal{C}$'. We start with a definition.

Definition 7.2. Let $h : \mathcal{C} \to \mathbb{R}$.
We say h is *Hamel independent* if and only if for all $n \in \mathbb{N}$, for all finite sequences $\alpha_0, \ldots, \alpha_n$ in $\mathcal{C}$ satisfying $\forall i, j\,[\, i \neq j \to \alpha_i \# \alpha_j\,]$, for all $t_0, \ldots, t_n \in \mathbb{Q}$,

$$\text{if } \neg\forall i\,[\, t_i = 0\,], \text{ then } t_0 h(\alpha_0) + \ldots + t_n h(\alpha_n) \# 0.$$

In the following theorem we construct a Hamel independent function $h : \mathcal{C} \to \mathbb{R}$. We will do so by first defining a function f from $\{0,1\}^*$, the collection of all finite sequences of zeros and ones, to the collection of all intervals with rational endpoints. Then we define, for each $\alpha \in \mathbb{C}$,

$$h(\alpha) \underset{\text{D}}{=} (f(\overline{\alpha}0), f(\overline{\alpha}1), f(\overline{\alpha}2), \ldots).$$

Hence f has to be defined in such a way that, first of all, for each $\alpha \in \mathcal{C}$, $h(\alpha)$ is a real number. Second, we have to guarantee that h is Hamel independent.

To formulate the following proof we introduce two operations on the collection of intervals with rational endpoints.

Definition 7.3. Let $a, b, c, d, e \in \mathbb{Q}$.
Addition of two intervals with rational endpoints is defined by:

$$[a, b] + [c, d] \underset{\text{D}}{=} [\, a + c,\ b + d\,],$$

and scalar multiplication is defined by:

$$\begin{aligned} e \cdot [a, b] &\underset{\text{D}}{=} [\, e \cdot a,\ e \cdot b\,] && \text{if } e \geq 0, \\ &\underset{\text{D}}{=} [\, e \cdot b,\ e \cdot a\,] && \text{if } e < 0. \end{aligned}$$

Theorem 7.4. *There exists a Hamel independent function $h : \mathcal{C} \to \mathbb{R}$.*

Proof. Let $q_0, q_1, \ldots$ be an enumeration of $\mathbb{Q}$.
For each n, let $s_0^n, \ldots, s_{2^n-1}^n$ be an enumeration of $\{0,1\}^n$.

We will define a function $f : \{0,1\}^* \to \{\, [\, a, b\,] \mid a, b \in \mathbb{Q} \mid a < b\,\}$ satisfying:

(i). $\forall s \in \{0,1\}^*\, \forall i \in \{0,1\}\, [\, f(s * \langle i \rangle) \subseteq f(s)\,]$

(ii). $\forall s \in \{0,1\}^*\, \forall i \in \{0,1\}\, [\, lg(f(s * \langle i \rangle)) \leq \frac{1}{2}\, lg(f(s))\,]$

(iii). $\forall n\, \forall i\, \forall p_0, \ldots, p_{2^n-1} \leq n\, [\, \neg\forall j\, [\, q_{p_j} = 0\,] \to (\sum_{j \neq i} q_{p_j} f(s_j^n)) \cap q_{p_i} f(s_i^n) = \emptyset\,]$

We will get to the exact definition of this function later. First note that properties (i) and (ii) of f guarantee that, for each $\alpha \in \mathcal{C}$, $(f(\overline{\alpha}0),\ f(\overline{\alpha}1),\ f(\overline{\alpha}2), \ldots)$ is a shrinking and shriveling sequence of intervals with rational endpoints. This means we can define a function $h : \mathcal{C} \to \mathbb{R}$ by

$$(h(\alpha))(n) \underset{\text{D}}{=} f(\overline{\alpha}n).$$

To see that the so-defined h is Hamel independent, let $\alpha_0, \ldots, \alpha_n$ be a finite sequence in $\mathcal{C}$ satisfying $\forall i, j\, [\, i \neq j \rightarrow \alpha_i \# \alpha_j\,]$ and let $t_0, \ldots, t_n \in \mathbb{Q}$ with $\neg \forall i\, [\, t_i = 0\,]$. We have to prove:

$$t_0 h(\alpha_0) + \ldots + t_n h(\alpha_n) \# 0$$

Determine $p_0, \ldots, p_n$ such that $q_{p_n} = -t_n$ and, for each $i < n$, $q_{p_i} = t_i$.
As $\forall i, j\, [\, i \neq j \rightarrow \alpha_i \# \alpha_j\,]$, we can determine $k \in \mathbb{N}$ such that

$$\forall i, j\, [\, i \neq j \rightarrow \overline{\alpha_i} k \neq \overline{\alpha_j} k\,]$$

Define $M \underset{\mathrm{D}}{=} max(\{\, p_i \mid 0 \leq i \leq n\, \} \cup \{k\})$.
By property (iii) of f,

$$\Big(\sum_{j<n} q_{p_j} f(\overline{\alpha_j} M)\Big) \cap q_{p_n} f(\overline{\alpha_n} M) = \emptyset.$$

Hence,

$$t_0 h(\alpha_0) + \ldots + t_{n-1} h(\alpha_{n-1}) \# - t_n h(\alpha_n).$$

And we conclude

$$t_0 h(\alpha_0) + \ldots + t_n h(\alpha_n) \# 0.$$

So we are done once we have defined a function f with the three properties listed above. We define f inductively, starting with:

$$f(\langle\, \rangle) \underset{\mathrm{D}}{=} [\, 1, 2\,].$$

Suppose f has been defined for all $s \in \{0,1\}^n$.
While defining f for all sequences of length $n+1$ we have to ensure:

$$\forall i\, \forall p_0, \ldots, p_{2^n - 1} \leq n+1\, [\, \neg \forall j\, [\, q_{p_j} = 0\,] \rightarrow \Big(\sum_{j \neq i} q_{p_j} f(s_j^{n+1})\Big) \cap q_{p_i} f(s_i^{n+1}) = \emptyset\,].$$

Let $m_0, \ldots, m_k$ be an enumeration of all finite sequences of the form

$$\langle i, r_0, \ldots, r_{2^{n+1}-1} \rangle,$$

where $0 \leq i < 2^{n+1}$, and for all $j < 2^{n+1}$, $r_j \in \{q_0, \ldots q_{n+1}\}$, and $r_i \neq 0$.

