ALGEBRAIC TOPOLOGY, EXERCISE SHEET 9, 19.11.2014

Exercise 1. Let X and Y be CW complexes and recall from Lecture 7 that the product $X \times Y$ has a natural CW structure as well.

(1) Endow $[0,1]$ with a nice CW structure and decide which of the maps

$$\begin{tikzcd} X \arrow[r, "i_0"] \arrow[dr, bend right, "="'] & X \times [0,1] \arrow[d, "p"] & X \arrow[l, "i_1"'] \arrow[dl, bend left, "="] \\ & X & \end{tikzcd}$$

are cellular. (The maps i_t are the inclusions $x \mapsto (x,t)$ and p is the projection $(x,t) \mapsto x$.)

(2) Let $i\colon (X^{(n)},*) \to (X,*)$ be the (pointed) inclusion of the n-skeleton. Show that the induced map at the level of homotopy groups $\pi_k(X^{(n)},*) \to \pi_k(X,*)$ is surjective for $k \leq n$ and injective for $k < n$. Thus, it is an isomorphism in dimensions $k < n$.

(3) Let X and Y be CW complexes with no $(n+1)$-cells. If X and Y are homotopy equivalent, show that $X^{(n)}$ and $Y^{(n)}$ are homotopy equivalent as well.

Exercise 2.

(1) Let (X,A) be a pair of spaces and let $n \geq 0$. Then the following are equivalent:
  (a) Every map $(D^n, S^{n-1}) \to (X,A)$ is homotopic relative to S^{n-1} to a map $D^n \to A$.
  (b) Every map $(D^n, S^{n-1}) \to (X,A)$ is homotopic through such maps to a map $D^n \to A$.
  (c) Every map $(D^n, S^{n-1}) \to (X,A)$ is homotopic through such maps to a constant map.
  (d) We have $\pi_n(X,A,a_0) = \pi_n(X,A) \cong 0$ for all $a_0 \in A$.

(2) Let (X,A) be a CW pair such that the m-cells of A and X are the same for all $m \leq n$. Show that $\pi_m(X,A,a_0) = \pi_m(X,A) \cong 0$ for all $a_0 \in A$ and $m \leq n$.

Exercise 3. For each $n \geq 0$, consider the embedding

$$S^n \to S^{n+1};\ x \mapsto (x,0)$$

realizing the n-sphere as the 'equator' of the $(n+1)$-sphere. Let $S^\infty = \bigcup_{n\geq 0} S^n$ be the union of all spheres, equipped with the weak topology.

(1) Show that S^∞ has a natural CW structure and that all inclusions $S^n \to S^\infty$ are cellular maps.

(2) Show that S^∞ is weakly contractible: for any $x \in S^\infty$ and any $n \geq 0$ we have that $\pi_n(S^\infty, x) = 0$.

Exercise 4. Construct a CW complex X with a 0-cell $x(n)$ for each natural number $n \geq 0$ and a 1-cell D^1_n, $n \geq 1$, which is glued to $x(0)$ at one end and to $x(n)$ on the other. For each natural number $n \geq 1$, let us also consider the segment

$$I_n = \{t \cdot e^{2\pi i/n}, \quad 0 \leq t \leq 1\} \quad \subseteq \quad \mathbb{C} \cong \mathbb{R}^2$$

which has boundary points 0 and $e^{2\pi i/n}$. From these we form the space $Y = \bigcup_{n\geq 1} I_n \subseteq \mathbb{C} \cong \mathbb{R}^2$ endowed with the subspace topology.

(1) Give a sketch proof of the fact that Y is a closed subspace of $\mathbb{C}$. Thus, Y is a compact space.
(2) Construct the obvious map $\psi\colon X \to Y = \bigcup_{n\geq 1} I_n$ which sends $x(0)$ to the origin $0 \in \mathbb{C}$ and $x(n)$ to $e^{2\pi i/n}$.
(3) Show that the map ψ is not a homeomorphism.