HOMOTOPICAL LOCALIZATIONS OF MODULE SPECTRA

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ABSTRACT. We prove that stable f-localizations (where f is any map of spectra) preserve ring spectrum structures and module spectrum structures, under suitable hypotheses, and we use this fact to describe all possible localizations of the integral Eilenberg–Mac Lane spectrum $H\mathbb{Z}$. As a consequence of this study, we infer that localizations of stable GEMs are stable GEMs, and it also follows that there is a proper class of nonequivalent stable localizations.

1. INTRODUCTION

A spectrum is called a stable GEM if it is homotopy equivalent to a wedge

$$\bigvee_{k\in\mathbb{Z}} \Sigma^k HA_k,$$

where each A_k is an abelian group and HA_k denotes an Eilenberg–Mac Lane spectrum with $\pi_0(HA_k) \cong A_k$ and $\pi_i(HA_k) = 0$ if $i \neq 0$. A spectrum is a stable GEM if and only if it admits an $H\mathbb{Z}$-module structure, where $H\mathbb{Z}$ is viewed as a ring spectrum in the usual way. Equivalently, X is a stable GEM if and only if X is a homotopy retract of $H\mathbb{Z} \wedge X$; more details about these claims are given in Section 5. Note the analogy with unstable homotopy, where a space X is a GEM (i.e., a weak product of abelian Eilenberg–Mac Lane spaces) if and only if it is a homotopy retract of the infinite symmetric product $\mathrm{SP}^\infty X$, a space whose homotopy groups are the integral homology groups of X.

Dror Farjoun [Dro96], and later Badzioch [Bad01], with different methods, proved that unstable f-localizations send GEMs to GEMs. In this article we prove that the same result is true in stable homotopy, by developing further certain ideas used by Bousfield in [Bou96] and [Bou99].

For this, one can choose to work in any suitable simplicial model category $\mathcal{S}$ whose homotopy category is equivalent to the homotopy category of CW-spectra. In order to ensure the existence of f-localizations, we ask that $\mathcal{S}$ be left proper, cofibrantly generated, and locally presentable. The Bousfield–Friedlander model category [BF78] or the model category of symmetric spectra built from simplicial sets [HSS00] satisfy these conditions. However, most of our arguments are homotopic in nature and make no use of any specific models for spectra.

Homotopical localization with respect to a map $f\colon A \to B$ of cofibrant spectra (f-localization, for short) is a functor L_f from $\mathcal{S}$ to itself, which sends weak equivalences to weak equivalences, is idempotent in the homotopy category and takes

values in the subcategory of fibrant spectra X such that the map of simplicial sets

$$\mathrm{Map}(f, X)\colon \mathrm{Map}(B, X) \longrightarrow \mathrm{Map}(A, X)$$

is a weak equivalence. All known forms of localizations in stable homotopy are f-localizations for suitable choices of the map f. Among these, homological localizations commute with the suspension operator. It is crucial to emphasize that not all f-localizations commute with suspension; we give necessary and sufficient conditions in Section 2.

We prove that if f is any map of spectra and E is a ring spectrum (i.e., a monoid in the homotopy category), then $L_f E$ is also a ring spectrum and the localization map $E \to L_f E$ is a ring map, if we assume either that both E and $L_f E$ are connective or that L_f commutes with suspension. If E is commutative, then $L_f E$ is also commutative. Similarly, if M is an E-module spectrum, then so is $L_f M$, and the localization map $M \to L_f M$ is an E-module map, provided that E is connective or L_f commutes with suspension. (A spectrum E is connective if its homotopy groups $\pi_i(E)$ vanish for $i < 0$.)

It will be shown elsewhere that a stronger result holds, namely that, if the model category $\mathcal{S}$ is closed symmetric monoidal and E is a monoid in $\mathcal{S}$, then $L_f E$ is homotopy equivalent to a monoid in $\mathcal{S}$, if we assume either that both E and $L_f E$ are connective or that L_f commutes with suspension. The same is true for commutative monoids. This is proved in the special case of homological localizations in Chapter VIII of [EKMM97].

It follows that f-localizations send $H\mathbb{Z}$-modules to $H\mathbb{Z}$-modules; that is, the class of stable GEMs is preserved by f-localizations. We show that, in fact, for every abelian group G,

$$L_f HG \simeq HA \vee \Sigma HB$$

for certain abelian groups A and B. When $G = \mathbb{Z}$, we show that $B = 0$ and the group A is a commutative ring with 1 such that $\mathrm{Hom}(A, A) \cong A$ via $\varphi \mapsto \varphi(1)$. Rings A with this property were called *rigid* in [CRT00]. It was shown in [DMV87] that there is a proper class of nonisomorphic rigid rings, and it is easy to see that for every rigid ring A there is a map f such that $L_f H\mathbb{Z} \simeq HA$. Remarkably, this shows that there is a proper class of nonequivalent f-localizations in the stable homotopy category. In contrast, recall from [DP01] that there is only a set of nonequivalent homological localizations. We do not know whether there is a proper class or a set of nonequivalent f-localizations in the stable homotopy category that commute with suspension. This question seems to be closely related to other unanswered questions about Bousfield classes, cohomological localizations, and localizing subcategories, some of which appear in the work of Hovey, Palmieri and Strickland, e.g. [Hov95], [HPS97], [Str04].

A more conceptual explanation of the fact that, for every abelian group G, the spectrum $L_f HG$ has at most two nonzero homotopy groups is that the homotopy category of HR-modules is equivalent to the homotopy category of ($\mathbb{Z}$-graded) chain complexes of R-modules, where R is any ring; see [EKMM97, IV.2]. However, this holds for *strict* HR-modules; that is, modules over HR in a closed symmetric monoidal category of spectra. For certain rings, including $R = \mathbb{Z}$, the distinction between strict HR-modules and homotopy HR-modules (i.e., HR-module spectra in the homotopy category) is unessential, but for other rings it is crucial. This phenomenon is discussed in [Gut03].

As an application of our results, we show that if $E = K$ (complex K-theory) or $E = E(n)$ (the Johnson–Wilson spectrum) for some n, then the E-localization of the sphere spectrum S has the following homology groups:

$$H_0(L_E S) \cong \mathbb{Q}, \qquad H_i(L_E S) = 0 \quad \text{for } i \neq 0.$$

A similar result (with possibly other subrings of the rationals) holds for every smashing localization, in the sense of [Rav84]. It follows as an easy consequence that, if L is a smashing localization such that $H_0(LS) \cong \mathbb{Q}$, then either L is ordinary $H\mathbb{Q}$-localization, or LS is unbounded.

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2. Localization of spectra with respect to a map

The notion of f-localization can be formulated in any category $\mathcal{S}$ with a simplicial model structure; see [GJ99] or [Hir03] for further details about methods and terminology. Given a map $f\colon A \to B$ between cofibrant objects in such a category, an object X is called f-*local* if it is fibrant and the induced map of simplicial sets

$$\mathrm{Map}(f, X)\colon \mathrm{Map}(B, X) \longrightarrow \mathrm{Map}(A, X)$$

is a weak equivalence. We label the following standard properties for later use.

Lemma 2.1. *If X is f-local and $Y \to X$ is a map with a left homotopy inverse where Y is fibrant, then Y is f-local.*

Proof. Factor $Y \to X$ into a cofibration $Y \to Z$ followed by a trivial fibration $Z \to X$. Then Z is f-local and Y is a retract of Z. It follows, as in [Hir03, 1.2.5], that $\mathrm{Map}(f, Y)\colon \mathrm{Map}(B, Y) \to \mathrm{Map}(A, Y)$ is a retract of $\mathrm{Map}(f, Z)$, and hence it is a weak equivalence. $\square$

Lemma 2.2. *Every homotopy limit of f-local objects is f-local.*

Proof. If I is a small category and $D\colon I \to \mathcal{S}$ is a diagram in $\mathcal{S}$ taking values in the subcategory of f-local objects, then

$$\mathrm{Map}(B, \mathrm{holim}_I D) \cong \mathrm{holim}_I \mathrm{Map}(B, D)$$

$$\simeq \mathrm{holim}_I \mathrm{Map}(A, D) \cong \mathrm{Map}(A, \mathrm{holim}_I D),$$

where $\mathrm{Map}(B, D)$ and $\mathrm{Map}(A, D)$ denote the corresponding diagrams of simplicial sets indexed by I. $\square$

A map $g\colon X \to Y$ is an f-*equivalence* if there is a cofibrant approximation $\tilde{g}\colon \tilde{X} \to \tilde{Y}$ such that

$$\mathrm{Map}(\tilde{g}, E)\colon \mathrm{Map}(\tilde{Y}, E) \longrightarrow \mathrm{Map}(\tilde{X}, E)$$

is a weak equivalence for every f-local object E. Similarly as in Lemmas 2.1 and 2.2, every homotopy retract of an f-equivalence is an f-equivalence, and every homotopy colimit of f-equivalences is an f-equivalence. In particular, every coproduct of f-equivalences is an f-equivalence.

