LECTURE 7: CW COMPLEXES AND BASIC CONSTRUCTIONS

In this section we will introduce CW complexes, which give us an important class of spaces which can be built inductively by gluing 'cells'. Here we will study basic notions and examples, some facts concerning the point set topology of these spaces, and also give elementary constructions. In later lectures we will study homotopical properties of CW complexes. Moreover, we will see that this theory allows us to perform interesting constructions.

By the very definition, a CW complex is given by a space which admits a filtration such that each next filtration step is obtained from the previous one by attaching cells. Let us begin by introducing this process. Let $e^n = \{(x_0, \ldots, x_{n-1}) \in \mathbb{R}^n \mid \sum x_i^2 \leq 1\}$ be a copy of the (closed) *n-ball*. Its boundary $\partial e^n = S^{n-1}$ is the $(n-1)$-*sphere* (for $n = 0$ we take $\partial e^0 = \emptyset$). If X is any space and $\chi\colon \partial e^n \to X$, one can form a new space $X \cup_\chi e^n$ as the *pushout*:

$$\begin{tikzcd} \partial e^n \arrow[r, "\chi"] \arrow[d, "i"'] & X \arrow[d] \\ e^n \arrow[r] & X \cup_\chi e^n \end{tikzcd}$$

More explicitly, $X \cup_\chi e^n$ is the space obtained from the disjoint union $X \sqcup e^n$ by identifying each $i(y) \in e^n$ with $\chi(y) \in X$ for all $y \in \partial e^n$, and equipping the resulting set with the quotient topology. The universal property of this quotient is a as follows.

Exercise 7.1.

(i) The maps $X \to X \cup_\chi e^n$ and $e^n \to X \cup_\chi e^n$ are continuous and make the above square commutative. Moreover, the triple consisting of the space $X \cup_\chi e^n$ and these two maps is initial with respect to this property. In other words, for all triples (W, g, h) consisting of a topological space W and continuous maps $g\colon X \to W$ and $h\colon e^n \to W$ such that the outer square in the following diagram commutes

$$\begin{tikzcd} \partial e^n \arrow[r] \arrow[d] & X \arrow[d] \arrow[ddr, bend left, "g"] & \\ e^n \arrow[r] \arrow[drr, bend right, "h"'] & X \cup_\chi e^n \arrow[dr, dashed, "\exists !"'] & \\ & & W \end{tikzcd}$$

then there is a unique dashed arrow $X \cup_\chi e^n \to W$ such that the two triangles commute.

(ii) Define more generally the notion of a pushout for two arbitrary maps $A \to X$ and $A \to Y$ of spaces with a common domain. Show that the pushout exists and is unique up to a unique isomorphism in a way which is compatible with the structure maps.

(iii) Recall the notion of a pullback from a previous lecture and compare the two notions. These two notions are dual to each other. Compare also the actual constructions of pushouts and pullbacks in the category of spaces and see in which sense they are dual.

(iv) The notion of a pushout makes sense in every category but does not necessarily exist. To familiarize yourself with the concept, show that the categories Set and Ab have pushouts by giving an explicit construction.

We refer to the space $X \cup_\chi e^n$ as being obtained from X by 'attaching an n-cell', and call $\chi\colon \partial e^n \to X$ the *attaching map*, and $e^n \to X \cup_\chi e^n$ the *characteristic map* of the 'cell' e^n. Note that this characteristic map restricts to a homeomorphism of the interior of e^n to its image in $X \cup_\chi e^n$, i.e., we have a relative homeomorphism $(e^n, \partial e^n) \to (X \cup_\chi e^n, X)$. The image of this homeomorphism is called the *open cell*, and the image of $e^n \to X \cup_\chi e^n$ the *closed cell* of this attachment.

Usually one attaches more than one cell, and writes e_σ for the cell with 'index σ', sometimes leaving the dimension implicit. If $\chi\colon \partial e_\sigma \to X$ is the attaching map, it is handy to freeze the index σ, and write $\chi_{\partial\sigma}$ for the attaching map, χ_σ for the characteristic map, and refer to e_σ or its image as the cell (with index) σ.