We define a sequence of functions $g_{-1}, \ldots, g_k : \{0,1\}^{n+1} \rightarrow \{\, [\, a, b\,] \mid a, b \in \mathbb{Q}\, \}$ such that for all $s \in \{0,1\}^n$, for $i \in \{0,1\}$:

$$f(s) = g_{-1}(s * \langle i \rangle) \supseteq g_0(s * \langle i \rangle) \supseteq \ldots \supseteq g_k(s * \langle i \rangle).$$

And for each $0 \leq l \leq k$, with $m_l = \langle i, r_0, \ldots, r_{2^{n+1}-1} \rangle$:

$$\Big(\sum_{j \neq i} r_j\, g_l(s_j^{n+1})\Big) \cap r_i\, g_l(s_i^{n+1}) = \emptyset. \tag{7.2}$$

We will get to the exact definition of the g_l in a moment. Assuming we have defined such a sequence of functions, define, for all $s \in \{0,1\}^{n+1}$,

$$f(s) \underset{\text{D}}{=} [\, g'_k(s),\, g'_k(s) + \frac{1}{2} \cdot lg(\, g_k(s)\,)\,].$$

Then $f(s) \subseteq g_k(s)$ and, because of (7.2), f satisfies (iii) for all sequences of length $n+1$.
Furtermore, for all $s \in \{0,1\}^n$, $i \in \{0,1\}$:

$$f(s * \langle i \rangle) \subseteq g_k(s * \langle i \rangle) \subseteq f(s).$$

And

$$lg(\, f(s * \langle i \rangle)\,) \leq \tfrac{1}{2}\, lg(\, g_k(s * \langle i \rangle)\,) \leq \tfrac{1}{2}\, lg(\, f(s)\,).$$

So f satisfies all requirements.

Now let us come to the precise definition of the g_l.
To begin with, define, for each $s \in \{0,1\}^n$, $i \in \{0,1\}$,

$$g_{-1}(s * \langle i \rangle) \underset{\text{D}}{=} f(s).$$

For $0 \leq l \leq k$, g_l is defined as follows.

Say $m_l = \langle i, r_0, \ldots, r_{2^{n+1}-1} \rangle$.
Define for all $j \neq i$:

$$g_l(s_j^{n+1}) \underset{\text{D}}{=} [\, g'_{l-1}(s_j^{n+1}),\, g'_{l-1}(s_j^{n+1}) + x_j\,],$$

where $x_j \underset{\text{D}}{=} min(\, lg(\, g_{l-1}(s_j^{n+1})\,),\, \frac{1}{r_j} \cdot \frac{1}{2^{n+1}} \cdot r_i \cdot lg(\, g_{l-1}(s_i^{n+1})\,)\,)$.

Then:

$$\begin{aligned} lg(\sum_{j \neq i} r_j \cdot g_l(s_j^{n+1})\,) &\leq \sum_{j \neq i} r_j \cdot \frac{1}{r_j} \cdot \frac{1}{2^{n+1}} \cdot r_i \cdot lg(\, g_{l-1}(s_i^{n+1})\,), \\ &\leq \frac{2^{n+1}-1}{2^{n+1}} \cdot r_i \cdot lg(\, g_{l-1}(s_i^{n+1})\,), \\ &< lg(\, r_i \cdot g_{l-1}(s_i^{n+1})\,). \end{aligned}$$

Hence we can find $a, b \in \mathbb{Q}$ such that

1. $[\, a, b\,] \subseteq r_i \cdot g_{l-1}(s_i^{n+1})$,
2. $(\sum_{j \neq i} r_j \cdot g_l(s_j^{n+1})) \cap [\, a, b\,] = \emptyset$.

Define $g_l(s_i^{n+1}) \underset{\text{D}}{=} \frac{1}{r_i} \cdot [\, a, b\,]$.

Then g_l satifies (7.2).
So we can define a function f satisfying (i), (ii) and (iii) and as we have seen, this enables us to define a Hamel independent function $h : \mathcal{C} \to \mathbb{R}$. □

7.2 A Hamel complete function from $\mathcal{C}$ to $\mathbb{R}$?

The result of the previous section makes us wonder whether we can also define a function $h : \mathcal{C} \to \mathbb{R}$ that is not only Hamel independent, but also complete.

Definition 7.5. Let $h : \mathcal{C} \to \mathbb{R}$.
We say h is *Hamel complete* if and only if for all $x \in \mathbb{R}$, there exist $n \in \mathbb{N}$, $\alpha_0, \ldots, \alpha_n \in \mathcal{C}$ satisfying $\forall i, j\,[\,i \neq j \;\to\; \alpha_i \;\#\; \alpha_j\,]$, $t_0, \ldots, t_n \in \mathbb{Q}$, such that

$$x = t_0 h(\alpha_0) + \ldots + t_n h(\alpha_n).$$

We say h is *positively Hamel incomplete* if and only if there exists $x \in \mathbb{R}$ such that, for all $n \in \mathbb{N}$, for all $\alpha_0, \ldots, \alpha_n \in \mathcal{C}$ satisfying $\forall i, j\,[\,i \neq j \;\to\; \alpha_i \;\#\; \alpha_j\,]$, for all $t_0, \ldots, t_n \in \mathbb{Q}$,

$$x \;\#\; t_0 h(\alpha_0) + \ldots + t_n h(\alpha_n).$$

There is no function from $\mathcal{C}$ to $\mathbb{R}$ that is both Hamel independent and Hamel complete. And even stronger: every Hamel independent function from $\mathcal{C}$ to $\mathbb{R}$ is positively Hamel incomplete.