An f-*localization* of an object X is an f-equivalence $l\colon X \to L_f X$, where $L_f X$ is f-local. Such a map l is initial in the homotopy category $\mathrm{Ho}\,\mathcal{S}$ among maps from X

to f-local objects, and it is terminal in $\mathrm{Ho}\mathcal{S}$ among f-equivalences with domain X. Each of these two universal properties ensures that, if an f-localization of X exists, then it is unique up to homotopy.

The existence of f-localization for every map f and all objects X is guaranteed if the simplicial model category $\mathcal{S}$ satisfies certain assumptions. Specifically, the following result is proved as indicated in [Bou77] or in [Hir03]. The notion of a *locally presentable* category was defined in [GU71]. The definitions of the terms *cofibrantly generated* and *left proper* can be found in [Hir03].

Theorem 2.3. *Let $\mathcal{S}$ be any locally presentable, cofibrantly generated, left proper, simplicial model category. Then for every map f there exists a functor L_f on $\mathcal{S}$ with a natural transformation $\mathrm{Id} \to L_f$ yielding an f-localization for all objects.* $\qquad\square$

(Such a functor need not be unique on $\mathcal{S}$, but it is unique up to homotopy, and a standard construction provides canonical models.) The Bousfield–Friedlander category of spectra [BF78] or the category of symmetric spectra built from simplicial sets [HSS00] satisfies the assumptions stated in Theorem 2.3. Moreover, for X cofibrant and Y fibrant, the simplicial set $\mathrm{Map}(X,Y)$ is an infinite loop space and, for any choice of basepoint, its homotopy groups satisfy

$$\pi_k(\mathrm{Map}(X,Y),*) \cong [\Sigma^k X, Y] \cong \pi_k(F(X,Y)) \quad \text{for } k > 0,$$

where $F(X,Y)$ denotes the function spectrum from X to Y. Thus, the homotopy groups of the simplicial set $\mathrm{Map}(X,Y)$ are isomorphic to those of the connective cover $F^c(X,Y)$ of the function spectrum from X to Y.

From now on, we will work with (derived) function spectra in the homotopy category, and omit any further reference to the chosen model category. Thus, for a map $f\colon A \to B$ in $\mathrm{Ho}\mathcal{S}$, a spectrum E is f-local if and only if the induced map of connective covers of function spectra

$$F^c(B,E) \longrightarrow F^c(A,E)$$

is an equivalence, i.e., induces isomorphisms of all homotopy groups. Likewise, a map $g\colon X \to Y$ in $\mathrm{Ho}\mathcal{S}$ is an f-equivalence if and only if the induced map

$$F^c(Y,E) \longrightarrow F^c(X,E)$$

is an equivalence for every f-local spectrum E. These are indeed the definitions adopted in [Bou96] and [Bou99].

Proposition 2.4. *Let f be any map.*
  (a) *If E is f-local, then $\Sigma^{-k}E$ is also f-local for $k \geq 0$.*
  (b) *If $g\colon X \to Y$ is an f-equivalence, then $\Sigma^k g$ is also an f-equivalence for $k \geq 0$.*

Proof. Part (a) is a special case of Lemma 2.2. A more direct argument can however be given as follows. If E is f-local, then f induces an equivalence

$$F^c(B,\Sigma^{-k}E) \simeq F^c(A,\Sigma^{-k}E) \quad \text{for } k \geq 0,$$

since $\pi_i(F^c(B,\Sigma^{-k}E)) \cong \pi_{i+k}(F^c(B,E))$ if $i \geq 0$ and $k \geq 0$. The proof of part (b) is similar. $\qquad\square$

Since $F^c(B,E) \simeq F^c(\Sigma^k B, \Sigma^k E)$ for all B, E and $k \in \mathbb{Z}$, we may also infer that if E is f-local then $\Sigma^k E$ is $\Sigma^k f$-local for every $k \in \mathbb{Z}$, and similarly for f-equivalences. From this fact we deduce the following result.

Proposition 2.5. *For every map f and every spectrum X there is an equivalence*

$$L_f\Sigma^{-k}X \simeq \Sigma^{-k}L_{\Sigma^k f}X \quad \text{for all } k \in \mathbb{Z}.$$

Proof. Since $l\colon X \to L_{\Sigma^k f}X$ is a $\Sigma^k f$-equivalence, $\Sigma^{-k}l$ is an f-equivalence. Moreover, $\Sigma^{-k}L_{\Sigma^k f}X$ is f-local since $L_{\Sigma^k f}X$ is $\Sigma^k f$-local, so our claim follows. $\square$

This is to be compared with the expression $L_f\Omega^k X \simeq \Omega^k L_{\Sigma^k f}X$ for spaces, which was proved in [Dro96].

Proposition 2.6. *Let $X \to Y \to Z$ be a cofibre sequence of spectra and let f be any map. If L_fX is contractible, then the map $Y \to Z$ is an f-equivalence.*

Proof. Consider the long exact sequence of abelian groups

$$\cdots \to [\Sigma X, E] \to [Z, E] \to [Y, E] \to [X, E] \to \cdots$$

for a spectrum E. If E is f-local, then $[\Sigma^k X, E] = 0$ for $k \geq 0$, by part (b) of Proposition 2.4. Therefore, $[Z, E] \cong [Y, E]$ for all f-local spectra E, and this implies that the map $Y \to Z$ is an f-equivalence. $\square$

For arbitrary spectra X and Y and any map f, there is a natural equivalence

$$L_fX \vee L_fY \simeq L_f(X \vee Y),$$

since the coproduct $X \vee Y \to L_fX \vee L_fY$ is an f-equivalence and $L_fX \vee L_fY$ is a product of f-local spectra, hence f-local (the same argument is valid for any finite wedge). Thus, L_f is an additive functor on HoS.

Localization with respect to a map of the form $f\colon A \to 0$ is called A-*nullification*, and it is denoted by P_A instead of L_f. The corresponding local spectra are called A-*null*. Thus, a spectrum X is A-null if and only if $F^c(A, X) \simeq 0$. For every map f there is a natural transformation $P_C \to L_f$, where C is the cofibre of f. This follows from the fact that every f-local spectrum is C-null.

As a consequence of Proposition 2.4, for every X there is a natural map

$$(2.1) \qquad\qquad \Sigma L_fX \longrightarrow L_f\Sigma X.$$

We say that L_f *commutes with suspension* if this natural map is an equivalence for all X. Thus, L_f commutes with suspension if and only if $\Sigma L_fX \simeq L_f\Sigma X$, since the latter implies that ΣL_fX is f-local, and the map (2.1) is then an f-equivalence between f-local spectra, hence an equivalence.

As we next show in detail, L_f commutes with suspension if and only if it is an exact functor on the stable homotopy category, or a triangulated functor in the sense of Neeman [Nee01].