Thus if we obtain Y from X by attaching a set J_n of n-cells, then, by considering J_n as a discrete space, we have a pushout diagram of the following form:

$$\begin{array}{ccc} J_n \times \partial e^n = \coprod_{\sigma\in J_n} \partial e^n_\sigma & \longrightarrow & X \\ \downarrow & & \downarrow \\ J_n \times e^n = \coprod_{\sigma\in J_n} e^n_\sigma & \longrightarrow & Y \end{array}$$

In particular, a subset of Y is open if and only if its preimages in X and each e^n are open, i.e., Y carries the quotient topology.

Definition 7.2. Let X be a topological space. A *CW decomposition* of X is a sequence of subspaces

$$X^{(0)} \subseteq X^{(1)} \subseteq X^{(2)} \subseteq \ldots, \qquad n \in \mathbb{N},$$

such that the following three conditions are satisfied:

(i) The space $X^{(0)}$ is discrete.
(ii) The space $X^{(n)}$ is obtained from $X^{(n-1)}$ by attaching a (possibly) infinite number of n-cells $\{e^n_\sigma\}_{\sigma\in J_n}$ via attaching maps $\chi_\sigma\colon \partial e^n_\sigma \to X^{(n-1)}$.
(iii) We have $X = \bigcup X^{(n)}$ with the *weak* topology (this means that a set $U \subseteq X$ is open if and only if $U \cap X^{(n)}$ is open in $X^{(n)}$ for all $n \geq 0$).

A CW decomposition is called *finite* if there are only finitely many cells involved. A *(finite) CW complex* is a space X equipped with a (finite) CW decomposition. Given a CW decomposition of a space X then the subspace $X^{(n)}$ is called the *n-skeleton* of X.

Remark 7.3.

(i) Note that by the very definition a CW complex is a space together with an additional structure given by the CW decomposition. Nevertheless, we will always only write X for a topological space endowed with a CW decomposition.
(ii) Condition (iii) in Definition 7.2 is only needed for infinite complexes.
(iii) From the definition of the weak topology it also follows that closed subsets of X can be detected by considering the intersections with all skeleta $X^{(n)}$.
(vi) The image of a characteristic maps $\chi_\sigma\colon e_\sigma \to X$ is called a *closed cell* in X, and the image of $\chi_\sigma\colon e^\circ_\sigma \to X$ an *open cell*. These need not be open in X! Every point of X lies in a unique open cell.

(v) Each $X^{(n)}$ is a closed subspace of $X^{(n+1)}$, and hence of X. (The open $(n+1)$-cells are open in $X^{(n+1)}$ but not necessarily in X).

Example 7.4.

(i) The interval $I = [0, 1]$ has a CW decomposition with two 0-cells and one 1-cell by identifying the boundary of the unique 1-cell with the two 0-cells as expected.

(ii) The circle S^1 has a CW decomposition with one 0-cell and one 1-cell and no other cells. Of course, it also has a CW composition with two 0-cells and two 1-cells.

(iii) More generally, if one identifies the boundary ∂e^n of the n-ball to a point, one obtains (a space homeomorphic to) the n-sphere. Thus the n-sphere has a CW decomposition with one 0-cell and one n-cell, and no other cells. One can also build up the n-sphere by starting with two points, then two half circles to form S^1, then two hemispheres to form S^2, and so on. Then S^n has a CW decomposition with exactly 2 i-cells for $i = 0, \dots, n$ (draw a picture for $n \leq 2$!). If we take the coordinates $(x_0, \dots, x_n)$ with $\sum x_i^2 = 1$ for S^n as before, these two i-cells are

$$e^i_+ = \{(x_0, \dots, x_i, 0 \dots, 0) \in S^n \mid x_i \geq 0\}$$

and

$$e^i_- = \{(x_0, \dots, x_i, 0 \dots, 0) \in S^n \mid x_i \leq 0\}.$$

(iv) The *real projective space* $\mathbb{RP}^n$, the space of lines through the origin in $\mathbb{R}^{n+1}$, can be constructed as the quotient $S^n/\mathbb{Z}_2$ where $\mathbb{Z}_2 = \mathbb{Z}/2\mathbb{Z}$ acts on the n-sphere by the antipodal map; in other words, by the quotient of S^n obtained by identifying x and $-x$. This identification maps the cell e^i_+ to e^i_-. Thus $\mathbb{RP}^n$ has a CW decomposition with exactly one i-cell for $i = 0, \dots, n$. Recall from the previous lecture, that the Grassmannian varieties $G_{k,n}(\mathbb{R})$ parametrize k-planes in $\mathbb{R}^n$. Thus, we have $\mathbb{R}P^n \cong G_{1,n+1}(\mathbb{R})$.