Theorem 7.6. *Every Hamel independent function $h : \mathcal{C} \to \mathbb{R}$ is positively Hamel incomplete.*

Proof. Let $h : \mathcal{C} \to \mathbb{R}$ be Hamel independent.
We will construct a real number x such that, for all $n \in \mathbb{N}$, for all $\alpha_0, \ldots, \alpha_n \in \mathcal{C}$ satisfying $\forall i, j\,[\,i \neq j \;\to\; \alpha_i \;\#\; \alpha_j\,]$, for all $t_0, \ldots, t_n \in \mathbb{Q}$,

$$x \;\#\; t_0 h(\alpha_0) + \ldots + t_n h(\alpha_n).$$

Let $m_0, m_1, m_2, \ldots$ be an enumeration of all the finite sequences of the form $\langle s_0, \ldots, s_k, q_0, \ldots, q_k \rangle$, where

1. for all $i \leq k$, $s_i \in \{0,1\}^*$ and $q_i \in \mathbb{Q}$,
2. $\forall i, j \leq k\,[\,i \neq j \;\to\; s_i \perp s_j\,]$,
3. $\exists \alpha \in \mathcal{C}\, \forall i \leq k\,[\,\neg\,(s_i \sqsubset \alpha)\,]$.

We will define a sequence of intervals with rational endpoints $[\,a_{-1}, b_{-1}\,]$, $[\,a_0, b_0\,]$, $[\,a_1, b_1\,], \ldots$ satisfying

(i). for all n, $[\,a_n, b_n\,] \subseteq [\,a_{n-1}, b_{n-1}\,]$,

(ii). for all n, $b_n - a_n \leq \frac{1}{2}\,(b_n - a_n)$,

(iii). for all n, if $m_n = \langle s_0, \ldots, s_k, q_0, \ldots, q_k \rangle$, then

$$\forall \beta \in \mathcal{C}\,[\,[\,a_n, b_n\,] \;\#\; q_0 f(s_0 * \beta^0) + \ldots + q_k f(s_k * \beta^k)\,].$$

Before we go into the exact definition of this sequence, we show how this gives us a real number x with the requested property.
First note that properties (i) and (ii) assure this is a shrinking and shriveling sequence of intervals with rational endpoints. So we can define a real number by $x \underset{D}{=} ([a_{-1}, b_{-1}], [a_0, b_0], [a_1, b_1], \ldots)$.

Property (iii) guarantees that the so-defined x satisfies the requirements. For suppose $\alpha_0, \ldots, \alpha_n$ are elements of $\mathcal{C}$ satisfying $\forall i, j [i \neq j \rightarrow \alpha_i \# \alpha_j]$ and $t_0, \ldots, t_n \in \mathbb{Q}$. We can determine N such that

$$\forall i, j \leq n [i \neq j \rightarrow \overline{\alpha_i}N \neq \overline{\alpha_j}N].$$

Note that there exists $\alpha \in \mathcal{C}$ such that

$$\forall i [\neg(\overline{\alpha_i}(N+1) \sqsubset \alpha)].$$

Hence we can determine $k \in \mathbb{N}$ such that

$$m_k = \langle \overline{\alpha_0}(N+1), \ldots, \overline{\alpha_n}(N+1), t_0, \ldots, t_n \rangle.$$

Then, by property (iii) of $[a_k, b_k]$, for all $\beta \in \mathcal{C}$,

$$[a_k, b_k] \# t_0 h(\overline{\alpha_0}(N+1) * \beta^0) + \ldots + t_n h(\overline{\alpha_n}(N+1) * \beta^n).$$

So in particular,

$$[a_k, b_k] \# t_0 h(\alpha_0) + \ldots + t_n h(\alpha_n).$$

As $x \in [a_k, b_k]$, this yields

$$x \# t_0 h(\alpha_0) + \ldots + t_n h(\alpha_n),$$

as required.

How do we define such a sequence of intervals? To begin with, we ensure $x \# 0$ by defining

$$[a_{-1}, b_{-1}] \underset{D}{=} [1, 2].$$

Now suppose $[a_{n-1}, b_{n-1}]$ has been defined.
Consider m_n and assume $m_n = \langle s_0, \ldots, s_k, q_0, \ldots, q_k \rangle$.
Property 2 of m_n ensures, for all $\beta, \gamma \in \mathcal{C}$, for all $i < j \leq k$,

$$s_i * \beta \# s_j * \gamma. \tag{7.3}$$

By property 3 we can determine $\alpha \in \mathcal{C}$ such that

$$\forall i \leq k [\neg(s_i \sqsubset \alpha)].$$

Note that this guarantees, for all $\beta \in \mathcal{C}$, for all $i \leq k$,

$$s_i * \beta \# \alpha. \tag{7.4}$$

We can find $q \in \mathbb{Q}$ such that

$$q \cdot h(\alpha) \in [\, a_{n-1}, b_{n-1}\,].$$

Combining the fact that h is Hamel independent with (7.3) and (7.4) yields

$$\forall \beta \in \mathcal{C}\, [\, q_0 h(s_0 * \beta^0) + \ldots + q_n h(s_n * \beta^n) - q \cdot h(\alpha) \# 0\,].$$

Hence,

$$\forall \beta \in \mathcal{C}\, \exists n\, [\, |\, q_0 h(s_0 * \beta^0) + \ldots + q_n h(s_n \beta^n) - q \cdot h(\alpha)\, | > \tfrac{1}{n}\,].$$

By applying the Fan Theorem, we can find $N \in \mathbb{N}$ such that

$$\forall \beta \in \mathcal{C}\, [\, |\, q_0 h(s_0 * \beta^0) + \ldots + q_n h(s_n \beta^n) - q \cdot h(\alpha)\, | > \tfrac{1}{N}\,].$$

Define $M \underset{\text{D}}{=} max(\, N, \frac{2}{b_{n-1} - a_{n-1}}\,)$.
We can find $a \in \mathbb{Q}$ such that

$$q \cdot h(\alpha) \in [\, a, a + \tfrac{1}{M}\,] \subseteq [\, a_{n-1}, b_{n-1}\,].$$

Then

$$\forall \beta \in \mathcal{C}\, [\, q_0 h(s_0 * \beta^0) + \ldots + q_n h(s_n * \beta^n) \# [\, a, \tfrac{1}{M}\,]\,].$$

Furthermore

$$\tfrac{1}{M} \leq \tfrac{1}{2}\,(b_{n-1} - a_{n-1}).$$

We define $[\, a_n, b_n\,] \underset{\text{D}}{=} [\, a, a + \frac{1}{M}\,]$.