Theorem 2.7. *Let $f\colon A \to B$ be a map of spectra. Then the following statements are equivalent:*

(i) *$\Sigma L_fX \simeq L_f\Sigma X$ for every spectrum X.*
(ii) *$\Sigma^k L_fX \simeq L_f\Sigma^k X$ for every spectrum X and every $k \in \mathbb{Z}$.*
(iii) *If E is any f-local spectrum, then $\Sigma^k E$ is f-local for all $k \in \mathbb{Z}$.*
(iv) *If g is any f-equivalence, then $\Sigma^k g$ is an f-equivalence for all $k \in \mathbb{Z}$.*
(v) *$L_fX \simeq L_{\Sigma^k f}X$ for every spectrum X and every $k \in \mathbb{Z}$.*
(vi) *The map $F(B, E) \to F(A, E)$ induced by f is an equivalence for every f-local spectrum E.*
(vii) *If E is f-local and X is any spectrum, then $F(X, E)$ is f-local.*

(viii) *If $g\colon X \to Y$ and $h\colon M \to N$ are f-equivalences, then $g \wedge h\colon X \wedge M \to Y \wedge N$ is also an f-equivalence.*

(ix) *If $X \to Y \to Z$ is any cofibre sequence of spectra, then $L_f X \to L_f Y \to L_f Z$ is also a cofibre sequence.*

Proof. Clearly (i) and (ii) are equivalent, and (ii) implies (iii). Note that (iii) implies that $\Sigma L_f X$ is f-local for all X, and hence (i) holds because (2.1) is an f-equivalence. It is straightforward that (iii) and (iv) are equivalent. From (iv) it follows that, given any $k \in \mathbb{Z}$, the map $\Sigma^k f$ is an f-equivalence, and f is also a $\Sigma^k f$-equivalence. This implies that the classes of f-local spectra and $\Sigma^k f$-local spectra coincide, thus yielding (v). Now (v) together with Proposition 2.5 implies (ii).

Statement (vi) is equivalent to the fact that f induces equivalences

$$F^c(B, \Sigma^k E) \simeq F^c(A, \Sigma^k E)$$

for all $k \in \mathbb{Z}$, and hence it follows from (iii). To prove (vii), we have to verify that

$$F^c(B, F(X, E)) \simeq F^c(A, F(X, E)),$$

which is equivalent to $F^c(X, F(B, E)) \simeq F^c(X, F(A, E))$, and this follows from (vi). Statement (viii) is proved from (vii) by taking any f-local spectrum E and observing that

$$F^c(Y \wedge N, E) \simeq F^c(Y, F(N, E)) \simeq F^c(X, F(N, E))$$
$$\simeq F^c(N, F(X, E)) \simeq F^c(M, F(X, E)) \simeq F^c(X \wedge M, E).$$

Now (iv) follows from (viii) by smashing g with the identity of $\Sigma^k S$, for $k \in \mathbb{Z}$. Hence, statements (i) to (viii) are equivalent.

We next deduce (ix) from (i). Given a cofibre sequence $X \to Y \to Z$, consider the ladder

$$
\begin{array}{ccccccccc}
\Sigma^{-1}Z & \longrightarrow & X & \longrightarrow & Y & \longrightarrow & Z & \longrightarrow & \Sigma X \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
L_f\Sigma^{-1}Z & \longrightarrow & L_f X & \longrightarrow & C & \longrightarrow & \Sigma L_f \Sigma^{-1}Z & \longrightarrow & \Sigma L_f X,
\end{array}
$$

where C is the cofibre of the map $L_f \Sigma^{-1} Z \to L_f X$. It follows from (i) that $\Sigma L_f \Sigma^{-1} Z \simeq L_f Z$, and $\Sigma L_f X \simeq L_f \Sigma X$. Thus, all vertical maps except perhaps $Y \to C$ are f-equivalences. The five-lemma implies then that $Y \to C$ is also an f-equivalence and that C is f-local. Therefore $C \simeq L_f Y$. Finally, in order to prove that (ix) implies (i), use the cofibre sequence $X \to 0 \to \Sigma X$, for any X. $\qquad\square$

Statement (vi) tells us that f-localization commutes with suspension if and only if the class of f-local spectra is characterized by means of the full function spectrum F, instead of its connective cover F^c. We also remark that, by the following fact, every f-localization that commutes with suspension is a nullification.

Corollary 2.8. *If a localization functor L_f commutes with suspension, then the natural transformation $P_C \to L_f$, where C is the cofibre of f, is an equivalence.*

Proof. There are natural transformations $L_{\Sigma f} \to P_C \to L_f$, which correspond to inclusions of the respective classes of local spectra. Since the composite $L_{\Sigma f} \to L_f$ is an equivalence by part (v) of Theorem 2.7, the arrow $P_C \to L_f$ is also an equivalence. $\qquad\square$

For a given map f, if we consider the wedge $g = \bigvee_{k \in \mathbb{Z}} \Sigma^k f$, then there is a natural transformation $L_f \to L_g$, and L_g commutes with suspension, by part (v) of Theorem 2.7, since $\Sigma g \simeq g$. This natural transformation $L_f \to L_g$ is an equivalence if and only if L_f commutes with suspension, since a spectrum is then g-local if and only if it is f-local. The same holds if we replace g by any map of the form $\bigvee_{k < N} \Sigma^k f$, where N is any integer.

3. Examples of f-localizations

We briefly discuss three well-known examples of localizations in the stable homotopy category, namely Postnikov sections, homological localizations and localizations at sets of primes, and display them as f-localizations for certain maps f.

3.1. **Postnikov sections.** The localization of a spectrum E with respect to the map $f : \Sigma^{k+1} S \to 0$, where S is the sphere spectrum and $k \in \mathbb{Z}$, is equivalent to the kth Postnikov section $E_{(k)}$ of E. Recall that $\pi_i(E_{(k)}) = 0$ for $i > k$, and that there is a map $E \to E_{(k)}$ inducing isomorphisms of homotopy groups in dimensions less than or equal to k. Postnikov sections do not commute with suspension. Indeed, if $\pi_k(E) \neq 0$, then $(\Sigma E)_{(k)} \not\simeq \Sigma E_{(k)}$.

3.2. **Homological localizations.** Homological localizations in stable homotopy were first constructed in [Bou79]. Let E be any spectrum. Then a spectrum X is called E-*acyclic* if $E_k(X) = 0$ for all $k \in \mathbb{Z}$ or, equivalently, if $E \wedge X \simeq 0$. A map of spectra $g : X \to Y$ is an E-*equivalence* if the map $g_* : E_k(X) \to E_k(Y)$ is an isomorphism for all $k \in \mathbb{Z}$, that is, if g induces an equivalence $E \wedge X \simeq E \wedge Y$. Thus, g is an E-equivalence if and only if its cofibre is E-acyclic. A spectrum Z is E-*local* if each E-equivalence $f : X \to Y$ induces an equivalence $F(Y, Z) \simeq F(X, Z)$ or, equivalently, if $F(A, Z) \simeq 0$ for each E-acyclic spectrum A.

An E-*localization* of a spectrum X is an E-equivalence $X \to L_E X$ from X to an E-local spectrum. E-localization is a nullification, as shown in [Bou79]. We label this fact for further reference:

Theorem 3.1. *Let E be any spectrum. Then there are an E-acyclic spectrum A and a natural equivalence $P_A X \simeq L_E X$ for every spectrum X.* $\square$

Homological localizations clearly commute with suspension.

3.3. **Localization at sets of primes.** Let G be any abelian group and let MG denote its Moore spectrum. Thus, $(H\mathbb{Z})_0(MG) \cong \pi_0(MG) \cong G$, $\pi_i(MG) = 0$ if $i < 0$, and $(H\mathbb{Z})_i(MG) = 0$ if $i \neq 0$. Here $H\mathbb{Z}$ denotes the integral Eilenberg–MacLane spectrum.

Lemma 3.2. *There is a natural exact sequence*

$$0 \longrightarrow \mathrm{Ext}(G, \pi_{k+1}(X)) \longrightarrow [\Sigma^k MG, X] \longrightarrow \mathrm{Hom}(G, \pi_k(X)) \longrightarrow 0$$

for each spectrum X, each abelian group G, and all k, where MG is the Moore spectrum associated with G.