(v) The *complex projective space* $\mathbb{CP}^n$ is the space of complex lines through the origin in $\mathbb{C}^{n+1}$. Such a line is determined by a point $(z_0, \dots, z_n) \neq 0$ on the line, and for any scalar $\lambda \in \mathbb{C} - \{0\}$ the tuple $(\lambda z_0, \dots, \lambda z_n)$ determines the same line for which we write $[z_0, \dots, z_n]$. The line can also be represented by a point $z = (z_0, \dots, z_n)$ with $|z| = 1$, so that z and λz represent the same line for all $\lambda \in S^1$. Thus $\mathbb{CP}^n = S^{2n+1}/S^1$ is a space of (real) dimension $2n$. There are inclusions

$$* = \mathbb{CP}^0 \subseteq \mathbb{CP}^1 \subseteq \mathbb{CP}^2 \subseteq \dots$$

where $\mathbb{CP}^{n-1} \subseteq \mathbb{CP}^n$ sends $[z_0, \dots, z_{n-1}]$ to $[z_0, \dots, z_{n-1}, 0]$. An arbitrary point in $\mathbb{CP}^n - \mathbb{CP}^{n-1}$ can be uniquely represented by $(z_0, \dots, z_{n-1}, t)$ where $t > 0$ is the real number $\sqrt{1 - \sum z_i \bar{z}_i}$. This defines a map

$$e^{2n} \to \mathbb{CP}^n \colon z = (z_0, \dots, z_{n-1}) \mapsto [z_0, \dots, z_{n-1}, t]$$

with $t = \sqrt{1 - ||z||}$. The boundary of e^{2n} (where $t = 0$) is sent to $\mathbb{CP}^{n-1}$. In this way, $\mathbb{CP}^n$ is obtained from $\mathbb{CP}^{n-1}$ by attaching one $2n$-cell. So $\mathbb{CP}^n$ has a CW structure with one cell in each even dimension $0, 2, \dots, 2n$. Similarly to the previous example, we have an identification $\mathbb{CP}^n \cong G_{k,n+1}(\mathbb{C})$.

(vi) Every compact manifold is homotopy equivalent to a CW complex. (This is a theorem which we only include to indicate the generality of the notion.)

(vii) As we will see in a later lecture, every topological space is *weakly homotopy equivalent* to a CW complex.

Exercise 7.5.

(i) The torus $\mathbb{T}$ can be obtained from the square by identifying opposite sides. Use an adapted CW decomposition of the square to also turn the torus into a CW complex.

(ii) Similarly we can obtain the Klein bottle from the unit square by identifying $(0,t) \sim (1,t)$ and $(s,0) \sim (1-s,1)$. Show that there is a similar CW decomposition of the Klein bottle.

(iii) Can you come up with CW decompositions of the torus and the Klein bottle which have the same number of cells in each dimension? In particular this shows the obvious fact that the number of cells does *not* determine the space.

Lemma 7.6. *Let X be a CW complex and let U be a subset of X. Then a subset $U \subset X$ is open if and only if $U \cap X^{(n)}$ is open for each n if and only if $\chi_\sigma^{-1}(U) \subseteq e_\sigma^n$ is open for each cell σ of X.*

Proof. The equivalence of the first two statements holds true by definition of CW complexes. It is immediate that the second condition implies the third one. We want to prove the converse implication by induction so let us begin by observing that $U \cap X^{(0)}$ is open in $X^{(0)}$ since $X^{(0)}$ is discrete. For the inductive step, let us assume that $U \cap X^{(n-1)}$ is open in $X^{(n-1)}$ for some $n \geq 1$. Recall that we then have a pushout diagram of the following form:

$$\begin{tikzcd} J_n \times \partial e^n = \coprod_{\sigma \in J_n} \partial e_\sigma^n \arrow[r] \arrow[d] & X^{(n-1)} \arrow[d] \\ J_n \times e^n = \coprod_{\sigma \in J_n} e_\sigma^n \arrow[r] & X^{(n)} \end{tikzcd}$$

By assumption $\chi_\sigma^{-1}(U) \subseteq e_\sigma^n$ is open for every $\sigma \in J_n$. But the above pushout diagram together with the induction assumption then tells us that also $U \cap X^{(n)}$ is open in $X^{(n)}$ concluding the proof. □

Thus, given a CW complex X with n-cells parametrized by index sets J_n, then taking all the attaching maps together we obtain a map

$$(\chi_\sigma)_{n,\sigma}\colon \coprod_n J_n \times e^n \cong \coprod_n \coprod_{\sigma \in J_n} e_\sigma^n \longrightarrow X.$$

The above lemma shows that X carries the quotient topology with respect to this map.