So we have defined an infinite sequence of intervals satisfying (i), (ii) and (iii). As we have seen, this sequence gives us a real number with the required property. □

Note that the same result holds for Hamel independent functions h from $\mathbb{N} \times \mathcal{C}$ to $\mathbb{R}$. The proof is very similar to the proof above. Only in this case, you consider the finite sequences m of the form $m = \langle s_0, \ldots, s_k, q_0, \ldots, q_k \rangle$, where $q_i \in \mathbb{Q}$ and $s_i \in \mathbb{N} \times \{0,1\}^*$ with $lg(s_i) \geq 1$, satisfying properties 2 and 3. As we require $lg(s) \geq 1$ we can still apply the Fan Theorem where we did above.

7.3 Connection with the Vitali relation

In this section we consider a weaker notion of being a 'Hamel independent function from $\mathbb{N} \times \mathcal{C} \to \mathbb{R}$', namely the following.

Definition 7.7. Let $h : \mathbb{N} \times \mathcal{C} \to \mathbb{R}$.
The function h is called *weakly Hamel independent* if and only if, for all $n \in \mathbb{N}$, for all $\alpha_0, \ldots, \alpha_n \in \mathbb{N} \times \mathcal{C}$ satisfying $\forall i, j\, [\, i \neq j \to h(\alpha_i) \# h(\alpha_j)\,]$, and for all $q_0, \ldots, q_n \in \mathbb{Q}$,

$$\text{if } \neg \forall i\, [\, q_i = 0\,], \text{ then } q_0 h(\alpha_0) + \ldots + q_n h(\alpha_n) \# 0.$$

This gives rise to a new notion of 'positively Hamel incomplete', slightly different from the one defined in the previous section. In this section we only work with weakly Hamel independent functions. Hence we choose not to add an extra adjective to distinguish the two notions, but ask the reader to remember that, from now on, the following notion is the notion we aim at if we call a function positively Hamel incomplete.

Definition 7.8. Let $h : \mathbb{N} \times \mathcal{C} \to \mathbb{R}$.
We call h *positively Hamel incomplete* iff there exists $x \in \mathbb{R}$ such that, for all $n \in \mathbb{N}$, for all $\alpha_0, \ldots, \alpha_n \in \mathbb{N} \times \mathcal{C}$ satisfying $\forall i, j\, [\, i \neq j \,\to\, h(\alpha_i) \# h(\alpha_j)\,]$ and for all $q_0, \ldots, q_n \in \mathbb{Q}$,

$$q_0 h(\alpha_0) + \ldots + q_n h(\alpha_n) \# x.$$

In [9] M.G. Nadkarni and V.S. Sunder mention a connection between Vitali independent sets and Hamel independent sets. We will give a connection between the weakly Hamel independent functions and the weakly Vitali independent functions defined in Section 6.2.2. Using this, we derive that every weakly Hamel independent function is positively Hamel incomplete.

7.3.1 Constructing weakly Vitali independent functions

How are the weakly Hamel- and weakly Vitali independent functions related? Well, for every weakly Hamel independent function $h : \mathbb{N} \times \mathcal{C} \to \mathbb{R}$, we will construct a function $g_h : \mathbb{N} \times \mathcal{C} \to \mathbb{R}$ such that $Ran(g_h)$ consists of all real numbers x of the form,

$$q_0 h(\alpha_0) + \ldots + q_n h(\alpha_n),$$

where $n \in \mathbb{N}$, $q_0, \ldots, q_n \in \mathbb{Q}$, and $\alpha_0, \ldots, \alpha_n \in \mathbb{N} \times \mathcal{C}$ satisfy

$$\bigwedge_i h(\alpha_i) \# h(\underline{0}) \quad \text{and} \quad \bigwedge_{i<j} h(\alpha_i) \# h(\alpha_j). \tag{7.5}$$

We will show that, for all $\alpha, \beta \in \mathbb{N} \times \mathcal{C}$, for all $q \in \mathbb{Q}$ with $q \neq 0$,

$$g_h(\alpha) + q \cdot h(\underline{0}) \# g_h(\beta).$$

Let $h : \mathbb{N} \times \mathcal{C} \to \mathbb{R}$ be weakly Hamel independent.
If h is a constant function, i.e. $Ran(h) = \{h(\underline{0})\}$, there is no $\alpha \in \mathbb{N} \times \mathcal{C}$ such that $h(\alpha) \# h(\underline{0})$ and we cannot define a function g_h as requested above (as its range should be the empty set). We will construct g_h in such a way that, if h is constant, $Ran(g_h) = \{h(\underline{0})\}$ as well. Therefore, we first define a function $f_h : \mathbb{N} \times \mathcal{C} \to \mathbb{R}$ satisfying

$$\neg\, \exists \beta\, [\, h(\beta) \# h(\underline{0})\,] \;\leftrightarrows\; \forall \alpha\, [\, f_h(\alpha) = h(\underline{0})\,], \tag{7.6}$$

and

$$\exists \beta\, [\, h(\beta) \# h(\underline{0})\,] \;\leftrightarrows\; \exists \beta\, \exists q\, \forall \alpha\, [\, f_h(\alpha) = q \cdot h(\beta) \,\wedge\, h(\beta) \# h(\underline{0})\,]. \tag{7.7}$$

For each $\alpha \in \mathbb{N}\times\mathcal{C}$ we construct a shrinking and shriveling sequence of intervals with rational endpoints $f_h(\alpha)(0),\ f_h(\alpha)(1),\ldots$ by

$$\begin{aligned} f_h(\alpha)(n) &\underset{\mathrm{D}}{=} h(\underline{0})(n) && \text{if } \forall s\in C_n\,[\,h(s*\underline{0})(n) \approx h(s*\underline{0})(n)\,],\\ &\underset{\mathrm{D}}{=} q\cdot h(t*\underline{0})(n+p) && \text{if } \exists s\in C_n\,[\,h(s*\underline{0})(n) \# h(\underline{0})(n)\,], \end{aligned}$$

where $C_n \underset{\mathrm{D}}{=} \{s \in \mathbb{N}\times\{0,1\}^{n-1} \mid s(0) \leq n\}$.
In this definition t, p and q are determined as follows:

Suppose there exists a natural number n such that

$$\exists s\in C_n\,[\,h(s*\underline{0})(n) \# h(\underline{0})(n)\,].$$

Let m be the smallest natural number with this property and define

$$t \underset{\mathrm{D}}{=} \mu s\in C_m\,[\,h(s*\underline{0})(m) \# h(\underline{0})(m)\,].$$

As h is weakly Hamel independent, $h(t*\underline{0}) \# 0$. Hence we can determine $q \in \mathbb{Q}$ such that

$$q\cdot h(t*\underline{0}) \in h(\underline{0})(m).$$

To ensure the sequence $f_h(\alpha)(0), f_h(\alpha)(1),\ldots$ is shrinking, define

$$p \underset{\mathrm{D}}{=} \mu k \in \mathbb{N}\,[\,q\cdot h(t*\underline{0})(m+k) \subseteq f(\underline{0})(m-1)\,].$$

The function f_h, thus defined, satisfies (7.6) and (7.7).

Next, we construct a decidable set $A \subseteq \bigcup_{n\in\mathbb{N}} (\mathbb{N}\times\{0,1\}^*)^n$ such that, for all $n \in \mathbb{N}$, for all $\alpha_0,\ldots,\alpha_n \in \mathbb{N}\times\mathcal{C}$,

$$\bigwedge_i h(\alpha_i) \# h(\underline{0}) \wedge \bigwedge_{i<j} h(\alpha_i) \# h(\alpha_j) \quad\leftrightarrows\quad \exists m\in\mathbb{N}\,[\,\langle\,\overline{\alpha_0}m,\ldots,\overline{\alpha_n}m\,\rangle \in A\,].$$

First note that, for all $\alpha,\beta \in \mathbb{N}\times\mathcal{C}$,

$$h(\alpha) \# h(\beta) \quad\leftrightarrows\quad \exists n\in\mathbb{N}\,[\,\neg\,(\,h(\alpha)(n) \approx h(\beta)(n)\,)\,].$$

Every continuous function from $\mathbb{N}\times\mathcal{C}$ to $\mathbb{R}$ is given by an element of Baire space, that is, there exists $\gamma \in \mathcal{N}$ such that, for all $\alpha \in \mathbb{N}\times\mathcal{C}$, for all $n \in \mathbb{N}$,

$$h(\alpha)(n) = (\gamma|\alpha)(n) = \gamma^n(\,\overline{\alpha}(\,\mu k\,[\,\gamma^n(\overline{\alpha}k) \neq 0\,]\,)\,) - 1.$$

This means, for all $\alpha,\beta \in \mathbb{N}\times\mathcal{C}$,

$$\begin{aligned} h(\alpha) \# h(\beta) \quad\leftrightarrows\quad & \exists n,\,k,\,l\in\mathbb{N}\,[\,\gamma^n(\overline{\alpha}k) > 0 \;\wedge\; \gamma^n(\overline{\beta}l) > 0 \;\wedge \\ & \qquad \neg\,(\,\gamma^n(\overline{\alpha}k) - 1 \;\approx\; \gamma^n(\overline{\beta}l) - 1\,)\,]. \end{aligned}$$

Define, for all $n \in \mathbb{N}$, for all $s, t \in \mathbb{N} \times \{0,1\}^*$,

$$P(n,s,t) \underset{\text{D}}{=} \gamma^n(s) > 0 \ \wedge \ \gamma^n(t) > 0 \ \wedge \ \neg(\gamma^n(s) - 1 \approx \gamma^n(t) - 1).$$

We define A by, for all $q, m \in \mathbb{N}$, for all $s_0, \ldots, s_q \in \mathbb{N} \times \{0,1\}^{m-1}$,

$$\begin{aligned} \langle s_0, \ldots, s_k \rangle \in A \quad \leftrightarrows \quad & \forall i \leq q \, \exists n,\, p,\, q \leq m \, [\, P(n,\, \overline{0}p,\, \overline{s_i}q) \,] \wedge \\ & \forall i < j \leq k \, \exists n,\, p,\, q \leq m \, [\, P(n,\, \overline{s_i}p,\, \overline{s_j}q) \,]. \end{aligned}$$

A is a decidable subset of $\mathbb{N} \times \{0,1\}^*$ and satisfies the requirement.

Now we are ready to define g_h.
Let $r_0, r_1, \ldots$ be an enumeration of $\mathbb{Q}$.
Let $\alpha \in \mathbb{N} \times \mathcal{C}$.
Remember that every natural number, so in particular $\alpha(0)$, codes a finite sequence of natural numbers. To simplify the notation, write $a = \alpha(0)(0)$, $b = \alpha(0)(1)$, $c = \alpha(0)(2)$, $d = \alpha(0)(3)$ and $\delta = (\alpha(1),\, \alpha(2), \ldots)$.
We define $g_h(\alpha)$ as follows,

if $\langle\, \overline{(\langle d(0) \rangle * \delta^0)}\, a, \ldots,\, \overline{(\langle d(b) \rangle * \delta^b)}\, a \,\rangle \in A$, then

$$g_h(\alpha) \underset{\text{D}}{=} r_{c(0)} \cdot h(\langle d(0) \rangle * \delta^0) + \ldots + r_{c(b)} \cdot h(\langle d(b) \rangle * \delta^b),$$

and if this decidable condition fails to be true,

$$g_h(\alpha) \underset{\text{D}}{=} f_h(\alpha).$$

We claim:

if there exists $\beta \in \mathbb{N} \times \mathbb{C}$ satisfying $h(\beta) \# h(\underline{0})$, then $Ran(g_h)$ is the set of all real numbers x that can be written as

$$q_0 h(\alpha_0) + \ldots + q_n h(\alpha_n)$$

where $n \in \mathbb{N}$, $q_0, \ldots, q_n \in \mathbb{Q}$ and $\alpha_0, \ldots, \alpha_n \in \mathbb{N} \times \mathcal{C}$ satisfying (7.5).