Proof. Pick a free presentation of the group G and use the corresponding cofibre sequence of Moore spectra; cf. [Bou79, (2.2)]. $\square$

Let $\mathbb{Z}_P$ denote the integers localized at a set of primes P (possibly empty). For every spectrum X, the P-*localization* of X is the natural map

$$1 \wedge \eta \colon X \longrightarrow X \wedge M\mathbb{Z}_P$$

where $\eta \colon S \to M\mathbb{Z}_P$ corresponds to the unit in $\mathbb{Z}_P = \pi_0(M\mathbb{Z}_P)$. A spectrum X is P-*local* if this map is an equivalence. We shall write X_P for $X \wedge M\mathbb{Z}_P$.

It follows from the definition that P-localization is a homological localization, namely $M\mathbb{Z}_P$-localization. Hence, P-localization commutes with suspension. We also remark that

$$X_P \simeq X \wedge S_P,$$

for all X; that is, P-localization is *smashing* (see the end of Section 5).

For each $k \in \mathbb{Z}$ we have

$$(3.1) \qquad \pi_k(X_P) \cong \pi_k(X) \otimes \pi_0(M\mathbb{Z}_P) \cong \pi_k(X) \otimes \mathbb{Z}_P,$$

and from this fact it follows that, if E is any spectrum, then

$$E_k(X_P) \cong \pi_k(E \wedge X \wedge M\mathbb{Z}_P) \cong E_k(X) \otimes \mathbb{Z}_P$$

for all X. Hence, P-localization of spectra induces P-localization of their homotopy and homology groups. If we consider cohomology, then Lemma 3.2 yields

$$E^k(X_P) \cong [X \wedge M\mathbb{Z}_P, \Sigma^k E] \cong [M\mathbb{Z}_P, F(X, \Sigma^k E)]$$
$$\cong \mathrm{Hom}(\mathbb{Z}_P, E^k(X)) \oplus \mathrm{Ext}(\mathbb{Z}_P, E^{k-1}(X)).$$

An explicit map f such that $L_f X \simeq X_P$ for all X can be displayed as follows.

Proposition 3.3. *Let P be a set of primes and let $g \colon \bigvee_{q \notin P} S \to \bigvee_{q \notin P} S$ be a wedge of maps inducing multiplication by q in $\pi_0(S)$ for each prime q not in P, and let $f = \bigvee_{n<0} \Sigma^n g$. Then $L_f X \simeq X \wedge M\mathbb{Z}_P$ for all X.*

Proof. The map f has been chosen so that f-local spectra are precisely those spectra whose homotopy groups are $\mathbb{Z}_P$-modules. For a spectrum X, the P-localization map $X \to X_P$ is an equivalence if and only if the homotopy groups of X are $\mathbb{Z}_P$-modules, by (3.1). Thus we conclude the proof by recalling that two localization functors with the same class of local spectra are necessarily equivalent. $\square$

Note that P-localization is also equivalent to nullification with respect to the cofibre of f, which is a wedge of Moore spectra, namely

$$\bigvee_{q \notin P,\, n<0} \Sigma^n M\mathbb{Z}/q.$$

One could also consider the effect of nullification with respect to $M = \bigvee_{q \notin P} M\mathbb{Z}/q$ (without the desuspensions). The resulting functor does not commute with suspension and does not change the homotopy groups of spectra in negative dimensions, since M is connective (see Remark 4.6). From Lemma 3.2 it follows that the M-null spectra are the spectra X such that $\pi_n(X)$ is a $\mathbb{Z}_P$-module for $n > 0$ and $\pi_0(X)$ is P'-torsion free, where P' denotes the complement of P. Hence, nullification with respect to M kills the P'-torsion on π_0 and P-localizes the homotopy groups in positive dimensions. In fact there is a sequence of localization functors,
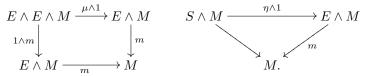
$$\cdots \to L_{\Sigma g} \to P_M \to L_g \to P_{\Sigma^{-1} M} \to L_{\Sigma^{-1} g} \to P_{\Sigma^{-2} M} \to \cdots,$$

converging to the identity in one direction and to P-localization in the other direction, similarly as in [CR97].

4. Localizations of ring spectra and module spectra

We recall the definition of ring spectra and module spectra in the homotopy category, as in [Ada74]. A spectrum E is called a *ring spectrum* if it is equipped with two maps $\mu\colon E \wedge E \to E$ and $\eta\colon S \to E$ such that the following diagrams commute up to homotopy:

$$
\begin{array}{ccc}
E \wedge E \wedge E & \xrightarrow{\ \mu\wedge1\ } & E \wedge E \\
{\scriptstyle 1\wedge\mu}\downarrow & & \downarrow{\scriptstyle \mu} \\
E \wedge E & \xrightarrow{\ \mu\ } & E
\end{array}
\qquad
\begin{array}{ccccc}
S \wedge E & \xrightarrow{\ \eta\wedge1\ } & E \wedge E & \xleftarrow{\ 1\wedge\eta\ } & E \wedge S \\
& \searrow & \downarrow{\scriptstyle \mu} & \swarrow & \\
& & E. & &
\end{array}
$$

It is said that E is *commutative* if $\mu \circ \tau \simeq \mu$, where $\tau\colon E \wedge E \to E \wedge E$ is the twist map. A spectrum M is called a *module spectrum* over a ring spectrum E, or an *E-module*, if it is equipped with a map $m\colon E \wedge M \to M$ such that the following diagrams commute up to homotopy:

$$
\begin{array}{ccc}
E \wedge E \wedge M & \xrightarrow{\ \mu\wedge1\ } & E \wedge M \\
{\scriptstyle 1\wedge m}\downarrow & & \downarrow{\scriptstyle m} \\
E \wedge M & \xrightarrow{\ m\ } & M
\end{array}
\qquad
\begin{array}{ccc}
S \wedge M & \xrightarrow{\ \ \ \eta\wedge1\ \ \ } & E \wedge M \\
& \searrow \qquad \swarrow{\scriptstyle m} & \\
& M. &
\end{array}
$$

Every ring spectrum E is an E-module spectrum with $m = \mu$.

A *ring map* between ring spectra (E,μ,η) and (E',μ',η') is a map $f\colon E \to E'$ such that $f \circ \mu \simeq \mu' \circ (f \wedge f)$ and $f \circ \eta \simeq \eta'$. An *E-module map* is defined similarly.

If R is a ring with unit and M is a left R-module, then the Eilenberg–Mac Lane spectrum HR is a ring spectrum, and HM is a module spectrum over HR. The structure maps on HR and HM come from the product $R \otimes R \to R$ and the unit $\mathbb{Z} \to R$ in the ring R, and from the structure homomorphism $R \otimes M \to M$ of M as an R-module. In particular, HR is an $H\mathbb{Z}$-module for every ring R. Moreover, every HR-module is an $H\mathbb{Z}$-module via the map $H\mathbb{Z} \to HR$ corresponding to the unit $\mathbb{Z} \to R$.

Remark 4.1. If M is an E-module spectrum, then, for every spectrum X, the graded abelian group $[\Sigma^* X, M]$ is a $\pi_*(E)$-module, as follows. For every map $\alpha\colon \Sigma^i S \to E$ and every map $f\colon \Sigma^j X \to M$ we obtain another map $\Sigma^{i+j} X \to M$ by smashing α with f and composing with the E-module structure map:

$$
\Sigma^{i+j} X \simeq \Sigma^i S \wedge \Sigma^j X \xrightarrow{\ \alpha\wedge f\ } E \wedge M \xrightarrow{\ m\ } M.
$$

In particular, if M is an HR-module spectrum, then $\pi_n(M)$ is an R-module for every n.

As we next prove, in the case of f-localization functors that commute with suspension, the f-localizations of ring spectra or module spectra acquire a compatible ring structure or module structure. In the rest of this section, we assume that f is a fixed map of spectra, and write L instead of L_f.

Theorem 4.2. *If the localization functor L commutes with suspension, then the following hold:*

(i) *If E is a ring spectrum, then the spectrum LE has a unique ring spectrum structure such that the localization map $l_E\colon E \to LE$ is a ring map. If E is commutative, then LE is also commutative.*

(ii) *If M is an E-module, then the spectrum LM has a unique E-module struc-*
ture such that the localization map $l_M \colon M \to LM$ is an E-module map.
Moreover, LM admits a unique LE-module structure extending the E-
module structure.