Corollary 7.7. *Let X be a CW complex, Y a topological space, and $g\colon X \to Y$ a map of sets. Then the following are equivalent:*

(i) *The map $g\colon X \to Y$ is continuous.*

(ii) *The restriction $g|\colon X^{(n)} \to Y$ is continuous for all $n \geq 0$.*

(iii) *The map $g \circ \chi_\sigma\colon e_\sigma^n \to Y$ is continuous for each cell e_σ^n.*

This corollary allows us to build continuous maps 'cell by cell'. Thus, not only CW complexes can be built inductively by attaching cells but the same holds also true for maps defined on a CW complex. There is also a similar result for homotopies.

Exercise 7.8. Let X be a CW complex, Y a topological space, and $H\colon X \times I \to Y$ a map of sets. Then H is continuous if and only if each composition

$$H \circ (\chi_\sigma \times id_I)\colon e_\sigma^n \times I \longrightarrow X \times I \longrightarrow Y$$

is continuous for each cell e_σ^n of X.

Before turning to CW subcomplexes and an adapted class of morphisms, let us establish some more fundamental properties of CW complexes.

Exercise 7.9. A CW complex is normal. Thus show that disjoint closed subsets have disjoint open neighborhoods and that points are closed.

In studying the topology of CW complexes, one often uses the following fact.

Proposition 7.10. *Any compact subset of a CW complex is contained in finitely many open cells.*

This proposition in fact immediately follows from the following statement, by choosing a point in every open cell that intersects non-trivially the given compact subset.

Lemma 7.11. *Let X be a CW complex and $A \subset X$ a subspace. If A has at most one point in each open cell then A is closed in X and the subspace topology on A is discrete.*

Proof. We check this by induction on n and for each $A \cap X^{(n)}$ as a subspace of $X^{(n)}$ (the closure then follows by definition of the weak topology on X). For $n = 0$ there is nothing to prove since $X^{(0)}$ is discrete. Suppose the statement has been proved for $A \cap X^{(n-1)} \subseteq X^{(n-1)}$. Write $A \cap X^{(n)} = B \sqcup C$ where $B = A \cap X^{(n-1)}$ and $C = A \cap (X^{(n)} - X^{(n-1)})$. Then C is open in A because the open n-cells are open in $X^{(n)}$, and for the same reason C is discrete. The set C is closed in $X^{(n)}$ because if $x \in \bar{C}$ then x lies in the same open cell as any point $c \in C$ close to x, hence $x = c$. So C is closed and discrete in $X^{(n)}$. Also B is closed and discrete in $X^{(n-1)}$ by induction hypothesis, hence in $X^{(n)}$ because $X^{(n-1)} \subseteq X^{(n)}$ is closed. Then $B \sqcup C$ has the same properties, which completes the induction step. □

Remark 7.12. This proposition allows us to explain the terminology 'CW complex'. In the original definition given by J.H.C. Whitehead, the following two properties played a more essential role:

- (C): The closure of every cell lies in a finite subcomplex ('closure finite').
- (W): A subset is open if and only if it is open in the n-skeleton for all n ('weak topology').

We now turn to an adapted class of morphisms between CW complexes.

Definition 7.13. A map $f\colon X \to Y$ between CW complexes is *cellular* if it satisfies $f(X^{(n)}) \subseteq Y^{(n)}$ for all n. It is immediate that we have a category CW of CW complexes and cellular maps.

Thus, such a cellular map induces commutative diagrams of the form:

$$\begin{tikzcd} X^{(n)} \arrow[r, "f|"] \arrow[d, "i"'] & Y^{(n)} \arrow[d, "i"] \\ X \arrow[r, "f"'] & Y \end{tikzcd}$$

Let us give some examples of cellular maps. We will see in a later lecture that this notion is rather generic.

Example 7.14.