Let us call the set of all these real numbers $\mathcal{V}_h$. It is clear that $Ran(g) \subseteq \mathcal{V}_h$.
Now suppose $x \in \mathcal{V}_h$.
Determine $q_0, \ldots, q_n \in \mathbb{Q}$ and $\alpha_0, \ldots, \alpha_n \in \mathbb{N} \times \mathcal{C}$ satisfying (7.5), such that

$$x = q_0 h(\alpha_0) + \ldots + q_n h(\alpha_n).$$

We can find $m \in \mathbb{N}$ such that

$$\langle\, \overline{\alpha_0}m, \ldots, \overline{\alpha_n}m \,\rangle \in A.$$

Determine $k_0, \ldots, k_n$ such that, for each $i \leq n$, $q_i = r_{k_i}$.

Define $\gamma \in \mathcal{C}$ such that, for all $i \leq n$,

$$\gamma^i = \alpha_i \circ S,$$

where S is the successor function defined in Definition 4.15.
Then

$$g_h(\langle m, n, \langle k_0, \ldots, k_n \rangle, \langle \alpha_0(0), \ldots, \alpha_n(0) \rangle \rangle * \delta) = x,$$

and therefore, $x \in Ran(g_h)$.
So $Ran(g_h) = \mathcal{V}_h$.

For the remainder of this section we only consider weakly Hamel independent functions h for which there exists $\beta \in \mathbb{N} \times \mathcal{C}$, with $h(\beta) \# h(\underline{0})$ (that is, functions with at least two different images).
Whenever we state: 'for every weakly Hamel-independent function h,...', we mean: 'for every weakly Hamel independent function h, for which there exists $\beta \in \mathbb{N} \times \mathcal{C}$ with $h(\beta) \# h(\underline{0})$,...'.

7.3.2 Weakly Vitali independence of g_h

The function g_h defined in the previous section is, in a sense, weakly Vitali independent as, for all $\alpha, \beta \in \mathbb{N} \times \mathcal{C}$, for all $q \in \mathbb{Q}_{\neq 0}$,

$$g_h(\alpha) + q \cdot h(\underline{0}) \# g_h(\beta). \tag{7.8}$$

The proof of this fact requires some effort.
First note, for all $\alpha, \beta \in \mathbb{N} \times \mathcal{C}$,

$$g_h(\alpha) = q_0 h(\alpha_0) + \ldots + q_n h(\alpha_n) \quad \text{and} \quad g_h(\beta) = t_0 h(\beta_0) + \ldots + t_m h(\beta_m),$$

for certain $q_0, \ldots, q_n \in \mathbb{Q}$ and $\alpha_0, \ldots, \alpha_n \in \mathbb{N} \times \mathcal{C}$ satisfying (7.5); and certain t_i, β_i satisfying similar conditions.
If we would be able to decide, for all i, j,

$$h(\alpha_i) = h(\beta_j) \vee h(\alpha_i) \# h(\beta_j), \tag{7.9}$$

then (7.8) would follow immediately from the fact that h is weakly Hamel independent. Unfortunately, we are not able to decide (7.9) in general. However, we are able to prove (7.8) using Lemma 6.16. We will do so using course-of-values induction.

Theorem 7.9. *Let $h : \mathbb{N} \times \mathcal{C} \to \mathbb{R}$.*
If h is weakly Hamel independent, then, for all α, $\beta \in \mathbb{N} \times \mathcal{C}$, for all $q \in \mathbb{Q}_{\neq 0}$,

$$g_h(\alpha) + q \cdot h(\underline{0}) \# g_h(\beta).$$

Proof. Let $q \in \mathbb{Q}_{\neq 0}$.
We have to show, for all $n, m \in \mathbb{N}$,

for all $q_0, \ldots, q_n \in \mathbb{Q}$ and for all $\alpha_0, \ldots, \alpha_n \in \mathbb{N} \times \mathcal{C}$ satisfying

$$\bigwedge_i h(\alpha_i) \# h(\underline{0}) \quad \text{and} \quad \bigwedge_{i<j} h(\alpha_i) \# h(\alpha_j), \tag{7.10}$$

for all $t_0, \ldots, t_m \in \mathbb{Q}$ and for all $\beta_0, \ldots, \beta_m, \in \mathbb{N} \times \mathcal{C}$ satisfying

$$\bigwedge_i h(\beta_i) \# h(\underline{0}) \quad \text{and} \quad \bigwedge_{i<j} h(\beta_i) \# h(\beta_j), \tag{7.11}$$

the following holds:

$$q \cdot h(\underline{0}) + q_0 h(\alpha_0) + \ldots + q_n h(\alpha_n) + -t_0 h(\beta_0) + \ldots + -t_m h(\beta_m) \# 0.$$

Call this statement $P(n, m)$.
We will prove, by course-of-values induction,

$$\forall n, m \in \mathbb{N} \, [\, P(n, m) \,].$$

Let $n, m \in \mathbb{N}$ and assume, for all $k, l \in \mathbb{N}$ with $k + l < n + m$, $P(k, l)$.
We prove $P(n, m)$.

Let $q_0, \ldots, q_n \in \mathbb{Q}$ and $\alpha_0, \ldots, \alpha_n \in \mathbb{N} \times \mathcal{C}$ satisfy (7.10), and let $t_0, \ldots, t_m \in \mathbb{Q}$ and $\beta_0, \ldots, \beta_m \in \mathbb{N} \times \mathcal{C}$ satisfy (7.11). We have to show:

$$q \cdot h(\underline{0}) + q_0 h(\alpha_0) + \ldots + q_n h(\alpha_n) + -t_0 h(\beta_0) + \ldots + -t_m h(\beta_m) \# 0.$$

Go through the following procedure:

For $j = 0, \ldots, m$, do the following:

For $i = 0, \ldots, n$, do the following:
By the induction hypothesis,

$$q \cdot h(\underline{0}) + \sum_{\substack{0 \leq r \leq n \\ r \neq i}} q_r h(\alpha_r) \; + (q_i - t_j) h(\alpha_i) \; + \sum_{\substack{0 \leq s \leq m \\ s \neq j}} -t_s h(\beta_s) \# 0.$$

Re-writing yields:

$$\underbrace{q \cdot h(\underline{0}) + \sum_{0 \leq r \leq n} q_r h(\alpha_r) \; + \sum_{\substack{0 \leq s \leq m \\ s \neq j}} -t_s h(\beta_s) \; + -t_j h(\alpha_i)}_{y} \; \# 0.$$

Then, by Lemma 6.16, either,

$$q \cdot h(\underline{0}) + \sum_{0 \le r \le n} q_r h(\alpha_r) + \sum_{0 \le s \le m} -t_s h(\beta_s) \# y,$$

in which case, $h(\beta_j) \# h(\beta_i)$, or,

$$q \cdot h(\underline{0}) + \sum_{0 \le r \le n} q_r h(\alpha_r) + \sum_{0 \le s \le m} -t_s h(\beta_s) \# 0,$$

and we conclude $P(n, m)$.