Proof. For the first part we need to construct a product $\overline{\mu}$ and a unit $\overline{\eta}$ on LE.
Let μ and η be the product and the unit of the ring spectrum E, respectively. We
have an equivalence $F(E, LE) \simeq F(LE, LE)$ because LE is local and the functor
L commutes with suspension by assumption. Then,

$$[E \wedge E, LE] \cong [E, F(E, LE)] \cong [E, F(LE, LE)] \cong [E \wedge LE, LE]$$
$$\cong [LE, F(E, LE)] \cong [LE, F(LE, LE)] \cong [LE \wedge LE, LE].$$

Hence, the product $\mu \colon E \wedge E \to E$ extends to a unique map $\overline{\mu} \colon LE \wedge LE \to LE$
rendering homotopy commutative the diagram

$$
\begin{array}{ccc}
E \wedge E & \xrightarrow{\ \mu\ } & E \\
{\scriptstyle l_E \wedge l_E}\downarrow & & \downarrow{\scriptstyle l_E} \\
LE \wedge LE & \xdashrightarrow[\ \overline{\mu}\ ]{} & LE.
\end{array}
$$

We define the unit $\overline{\eta}$ as the composition $l_E \circ \eta$. The commutativity of the diagrams
for $\overline{\mu}$ and $\overline{\eta}$ follows from the commutativity of the diagrams for μ and η and the
universal property of L (using part (viii) of Theorem 2.7).

The commutativity of LE when E is commutative and all the statements in
part (ii) are proved in the same way. $\qquad\square$

As we next show, localization functors not commuting with suspension need not
preserve ring structures nor module structures in general. The following lemma is
useful to prove that certain spectra fail to be ring spectra or module spectra.

Lemma 4.3. *Let E and F be ring spectra, and let M be an E-module spectrum.*
If $F_0(E) = 0$, then $F_i(M) = 0$ for all $i \in \mathbb{Z}$.

Proof. Since E and F are ring spectra, $E \wedge F$ is also a ring spectrum. The assump-
tion $F_0(E) = 0$ says that the unit map $S \to E \wedge F$ is null, and this implies that
$E \wedge F \simeq 0$. If M is an E-module, then the composite

$$S \wedge M \longrightarrow E \wedge M \longrightarrow M,$$

where the first arrow comes from the unit of E and the second arrow is the structure
map of M as an E-module, is an equivalence. By smashing this sequence with F,
we find that $F \wedge M \simeq 0$, as claimed. $\qquad\square$

Example 4.4. Given a natural number n and a fixed prime p, let $K(n)$ denote the
ring spectrum corresponding to nth Morava K-theory. If we consider its 0th Post-
nikov section $K(n)_{(0)}$, then, as observed in p. 95 of [Rud98], $(H\mathbb{Z}/p)_0\big(K(n)_{(0)}\big) = 0$
and $(H\mathbb{Z}/p)_i\big(K(n)_{(0)}\big) \neq 0$ for some $i > 0$. This implies that $K(n)_{(0)}$ cannot be
a ring spectrum, by Lemma 4.3. We give another approach with further details to
this example, in view of its relevance. There is a cofibre sequence

$$\Sigma^d k(n) \longrightarrow K(n) \longrightarrow K(n)_{(0)},$$

where $d = 2(p^n - 1)$ and $k(n)$ denotes as usual the connective cover of $K(n)$.
Suppose that $K(n)_{(0)}$ were a ring spectrum. Then $K(n)_{(0)} \wedge H\mathbb{Z}/p$ would also be

a ring spectrum. Since $K(n) \wedge H\mathbb{Z}/p \simeq 0$, it follows that $(H\mathbb{Z}/p)_0 \left(K(n)_{(0)} \right) = 0$. Therefore the unit $S \to K(n)_{(0)} \wedge H\mathbb{Z}/p$ would be null, and this would imply that $K(n)_{(0)} \wedge H\mathbb{Z}/p \simeq 0$. But this is false, since the mod p cohomology of $k(n)$ is a nonzero quotient of the Steenrod algebra; see, e.g., p. 545 of [Rud98].

A similar argument, now considering $K(n)$ as a $K(n)$-module, and using that $(H\mathbb{Z}/p)_0(K(n)) = 0$ and Lemma 4.3, shows that $K(n)_{(0)}$ is not a $K(n)$-module.

This difficulty can be repaired by imposing suitable connectivity conditions. The following result extends an observation made by Bousfield in [Bou99].

Theorem 4.5. *Let L be any localization. Then the following hold:*

  (i) *If E is a connective ring spectrum and LE is connective, then the spectrum LE has a unique ring structure such that the localization map $l_E \colon E \to LE$ is a ring map. If E is commutative, then LE is also commutative.*

  (ii) *If M is an E-module, where E is a connective ring spectrum, then LM has a unique E-module structure such that the localization map $l_M \colon M \to LM$ is an E-module map. Moreover, if LE is connective, then LM also admits a unique LE-module structure extending the E-module structure.*

Proof. Using that E is a connective spectrum, we have equivalences
$$F^c(E, F^c(X, Y)) \simeq F^c(E, F(X, Y)) \simeq F^c(E \wedge X, Y)$$
that give a bijection $[E, F^c(X, Y)] \cong [E \wedge X, Y]$. Then one proceeds as in the proof of Theorem 4.2. $\qquad\square$

Remark 4.6. The assumption that LE is connective in part (i) of Theorem 4.5 is automatically fulfilled if E, A and B are connective, by the following argument: A spectrum X is connective if and only if $P_S X \simeq 0$, where S is the sphere spectrum. Therefore, if A and B are connective, then $f \colon A \to B$ is a P_S-equivalence. This implies that every f-equivalence is also a P_S-equivalence, and in particular the localization map $E \to LE$ is a P_S-equivalence, from which it follows that L does not change the homotopy groups in negative dimensions.

5. Localization of stable GEMs

Using results due to Bousfield [Bou99] and Rudyak [Rud98], we next show that the stable GEMs are precisely the $H\mathbb{Z}$-modules. Thus, the results in the previous section imply that every f-localization sends stable GEMs to stable GEMs. In addition, we will describe all the localizations of the ring spectrum $H\mathbb{Z}$.

Definition 5.1. Let R be an arbitrary ring with unit. A spectrum E is called a *stable R-GEM* if it is equivalent to a wedge of suspensions of Eilenberg–Mac Lane spectra, $E \simeq \bigvee_{k \in \mathbb{Z}} \Sigma^k HA_k$, where each A_k is an R-module (hence, each HA_k is an HR-module spectrum). If $R = \mathbb{Z}$, then stable $\mathbb{Z}$-GEMs are called *stable GEMs*.

Note that, if E is a ring spectrum, then for every spectrum X, the smash product $E \wedge X$ has an E-module structure given by $\mu \wedge 1 \colon E \wedge E \wedge X \to E \wedge X$.

For any ring spectrum (E, μ, η), the triple $(E \wedge -, \eta \wedge 1, \mu \wedge 1)$ is a monad on the stable homotopy category $\mathrm{Ho}\mathbb{S}$. The algebras over this monad are pairs (M, m), where M is a spectrum and $m \colon E \wedge M \to M$ is a map that renders certain diagrams commutative; see [Bor94]. These diagrams endow M precisely with the structure of an E-module spectrum. Thus, the category of algebras (also called the Eilenberg–Moore category) associated with the monad $(E \wedge -, \eta \wedge 1, \mu \wedge 1)$ is the category

whose objects are the E-module spectra and whose morphisms are the E-module maps in HoS. We denote this category by $\mathrm{HoS}_{E\text{-mod}}$ and denote by $[M, N]_{E\text{-mod}}$ the corresponding group of E-module maps from M to N.

Every monad factors as a pair of adjoint functors through its Eilenberg–Moore category; see [Bor94]. In our case, this gives the following result. We are indebted to Gustavo Granja for pointing out this result to us, and to the referee for observing that it comes from the fact that $\mathrm{HoS}_{E\text{-mod}}$ is an Eilenberg–Moore category, so we may omit the proof.