(i) The vector space $\mathbb{R}^n$ maps injectively to $\mathbb{R}^{n+1}$ by adding a zero as the last coordinate, i.e., we have a map

$$i_n\colon \mathbb{R}^n \to \mathbb{R}^{n+1}\colon (t_1, \ldots, t_n) \mapsto (t_1, \ldots, t_n, 0).$$

These maps restrict to maps of spheres as follows

$$j_n = i_{n+1}|\colon S^n \longrightarrow S^{n+1}$$

and these maps are cellular with respect to the CW decompositions on the spheres with precisely two cells in each dimension lower or equal to the dimension of the respective sphere (but not with respect to the other one).

(ii) Since the inclusions $i_n\colon \mathbb{R}^n \to \mathbb{R}^{n+1}$ are compatible with the actions by $\mathbb{R}^\times$, we obtained induced maps $j'_n\colon \mathbb{R}P^{n-1} \to \mathbb{R}P^n$ which are easily seen to be cellular with respect to the CW decomposition of Example 7.4. The maps are also obtained from the maps j_n of the last example by passing to the quotient of the $\mathbb{Z}/2\mathbb{Z}$-action and these quotient maps are also cellular. Thus, we have a diagram of cellular maps:

$$\begin{tikzcd}
S^0 \arrow[r, "j_0"] \arrow[d, "q_0"'] & S^1 \arrow[r, "j_1"] \arrow[d, "q_1"'] & S^2 \arrow[r, "j_2"] \arrow[d, "q_2"'] & \ldots \\
\mathbb{R}P^0 \arrow[r, "j'_0"'] & \mathbb{R}P^1 \arrow[r, "j'_1"'] & \mathbb{R}P^2 \arrow[r, "j'_2"'] & \ldots
\end{tikzcd}$$

Similarly, in the case of complex numbers, we have cellular maps:

$$\mathbb{C}P^0 \longrightarrow \mathbb{C}P^1 \longrightarrow \mathbb{C}P^2 \longrightarrow \ldots$$

(iii) In Example 7.4 we introduced two CW decompositions on the n-sphere. Let us write S^n for the one with two cells in each dimension $d \leq n$ while we write $\widehat{S^n}$ for the one with precisely one 0-cell and one n-cell. Then the identity map $id\colon \widehat{S^n} \to S^n$ is cellular, while this is not the case for $id\colon S^n \to \widehat{S^n}$ if $n \geq 2$.

We now turn to subcomplexes of CW complexes.

Proposition 7.15. *Let X be a CW complex and let $Y \subseteq X$ be a closed subspace such that the intersection $Y \cap (X^{(n)} - X^{(n-1)})$ is the union of open n-cells. The filtration*

$$Y^{(0)} \subseteq Y^{(1)} \subseteq \ldots \subseteq Y$$

given by $Y^{(n)} = Y \cap X^{(n)}$ then defines a CW decomposition on Y. Moreover, the inclusion $Y \to X$ is then a cellular map.

This proposition allows us to introduce pointed CW complexes and pairs of CW complexes.

Definition 7.16. In the notation of the above proposition, we refer to Y as a *CW subcomplex* of X and to (X, Y) as a *CW pair*. A *pointed CW complex* (X, x_0) is a CW complex X together with a chosen base point $x_0 \in X^{(0)}$.

In the obvious way, this gives us the category of pointed CW complexes and CW pairs whose definitions are left to the reader.

Example 7.17.

(i) For an arbitrary CW complex X, we have CW pairs $(X, X^{(n)})$ for all n and similarly $(X^{(n)}, X^{(m)})$ for $n \geq m$.

(ii) We have CW pairs (S^n, S^m), $(\mathbb{R}P^n, \mathbb{R}P^m)$ and similarly in the complex case for $n \geq m$. If we endow the unions

$$S^\infty = \bigcup_n S^n, \qquad \mathbb{R}P^\infty = \bigcup_n \mathbb{R}P^n, \qquad \text{and} \qquad \mathbb{C}P^\infty = \bigcup_n \mathbb{C}P^n$$

with the weak topology then each of the three spaces carries canonically a CW structure. Moreover, we have CW pairs (S^∞, S^n), $(\mathbb{R}P^\infty, \mathbb{R}P^n)$. and $(\mathbb{C}P^\infty, \mathbb{C}P^n)$ for all n.

Exercise 7.18. Let (X, Y) be a CW pair. Then the quotient space X/Y can be turned in a CW complex such that the quotient map $X \to X/Y$ is cellular.