After this procedure, there are two possibilities:

(i). $\forall i, j\, [\, h(\alpha_i) \# h(\beta_j)\,]$.
Then, as h is weakly Hamel independent,

$$q \cdot h(\underline{0}) + \sum_{0 \le r \le n} q_r h(\alpha_r) + \sum_{0 \le s \le m} -t_s h(\beta_s) \# 0$$

and $P(n, m)$.

(ii). We have concluded $P(n, m)$.

So, in either case, $P(n, m)$.
By course-of-values induction we have proven,

$$\forall n, m \in \mathbb{N}\, [\, P(n, m)\,].$$

It follows that, for all $\alpha, \beta \in \mathbb{N} \times \mathcal{C}$, for all $q \in \mathbb{Q}_{\neq 0}$,

$$g_h(\alpha) + q \# g_h(\beta).$$

□

7.3.3 Incompleteness of weakly Hamel independent functions

In Section 6.2.2 we have already seen that every weakly Vitali independent function from $\mathcal{C}$ to $\mathbb{R}$ is positively Vitali incomplete. The same holds for weakly Vitali independent functions from $\mathbb{N} \times \mathcal{C}$ to $\mathbb{R}$.

Theorem 7.10. *Every weakly Vitali independent function from $\mathbb{N} \times \mathcal{C}$ to $\mathbb{R}$ is positively Vitali incomplete.*

Proof. Let $g : \mathbb{N} \times \mathcal{C} \to \mathbb{R}$ be weakly Vitali independent.
Then, for all $\alpha, \beta \in \mathbb{N} \times \mathcal{C}$, for all $q \in \mathbb{Q}_{\neq 0}$,

$$g(\alpha) + q \# g(\beta). \tag{7.12}$$

We have to construct a real number x such that, for all $\alpha \in \mathbb{N} \times \mathcal{C}$, for all $q \in \mathbb{Q}$,

$$x \# g(\alpha) + q.$$

Let $q_0, q_1, \ldots$ be an enumeration of $\mathbb{Q}$.
We will inductively define a shrinking and shriveling sequence of intervals with rational endpoints, such that in step n we ensure, for all $\alpha \in \mathbb{N} \times \mathcal{C}$ with $\alpha(0) \leq n$, for all q_i with $i \leq n$, $g(\alpha) + q_i \# [\, a_n, b_n\,]$.

To begin with, we define

$$[\, a_{-1}, b_{-1}\,] \underset{\text{D}}{=} [\,1, 2\,].$$

Now let $n \in \mathbb{N}$ and suppose $[\, a_{n-1}, b_{n-1}\,]$ has already been defined.
We consider all elements of $\mathbb{N} \times \mathcal{C}$ with $\alpha(0) \leq n$ and define a shrinking finite sequence of intervals with rational endpoints $[\, c_{-1}, d_{-1}\,], [\, c_0, d_0\,], \ldots, [\, c_n, d_n\,]$.
First, we define

$$[\, c_{-1}, d_{-1}\,] \underset{\text{D}}{=} [\, a_{n-1}, b_{n-1}\,].$$

For $0 \leq i \leq n$, we do the following.

Determine $r \in \mathbb{Q}$ such that $r \neq q_i$ and $g(\underline{0}) + r \in [\, c_{n-1}, d_{n-1}\,]$.
As $r - q_i \neq 0$, it follows from (7.12) that,

$$\forall \beta \in \mathbb{N}_{\leq n} \times \mathcal{C}\, [\, g(\underline{0}) + (r - q_i) \# g(\beta)\,].$$

Hence, by the Fan Theorem, we can determine $N \in \mathbb{N}$, such that

$$\forall \beta \in \mathbb{N}_{\leq n} \times \mathcal{C}\, [\, |\, g(\underline{0}) + r - (g(\beta) + q_i)\,| > \tfrac{1}{N}\,].$$

Determine $c_i, d_i \in \mathbb{Q}$ such that

(i). $g(\underline{0}) + r \in [\, c_i, d_i\,] \subseteq [\, c_{i-1}, d_{i-1}\,]$,
(ii). $0 < d_i - c_i < \frac{1}{N}$.

Then, for all $\alpha \in \mathbb{N}_{\leq n} \times \mathcal{C}$,

$$g(\alpha) + q_i \# [\, c_i, d_i\,].$$

To ensure the sequence is shriveling, define

$$[\, a_n, b_n\,] \underset{\text{D}}{=} [\, c_n, c_n + \tfrac{1}{2}(d_n - c_n)\,].$$

Define $x = (\,[\, a_0, b_0\,], [\, a_1, b_1\,], \ldots)$.
This real number x 'proves' that g is positively Vitali incomplete. For let $\alpha \in \mathbb{N} \times \mathcal{C}$ and $q \in \mathbb{Q}$. We can determine $n \in \mathbb{N}$ such that $q_n = q$.
Define $m \underset{\text{D}}{=} max(\, n, \alpha(0)\,)$, then

$$g(\alpha) + q \# [\, a_m, b_m\,].$$

Hence, as $x \in [\, a_m, b_m\,]$,

$$g(\alpha) + q \# x.$$

□

In a similar way one can prove that, for all $y \in \mathbb{R}$, for any function $g : \mathbb{N}\times\mathcal{C} \to \mathbb{R}$ satisfying

$$\forall \alpha, \beta \in \mathbb{N} \times \mathcal{C}\, \forall q \in \mathbb{Q}_{\neq 0}\, [\, g(\alpha) + q \cdot y \# g(\beta)\,],$$

there exists a real number x such that

$$\forall \alpha \in \mathbb{N} \times \mathcal{C}\, \forall q \in \mathbb{Q}\, [\, x \# g(\alpha) + q \cdot y\,].$$

Using Theorem 7.10, and the connection between the weakly Vitali- and the weakly Hamel independent functions, we prove that every weakly Hamel independent function is positively Hamel incomplete. We start with a lemma which is a generalization of Lemma 6.16.