Lemma 5.2. *Let E be any ring spectrum. Then the functor $\mathrm{HoS} \to \mathrm{HoS}_{E\text{-mod}}$ assigning to each spectrum X the spectrum $E \wedge X$ is left adjoint to the forgetful functor $\mathrm{HoS}_{E\text{-mod}} \to \mathrm{HoS}$. That is, for every spectrum X and every E-module M, there is a natural isomorphism*

$$[X, M] \cong [E \wedge X, M]_{E\text{-mod}}$$

induced by the unit of E. □

Proposition 5.3. *For every $H\mathbb{Z}$-module M, there is a map of $H\mathbb{Z}$-modules*

$$H\mathbb{Z} \wedge \left(\bigvee_{k \in \mathbb{Z}} \Sigma^k MG_k \right) \xrightarrow{\tilde{\alpha}} M$$

which is an equivalence, where $G_k = \pi_k(M)$.

Proof. An argument was sketched in [Ada74, p. 307]. For each $k \in \mathbb{Z}$, take one map $\alpha_k \in [\Sigma^k MG_k, M]$ mapping to the identity in $\mathrm{Hom}(G_k, G_k)$. Let $\alpha = \bigvee_{k \in \mathbb{Z}} \alpha_k$ be the wedge of all α_k constructed in this way. This yields, by Lemma 5.2, an $H\mathbb{Z}$-module map

$$\tilde{\alpha} \colon H\mathbb{Z} \wedge \left(\bigvee_{k \in \mathbb{Z}} \Sigma^k MG_k \right) \longrightarrow M,$$

namely $\tilde{\alpha} = m \circ (1 \wedge \alpha)$. Now the map $\tilde{\alpha}$ induces an isomorphism on π_k for all $k \in \mathbb{Z}$, because $\eta \wedge 1$ and $1 \wedge \alpha$ are isomorphisms on π_k. So $\tilde{\alpha}$ is an equivalence. □

Proposition 5.3 tells us that $H\mathbb{Z}$-module spectra and stable GEMs are the same thing, because $H\mathbb{Z} \wedge MG \simeq HG$ for any abelian group G, and therefore, if M is an $H\mathbb{Z}$-module, then $M \simeq \bigvee_{k \in \mathbb{Z}} \Sigma^k H\pi_k(M)$. Similarly, the HR-modules are precisely the stable R-GEMs, because each HR-module spectrum is an $H\mathbb{Z}$-module spectrum, and the homotopy groups of HR-module spectra are R-modules (see Remark 4.1).

Corollary 5.4. *Let A and B be abelian groups. Then*

$$[HA, HB]_{H\mathbb{Z}\text{-mod}} \cong \mathrm{Hom}(A, B),$$

$$[HA, \Sigma HB]_{H\mathbb{Z}\text{-mod}} \cong \mathrm{Ext}(A, B),$$

$$[HA, \Sigma^k HB]_{H\mathbb{Z}\text{-mod}} = 0 \quad \textit{if } k \neq 0, 1.$$

Proof. As a special case of Proposition 5.3, $HA \simeq H\mathbb{Z} \wedge MA$ as $H\mathbb{Z}$-modules. By Lemma 5.2, there is a natural bijection

$$[MA, \Sigma^k HB] \cong [HA, \Sigma^k HB]_{H\mathbb{Z}\text{-mod}},$$

and the result follows using Lemma 3.2. □

In the rest of this section, L denotes f-localization with respect to a fixed but arbitrary map f. Using the results on f-localizations of E-modules of Section 4, we have the following.

Theorem 5.5. *Let R be any ring with unit. If E is a stable R-GEM, then LE is a stable R-GEM and the localization map $l_E \colon E \to LE$ is an HR-module map.*

Proof. A stable R-GEM is the same as an HR-module, and HR is a connective spectrum. Hence we may apply Theorem 4.5. $\qquad\square$

Next, we are going to study the case when the spectrum E is an iterated suspension of a single Eilenberg–Mac Lane spectrum, i.e., $E \simeq \Sigma^n HG$, where G is an R-module and $n \in \mathbb{Z}$. By Theorem 5.5, we know that $L\Sigma^n HG \simeq \bigvee_{k \in \mathbb{Z}} \Sigma^k HG_k$ with each G_k an R-module. In fact, most of the R-modules G_k are zero, as we next explain. Consider the following sequence of $H\mathbb{Z}$-module maps, where β is a homotopy inverse of the map given by Proposition 5.3, and p_i is the projection onto the ith factor:

$$\Sigma^n HG \xrightarrow{\ l\ } L\Sigma^n HG \xrightarrow{\ \beta\ } \bigvee_{k \in \mathbb{Z}} \Sigma^k HG_k \xrightarrow{\ p_i\ } \Sigma^i HG_i.$$

By Corollary 5.4,

$$[\Sigma^n HG, \Sigma^i HG_i]_{H\mathbb{Z}\text{-mod}} = 0$$

unless $i = n$ or $i = n + 1$. The universal property of localization and the fact that $\Sigma^i HG_i$ is f-local because it is a retract of $\bigvee_{k \in \mathbb{Z}} \Sigma^k HG_k$ (see Lemma 2.1) tell us that $G_i = 0$ if $i \neq n$ or $i \neq n + 1$.

Therefore, the localization of any iterated suspension of an Eilenberg–Mac Lane spectrum has at most two nonzero homotopy groups:

Theorem 5.6. *Let G be any abelian group and $n \in \mathbb{Z}$. Then*

$$L\Sigma^n HG \simeq \Sigma^n HG_1 \vee \Sigma^{n+1} HG_2$$

as $H\mathbb{Z}$-modules, for some abelian groups G_1, G_2. If G is an R-module for some ring R, then G_1 and G_2 are also R-modules. $\qquad\square$

Corollary 5.7. *If $E \simeq \bigvee_{i \in \mathbb{Z}} \Sigma^i HA_i$ then $LE \simeq \bigvee_{i \in \mathbb{Z}} L\Sigma^i HA_i$.*

Proof. The spectrum $\bigvee_{i \in \mathbb{Z}} L\Sigma^i HA_i$ is f-local because it is equivalent to the product $\prod_{i \in \mathbb{Z}} L\Sigma^i HA_i$, by Proposition III.3.14 in [Ada74] (here we use Theorem 5.6 to ensure that, for each value of i, only a finite number of homotopy groups —in fact, at most two— are nonzero), and the map

$$\bigvee_{i \in \mathbb{Z}} \Sigma^i HA_i \longrightarrow \bigvee_{i \in \mathbb{Z}} L\Sigma^i HA_i$$

is an f-equivalence, because it is a wedge of f-equivalences. $\qquad\square$

There are some special cases in which the localization of an Eilenberg–Mac Lane spectrum is a single Eilenberg–Mac Lane spectrum.

Theorem 5.8. *If G is free abelian and n is any integer, then $L\Sigma^n HG \simeq \Sigma^n HA$ for some abelian group A.*

Proof. From Theorem 5.6, we know that $L\Sigma^n HG$ has at most two nonzero homotopy groups, say A and B. By Corollary 5.4,

$$[\Sigma^n HG, \Sigma^{n+1} HB]_{H\mathbb{Z}\text{-mod}} \cong \text{Ext}(G, B).$$

If G is free, then $\text{Ext}(G, B) = 0$, and this tells us that the projection $\Sigma^n HG \to \Sigma^{n+1} HB$ is nullhomotopic. Moreover, $\Sigma^{n+1} HB$ is f-local, because it is a retract of $L\Sigma^n HG$. Then the universal property of localization forces that $B = 0$. $\square$

Theorem 5.9. *If p is a prime, k is a positive integer and n is any integer, then $L\Sigma^n H\mathbb{Z}/p^k \simeq \Sigma^n H\mathbb{Z}/p^i$ with $0 \le i \le k$.*

Proof. We assume that $n = 0$ for simplicity, and start by considering the case $k = 1$. By Theorem 5.6,

$$L H\mathbb{Z}/p \simeq HA_1 \vee \Sigma HA_2,$$

where A_1 and A_2 are $\mathbb{Z}/p$-modules. Since retracts and desuspensions of f-local spectra are f-local, it follows that either $A_1 = 0$ and $A_2 = 0$, or $H\mathbb{Z}/p$ is f-local. If $A_1 = 0$ and $A_2 = 0$, then $LH\mathbb{Z}/p$ is contractible and the cofibre sequences

$$H\mathbb{Z}/p \longrightarrow H\mathbb{Z}/p^r \longrightarrow H\mathbb{Z}/p^{r-1}$$

imply inductively that $LH\mathbb{Z}/p^r$ is contractible for all r, by Proposition 2.6.