We will now establish a few more closure properties of CW complexes. Let us begin with a more difficult one, namely the product. Recall that we observed that each CW complex is obtained from a disjoint union of cells by passing to a quotient space. Namely, for a CW complex X we have a quotient map:

$$\bigsqcup_n J_n \times e^n \longrightarrow X$$

Given two CW complexes X and Y one might now try to take two such presentations

$$\bigsqcup_n J_n(X) \times e^n \longrightarrow X \qquad \text{and} \qquad \bigsqcup_m J_m(Y) \times e^m \longrightarrow Y$$

and use homeomorphisms $e^n \times e^m \cong e^{n+m}$ to obtain a map

$$\bigsqcup_k J_k(X \times Y) \times e^k \longrightarrow X \times Y$$

where $J_k(X \times Y) = \sqcup_{n+m=k} J_n(X) \times J_m(Y)$. However, this map is, in general, not a quotient map. More conceptually, the problem is that the formation of products and quotients in the category of spaces are not compatible in general. Nevertheless, under certain 'finiteness conditions' one can obtain a positive result. We will give a proof of this result in a later lecture.

Proposition 7.19. *Let X, K be CW complexes such that K is finite. Then the product $X \times K$ is again a CW complex with the above CW decomposition.*

Proof. Will be given in a later lecture. □

Using the last proposition we can establish many more closure properties for the class of CW complexes.

Corollary 7.20.

(i) *The coproduct of two CW complexes is again a CW complex such that the inclusions of the respective summands are cellular.*

(ii) *Given a CW complex X then the cylinder $X \times I$ is again a CW complex. For each n-cell e^n_σ of X we obtain three cells for $X \times I$, namely two n-cells $e^n_\sigma \times \{0\}, e^n_\sigma \times \{1\}$, and an $(n+1)$-cell $e^n_\sigma \times e^1$. Moreover, the cylinder comes with cellular maps $i_0, i_1 \colon X \to X \times I$ and $p \colon X \times I \to X$.*

(iii) *Given a CW complex X, then the unreduced suspension $SX = (X \times I)/(X \times \partial I)$ is again a CW complex.*

(iv) *We have similar variants for the context of pointed CW complexes. The wedge product of pointed CW complexes is again a CW complex. Similarly, the reduced cylinder of a pointed CW complex is again a pointed CW complex. More generally, if K is a finite pointed CW complex and if X is a pointed CW complex, then so is the smash product $X \wedge K$. In particular, the (reduced) suspension of a pointed CW complex is again a pointed CW complex.*

Proof. The first statement is immediate while the other ones follow immediately from Exercise 7.18, Proposition 7.19, and Example 7.4. □

In the definition of a CW complex X, the first condition we imposed was that $X^{(0)}$ is to be a discrete space and then that the higher skeleta are obtained from the lower ones by attaching n-cells for $n \geq 1$. We can also think of $X^{(0)}$ as being obtained from the empty space by attaching 0-cells; in fact, using the convention that $\partial e^0 = \emptyset$ we have a pushout:

$$\begin{tikzcd} X^{(0)} \times \partial e^0 = \coprod_{\sigma \in X_0} \partial e^0_\sigma \arrow[r, "\cong"] \arrow[d] & X^{(-1)} = \emptyset \arrow[d] \\ X^{(0)} \times e^0 = \coprod_{\sigma \in J_0} e^0_\sigma \arrow[r, "\cong"'] & X^{(0)} \end{tikzcd}$$

This observation is more than only a rather picky remark since it motivates the following generalization of the notion of CW complex.

Definition 7.21. Let (X, A) be a pair of spaces. Then X is a *CW complex relative to* A, if there is a filtration of X,

$$A = X^{(-1)} \subseteq X^{(0)} \subseteq X^{(1)} \subseteq \ldots \subseteq X,$$

such that the following two properties are satisfied:

(i) The space $X^{(n)}$ is obtained from $X^{(n-1)}$ by attaching n-cells for $n \geq 0$.
(ii) The space X is the union $\bigcup_{n \geq -1} X^{(n)}$ endowed with the weak topology.

In this situation, the pair (X, A) is called a *relative CW complex*.

Example 7.22.

(i) Let X be a CW complex and $x_0 \in X_0$. Then we have a relative CW complex (X, x_0).
(ii) More generally, every CW pair is a relative CW complex.

One point of the notion of a relative CW complex (X, A) is that the associated inclusion map $A \to X$ is not an arbitrary map but has nice properties. In a way, these properties are dual to the ones of fibrations. We will come back to this in the next lecture where we will, in particular, talk about *cofibrations*.