Lemma 7.11. *Let $n \in \mathbb{N}$.*
For all $y, x_0, \ldots, x_n \in \mathbb{R}$ satisfying,

$$\forall i, j \leq n\, [\, i \neq j \;\to\; x_i \# x_j\,],$$

there exists $k \leq n$ such that,

$$\forall i \leq n\, [\, i \neq k \;\to\; x_i \# y\,].$$

Proof. We prove this lemma by course-of-values induction.
It is clear that it is true for $n = 0$ (pick $k = 0$, there is no j with $j \neq k$).

Now assume the statement holds for n and consider $n + 1$.
Let $y, x_0, \ldots, x_n \in \mathbb{R}$ such that, for all $i, j \leq n + 1$ with $i \neq j$, $x_i \# x_j$.
Applying the induction hypothesis yields $k \leq n$, such that,

$$\forall i \leq n\, [\, i \neq k \;\to\; x_j \# y\,].$$

By assumption $x_k \# x_{n+1}$. Therefore, using Lemma 6.16,

$$y \# x_k \;\vee\; y \# x_{n+1}.$$

This means either,

$$\forall i \leq n + 1\, [\, i \neq n + 1 \;\to\; x_j \# y\,].$$

or,

$$\forall i \leq n + 1\, [\, i \neq k \;\to\; x_j \# y\,].$$

In either case we see the statement holds for $n + 1$.
Hence, by course-of-values induction, we conclude that the statement holds for all natural numbers. □

Theorem 7.12. *Every weakly Hamel independent function from $\mathbb{N} \times \mathcal{C}$ to $\mathbb{R}$ is positively Hamel incomplete.*

Proof. Let $h : \mathbb{N} \times \mathcal{C} \to \mathbb{R}$ be weakly Hamel independent.
Then, by Theorem 7.9, for all $\alpha, \beta \in \mathbb{N} \times \mathcal{C}$, for all $q \in \mathbb{Q}_{\neq 0}$,

$$g_h(\alpha) + q \cdot h(\underline{0}) \# g_h(\beta).$$

Hence, according to (the comment after) Theorem 7.10, we are able to determine a real number x such that, for all $\alpha \in \mathbb{N} \times \mathcal{C}$, for all $q \in \mathbb{Q}$,

$$x \# g_h(\alpha) + q \cdot h(\underline{0}).$$

We will show that this x 'proves' that h is positively Hamel incomplete.
Let $n \in \mathbb{N}$, $q_0, \dots q_n \in \mathbb{Q}$ and $\alpha_0, \dots \alpha_n \in \mathbb{N} \times \mathcal{C}$ with, for all $i < j$, $h(\alpha_i) \# h(\alpha_j)$.
We claim:

$$x \# q_0 h(\alpha_0) + \dots + q_n h(\alpha_n).$$

As, for all $i < j$, $h(\alpha_i) \# h(\alpha_j)$, we can apply Lemma 7.3.3 (with $y = h(\underline{0})$) and find k such that

$$\forall i \leq n\, [\, i \neq k \;\to\; h(\alpha_i) \# h(\underline{0})\,].$$

We may assume, without loss of generality, $k = 0$.
Then $q_1 h(\alpha_1) + \dots + q_n h(\alpha_n) \in Ran(g_h)$ and therefore,

$$x \# q_0 h(\underline{0}) + q_1 h(\alpha_0) + \dots + q_n h(\alpha_n).$$

Using Lemma 6.16, we see that either,

$$q_0 h(\alpha_0) + \dots + q_n h(\alpha_n) \# x,$$

and we have proven our claim, or,

$$q_0 h(\alpha_0) + \dots + q_n h(\alpha_n) \# q_0 h(\underline{0}) + q_1 h(\alpha_0) + \dots + q_n h(\alpha_n),$$

in which case $h(\alpha_0) \# h(\underline{0})$.
As we already know, for all $i \neq 0$, $h(\alpha_i) \# h(\underline{0})$, this means

$$q_0 h(\alpha_0) + \dots + q_n h(\alpha_n) \in Ran(g_h)$$

and we can also conclude:

$$q_0 h(\alpha_0) + \dots + q_n h(\alpha_n) \# x.$$

∎

7.4 Further research

In [9] Nadkarni and Sunder prove that, classically, a Hamel basis cannot be an analytic subset of $\mathbb{R}$, that is, there exists no continuous function $h : \mathcal{N} \to \mathbb{R}$ such that $Ran(h)$ is a basis for $\mathbb{R}$ over $\mathbb{Q}$. Their proof relies on the fact that every analytic subset of the reals is Lebesgue measurable. Nadkarni and Sunder show that, if there were an analytic Hamel basis, then one could construct a Vitali set that is an analytic subset of $\mathbb{R}$. As no Vitali set is Lesbesgue measurable, they conclude there is no analytic Hamel basis.
In view of these classical results, we would like to prove intuitionistically that every (weakly) Hamel independent function from $\mathcal{N}$ to $\mathbb{R}$ is positively Hamel incomplete. Unfortunately, this problem is still open. There exists a connection between weakly Hamel independent functions from $\mathcal{N}$ to $\mathbb{R}$ and weakly Vitali independent functions from $\mathcal{N}$ to $\mathbb{R}$, similar to the connection defined in the previous section for such functions from $\mathbb{N} \times \mathcal{C}$ to $\mathbb{R}$. Therefore, it would be sufficient to show that every (weakly) Vitali independent function from $\mathcal{N}$ to $\mathbb{R}$ is positively Vitali incomplete.

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