Now let k be any positive integer. By the preceding discussion, we may assume that $H\mathbb{Z}/p$ is f-local. Using Theorem 5.6 again, we may write

$$LH\mathbb{Z}/p^k \simeq HB_1 \vee \Sigma HB_2$$

where B_1 and B_2 are $\mathbb{Z}/p^k$-modules (i.e., abelian groups with an exponent dividing p^k). Therefore, by [Kap69], B_1 and B_2 are direct sums of subgroups isomorphic to $\mathbb{Z}/p^j$ with $j \le k$. If $B_2 = 0$, then $\text{Hom}(B_1, B_1) \cong \text{Hom}(\mathbb{Z}/p^k, B_1)$, by the universal property of L, and this forces B_1 to consist of only one summand and $\mathbb{Z}/p^k \to B_1$ to be surjective (as in Theorem 3.4 in [Cas00]). Finally, we suppose that B_2 is nonzero and will arrive at a contradiction. Let m be the smallest integer such that $\mathbb{Z}/p^m$ is a direct summand of B_2. Then $\Sigma H\mathbb{Z}/p^m$ and $H\mathbb{Z}/p^m$ are f-local. Now the cofibre sequences

$$H\mathbb{Z}/p^{(r+1)m} \longrightarrow H\mathbb{Z}/p^{rm} \longrightarrow \Sigma H\mathbb{Z}/p^m$$

imply inductively that $H\mathbb{Z}/p^{rm}$ is f-local for $r \ge 1$, by Lemma 2.2. Choose r such that $rm > k$ and infer that $H\mathbb{Z}/p^k$ is f-local, by using downwards the cofibre sequences

$$H\mathbb{Z}/p^{i-1} \longrightarrow H\mathbb{Z}/p^i \longrightarrow H\mathbb{Z}/p.$$

This is indeed a contradiction. $\square$

Corollary 5.10. *If G is a finitely generated abelian group and n is any given integer, then $L\Sigma^n HG \simeq \Sigma^n HA$ for some abelian group A.*

Proof. This follows from Theorem 5.8 and Theorem 5.9, since L commutes with finite products. $\square$

Bousfield has informed us that the conclusion that $L\Sigma^n HG$ has at most one nonzero homotopy group holds, more generally, if G is a reduced abelian group; that is, if the only divisible subgroup of G is the trivial subgroup. The argument will not be given in this article.

If we now project $L\Sigma^n HG$ onto the first summand

$$\Sigma^n HG \xrightarrow{\ l\ } \Sigma^n HA \vee \Sigma^{n+1} HB$$

with maps $p_1 \circ l$ going diagonally and p_1 going down to $\Sigma^n HA$.

we can obtain information about the group A. The universal property of L yields an isomorphism of abelian groups

$$[\Sigma^n HA, \Sigma^n HA] \times [\Sigma^{n+1} HB, \Sigma^n HA] \cong [HG, HA],$$

where $[\Sigma^{n+1} HB, \Sigma^n HA] = 0$ because

$$[\Sigma HB, HA] \cong (HA)^0(\Sigma HB) \cong \operatorname{Hom}(\pi_0(\Sigma HB), A) = 0.$$

Hence we get

$$\operatorname{Hom}(A, A) \cong [\Sigma^n HA, \Sigma^n HA] \cong [HG, HA] \cong \operatorname{Hom}(G, A).$$

In the case when $G = \mathbb{Z}$, this says that $\operatorname{Hom}(A, A) \cong A$. Therefore, if A is nonzero, then A admits a ring structure, with a multiplication coming from composition in $\operatorname{Hom}(A, A)$ and a unit coming from the identity homomorphism. Moreover, the isomorphism $\operatorname{Hom}(A, A) \cong A$ is given by evaluation at the unit.

Definition 5.11. A ring A with unit such that $\operatorname{Hom}(A, A) \cong A$ via $\varphi \mapsto \varphi(1)$ is called *rigid*.

This terminology was first used in [CRT00]. However, rigid rings had previously been studied in a different context, under the name of *E-rings*. The most obvious examples of such rings are $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{Z}/p$, or the p-adics $\hat{\mathbb{Z}}_p$, for any p. There are many other examples: for instance, each product $\prod_{p \in P} \hat{\mathbb{Z}}_p$ is rigid, for any set of (distinct) primes P. In fact, as shown in [DMV87], there are rigid rings of arbitrarily large cardinality. However, neither the group $\mathbb{Z}/p^\infty$ of pth roots of unity nor the p-adic field $\hat{\mathbb{Q}}_p$ admits the structure of a rigid ring. Therefore, there is no localization L whatsoever such that $\pi_0(LH\mathbb{Z})$ is isomorphic to $\mathbb{Z}/p^\infty$ or $\hat{\mathbb{Q}}_p$.

Rigid rings are commutative. Solid rings in the sense of [BK72b] are rigid (yet, all solid rings are countable). Proofs of these claims and further details about rigid rings can be found in [CRT00].

All rigid rings occur as homotopy groups of localizations of $H\mathbb{Z}$, since, if A is any rigid ring, then $L_f H\mathbb{Z} \simeq HA$, where f is the map $H\mathbb{Z} \to HA$ induced by the unit homomorphism $\mathbb{Z} \to A$. Alternatively, if we choose $g \colon S \to MA$, then the map $H\mathbb{Z} \to HA$ is a g-equivalence, and HA is g-local, so $L_g H\mathbb{Z} \simeq HA$ as well.

We can summarize the results obtained for f-localizations of $H\mathbb{Z}$ in the following theorem.

Theorem 5.12. *Let f be any map of spectra. Then the f-localization of $H\mathbb{Z}$ has at most one nonzero homotopy group, i.e., $L_f H\mathbb{Z} \simeq HA$. Moreover, the group A has a rigid ring structure if $A \neq 0$. All rigid rings appear this way.* $\square$

Corollary 5.13. *There is a proper class of non-equivalent f-localizations.*

Proof. According to results in [DMV87], there is a proper class of non-isomorphic rigid rings. $\square$

We conclude with another application. We say that a localization L is *smashing* if the natural map $1 \wedge l_S \colon X \to X \wedge LS$ is an L-localization for all spectra X, where $l_S \colon S \to LS$ is the localization of the sphere spectrum. It follows from this definition that every smashing localization is homological (namely, $L \simeq L_E$, where $E = LS$) and hence it commutes with suspension. Moreover, LS is a commutative ring spectrum, by Theorem 4.2 (this was already observed by Bousfield in [Bou79]), and the multiplication map $LS \wedge LS \to LS$ is an equivalence, since it is a left inverse of $1 \wedge l_S \colon LS \to LS \wedge LS$. Conversely, if E is a ring spectrum and the multiplication map $E \wedge E \to E$ is an equivalence, then L_E is smashing, by [Rav84, 1.29].

As also shown in [Rav84, 1.27], a homological localization L_E is smashing if and only if it commutes with direct limits. This happens, for example, if E is the spectrum K of (complex) K-theory or the Johnson–Wilson spectrum $E(n)$ for any n.

Theorem 5.14. *If L is smashing, then $H_n(LS) = 0$ if $n \neq 0$, and it is either zero or a subring of the rationals if $n = 0$.*

Proof. We have

$$H_n(LS) = \pi_n(H\mathbb{Z} \wedge LS) \cong \pi_n(LH\mathbb{Z}) \cong \pi_n(HA)$$

for some rigid ring A, by Theorem 5.12. Hence, $H_n(LS) = 0$ if $n \neq 0$. From the fact that $LS \wedge LS \simeq LS$ and the Künneth theorem, we infer that

$$H_0(LS) \otimes H_0(LS) \cong H_0(LS)$$

and

$$\mathrm{Tor}(H_0(LS), H_0(LS)) = 0.$$

Hence, $H_0(LS)$ is a torsion-free solid ring. Thus, it is a subring of the rationals, by the classification of solid rings; see [BK72a]. $\qquad\square$

Corollary 5.15. *If L is a smashing localization such that $H_0(LS) \cong \mathbb{Q}$, then either L is ordinary $H\mathbb{Q}$-localization, or the homotopy groups $\pi_i(LS)$ are nonzero for infinitely many negative values of i.*

Proof. The assumption made implies that $LS \wedge H\mathbb{Q}$ is nonzero, and this implies that $H\mathbb{Q}$ is L-local. Now the unit $S \to H\mathbb{Q}$ factors through LS, yielding a map $LS \to H\mathbb{Q}$ which, according to Theorem 5.14, is an integral homology equivalence. If LS is bounded below, then it follows that $LS \simeq H\mathbb{Q}$, from which we infer that $LX \simeq X \wedge LS \simeq X \wedge H\mathbb{Q}$ for all X. $\qquad\square$

The ring $H_0(LS)$ happens to be $\mathbb{Q}$, for instance, if L is localization with respect to $E = K$ or $E = E(n)$ for any n. In each of these cases, the spectrum $H\mathbb{Q}$ is E-local, since it is a retract of $E \wedge H\mathbb{Q}$ (which is nonzero). Hence, it suffices to show that the natural map $H\mathbb{Z} \to H\mathbb{Q}$ is an E-equivalence. For this, we may use the fact that $E_k(H\mathbb{Z}) = \lim_i E_{k+i}(K(\mathbb{Z}, i))$. For $E = K$, the result follows from [AH68]. For $E = E(n)$, it is a consequence of [Bou82, Example 7.5].

The above remarks yield an easy proof of the well-known fact that the spectra $L_K S$ and $L_{E(n)} S$ are not bounded below.

References

[Ada74] J. F. Adams, Stable Homotopy and Generalised Homology, University of Chicago Press, Chicago, 1974.

[AH68] D. W. Anderson and L. Hodgkin, The K-theory of Eilenberg–Mac Lane complexes, Topology 7 (1968), pp. 317–329.

[Bad01] B. Badzioch, Recognition principle for generalized Eilenberg–Mac Lane spaces, In: Cohomological Methods in Homotopy Theory (Barcelona, 1998), Progress in Math., vol. 196, Birkhäuser, Basel, 2001, pp. 21–26.

[Bor94] F. Borceux, Handbook of Categorical Algebra II: Categories and Structures, Encyclopedia of Mathematics and its Applications, vol. 51, Cambridge University Press, Cambridge, 1994.

[Bou77] A. K. Bousfield, Constructions of factorization systems in categories, J. Pure Appl. Algebra 9 (1976/77), no. 2, pp. 207–220.

[Bou79] A. K. Bousfield, The localization of spectra with respect to homology, Topology 18 (1979), no. 4, pp. 257–281.

[Bou82] A. K. Bousfield, On homology equivalences and homological localizations of spaces, Amer. J. Math. 104 (1982), no. 5, pp. 1025–1042.

[Bou96] A. K. Bousfield, Unstable localization and periodicity, In: Algebraic Topology: New Trends in Localization and Periodicity (Sant Feliu, 1994), Progress in Math., vol. 136, Birkhäuser, Basel, 1996, pp. 33–50.

[Bou99] A. K. Bousfield, On $K(n)$-equivalences of spaces, In: Homotopy Invariant Algebraic Structures (Baltimore, 1998), Contemp. Math., vol. 239, Amer. Math. Soc., Providence, 1999, pp. 85–89.

[BF78] A. K. Bousfield and E. M. Friedlander, Homotopy theory of Γ-spaces, spectra, and bisimplicial sets, In: Geometric Applications of Homotopy Theory (Evanston, 1977), II, Lecture Notes in Math., vol. 658, Springer-Verlag, Berlin Heidelberg New York, 1978, pp. 80–130.

[BK72a] A. K. Bousfield and D. M. Kan, The core of a ring, J. Pure Appl. Algebra 2 (1972), pp. 73–81.

[BK72b] A. K. Bousfield and D. M. Kan, Homotopy Limits, Completions and Localizations, Lecture Notes in Math., vol. 304, Springer-Verlag, Berlin Heidelberg New York, 1972.

[Cas00] C. Casacuberta, On structures preserved by idempotent transformations of groups and homotopy types, In: Crystallographic Groups and Their Generalizations (Kortrijk, 1999), Contemp. Math., vol. 262, Amer. Math. Soc., Providence, 2000, pp. 39–68.

[CR97] C. Casacuberta and J. L. Rodríguez, On towers approximating homological localizations, J. London Math. Soc. 56 (1997), pp. 645–656.

[CRT00] C. Casacuberta, J. L. Rodríguez, and J.-Y. Tai, Localizations of abelian Eilenberg–Mac Lane spaces of finite type, preprint, 2000.

[Dro96] E. Dror Farjoun, Cellular Spaces, Null Spaces and Homotopy Localization, Lecture Notes in Math., vol. 1622, Springer-Verlag, Berlin Heidelberg New York, 1996.

[DMV87] M. Dugas, A. Mader, and C. Vinsonhaler, Large E-rings exist, J. Algebra 108 (1987), pp. 88–101.

[DP01] W. G. Dwyer, J. H. Palmieri, Ohkawa's theorem: there is a set of Bousfield classes, Proc. Amer. Math. Soc. 129 (2001), no. 3, pp. 881–886.

[EKMM97] A. D. Elmendorf, I. Kriz, M. A. Mandell, and J. P. May, Rings, Modules, and Algebras in Stable Homotopy Theory, Math. Surveys Monographs, vol. 47, Amer. Math. Soc., Providence, 1997.

[GU71] P. Gabriel and F. Ulmer, Lokal Präsentierbare Kategorien, Lecture Notes in Math., vol. 221, Springer-Verlag, Berlin Heidelberg New York, 1971.

[GJ99] P. G. Goerss and J. F. Jardine, Simplicial Homotopy Theory, Progress in Math., vol. 174, Birkhäuser, Basel, 1999.

[Gut03] J. J. Gutiérrez, Strict modules and homotopy modules in stable homotopy, preprint, 2003.

[Hir03] P. S. Hirschhorn, Model Categories and Their Localizations, Math. Surveys Monographs, vol. 99, Amer. Math. Soc., Providence, 2003.

[Hov95] M. Hovey, Cohomological Bousfield classes, J. Pure Appl. Algebra 103 (1995), pp. 45–59.

[HPS97] M. Hovey, J. H. Palmieri, and N. P. Strickland, Axiomatic Stable Homotopy Theory, Mem. Amer. Math. Soc. 128 (1997), no. 610.

[HSS00] M. Hovey, B. Shipley, and J. Smith, Symmetric spectra, J. Amer. Math. Soc. 13 (2000), no. 1, pp. 149-208.

[Kap69] I. Kaplansky, Infinite Abelian Groups, The Univ. of Michigan Press, Ann Arbor, 1969.

[Nee01] A. Neeman, Triangulated Categories, Ann. of Math. Studies, vol. 148, Princeton University Press, Princeton, 2001.

[Rav84] D. C. Ravenel, Localization with respect to certain periodic homology theories, Amer. J. Math. 106 (1984), pp. 351–414.

[Rud98] Y. B. Rudyak, On Thom Spectra, Orientability, and Cobordism, Springer Monographs in Math., Springer-Verlag, Berlin Heidelberg New York, 1998.

[Str04] N. P. Strickland, Axiomatic stable homotopy —a survey, In: Axiomatic, Enriched and Motivic Homotopy Theory (Cambridge, 2002), NATO Science Series II, Mathematics, Physics and Chemistry, vol. 131, Kluwer Academic Publishers, Dordrecht, 2004, pp. 69–98.

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