ALGEBRAIC TOPOLOGY

MASTERMATH (FALL 2014)

Written exam, 21/01/2015, 3 hours

State clearly your name, student number and university **on every page**, and also number the pages. Begin each answer on a separate sheet and try to write legibly. You are not allowed to use laptops, cell phones or similar devices during the exam. Do not get stuck for too long on parts you cannot do, but try first those parts that you think are easy or doable. Good luck!

Exercise 1. *In this exercise all spaces and maps should be taken to be pointed.*

(i) Give the definition of an H-space.

(ii) Recall that for every pointed space X the fundamental group $\pi_1(X)$ acts on the higher homotopy groups $\pi_n(X)$. Describe the definition of this action and show that for $n = 1$ the action is given by conjugation.

(iii) Now let X be an H-space. Prove that $\pi_1(X)$ is abelian and that $\pi_1(X)$ acts trivially on $\pi_n(X)$, if we take the basepoint of X to be the unit of the H-space structure.

Exercise 2. Which of the following statements are always true and which are false? Justify your answer briefly in a couple of lines.

(i) If $k > n$, then $\pi_k(S^n, x_0) = 0$.

(ii) If $k < n$, then $\pi_k(S^n, x_0) = 0$.

(iii) If $f\colon X \to Y$ and $g\colon Y \to Z$ are Serre fibrations, then so is $g \circ f\colon X \to Z$.

(iv) If $f\colon X \to Y$ and $g\colon Y \to Z$ are Hurewicz fibrations, then so is $g \circ f\colon X \to Z$.

(v) If $Y = X \cup_f e^n$ is obtained by attaching an n-cell to X along a map $f\colon \partial e^n \to X$, and X is a CW-complex, then Y is also a CW-complex.

(vi) If $Y = X \cup_f e^n$ is obtained by attaching an n-cell to X along a map $f\colon \partial e^n \to X$, and X is a CW-complex of dimension less or equal to $n-1$, then Y is also a CW-complex.

(vii) Let X be a CW-complex. If $f\colon X \to Y$ is a map of connected spaces and

$$\pi_n(f)\colon \pi_n(X, x_0) \to \pi_n(Y, f(x_0))$$

is zero for every choice of basepoint x_0, then f is homotopic to a constant map.

Exercise 3. Let $f\colon (Y, y_0) \to (X, x_0)$ and $g\colon (Z, z_0) \to (X, x_0)$ be two maps of pointed spaces. Define the *homotopy pullback* of Y and Z over X to be the pullback

$$\begin{tikzcd}
Y \times^h_X Z \arrow[r] \arrow[d] & X^I \arrow[d, "(\epsilon_0,\epsilon_1)"] \\
Y \times Z \arrow[r, "f \times g"'] & X \times X
\end{tikzcd}$$

so that $Y \times^h_X Z \subseteq X^I \times Y \times Z$ consists of tuples (γ, y, z) so that $\gamma(0) = f(y)$ and $\gamma(1) = g(z)$.

The above diagram is a diagram of pointed spaces if we choose the basepoint of X^I to be the constant path κ_{x_0} and if we choose the basepoint of $Y \times^h_X Z$ to be the tuple (κ_{x_0}, y_0, z_0).

(i) Prove that the map $Y \times^h_X Z \to Y \times Z$ is a Serre fibration with fiber $\Omega(X, x_0)$.

(ii) Show that for any two maps of pointed spaces $f\colon Y \to X$ and $g\colon Z \to X$, there is a long exact sequence

$$\cdots \longrightarrow \pi_{n+1}(X) \longrightarrow \pi_n(Y \times^h_X Z) \longrightarrow \pi_n(Y) \times \pi_n(Z) \longrightarrow \pi_n(X) \longrightarrow \cdots$$
$$\cdots \longrightarrow \pi_1(X) \longrightarrow \pi_0(Y \times^h_X Z) \longrightarrow \pi_0(Y) \times \pi_0(Z)$$

such that the map $\pi_1(Y) \times \pi_1(Z) \to \pi_1(X)$ sends (α, β) to $g_*(\beta) \cdot f_*(\alpha)^{-1}$.

Exercise 4. *In this exercise, all spaces are assumed to be pointed, with their basepoints left implicit.* Recall that the Postnikov tower of a connected space X is a sequence of fibrations

$$\cdots \xrightarrow{\psi_n} P_n(X) \xrightarrow{\psi_{n-1}} \cdots \xrightarrow{\psi_2} P_2(X) \xrightarrow{\psi_1} P_1(X)$$

and maps $f_n \colon X \to P_n(X)$ such that $\psi_n \circ f_{n+1} = f_n$, with the property that $\pi_k(P_n(X)) = 0$ for $k > n$ and $\pi_k(f_n) \colon \pi_k(X) \to \pi_k(P_n(X))$ is an isomorphism for $k \leq n$.

(i) Sketch the construction of this tower in no more than 20 lines.
(ii) Explain why the fiber F_n of ψ_{n-1} is an Eilenberg–Mac Lane space $K(\pi, n)$, where $\pi = \pi_n(X)$.

Exercise 5. Let M be the Möbius strip, obtained as a quotient of $[0,1] \times [-1,1]$ by identifying $(0,t) \sim (1,-t)$. The Möbius strip comes equipped with two inclusions of the circle

$$c \colon S^1 \longrightarrow M, \qquad \partial \colon S^1 \longrightarrow M,$$

where the image of c is the central circle $[0,1] \times \{0\}/\sim$ and the image of ∂ is the boundary circle $[0,1] \times \{\pm 1\}/\sim$.

(i) Give a CW-structure for the Möbius strip such that both maps c and ∂ are cellular maps, if we consider the circle as a CW-complex with one 0-cell and one 1-cell. *Provide a picture to clarify your answer.*
(ii) Show that c is the inclusion of a strong deformation retract. If $p \colon M \to S^1$ is the associated retraction, prove that
$$S^1 \xrightarrow{\partial} M \xrightarrow{p} S^1$$
provides a factorization of the map $\beta_2 \colon S^1 \to S^1$ defined as $\beta_2(e^{i\theta}) = e^{2i\theta}$ as a cofibration followed by a homotopy equivalence.

We will inductively define spaces $M_{(n)}$ by taking n Möbius strips and gluing the central circle of the k-th Möbius strip to the boundary circle of the $(k+1)$-st Möbius strip. The inclusion of the central circle of the n-th Möbius strip then provides an inclusion $c_{(n)} \colon S^1 \to M_{(n)}$.

More precisely, let $c_{(1)} \colon S^1 \to M_{(1)}$ be the inclusion of the central circle of the Möbius strip. Assuming we have defined $c_{(n)} \colon S^1 \to M_{(n)}$, we define $M_{(n+1)}$ as the pushout

$$\begin{tikzcd} S^1 \arrow[r, "\partial"] \arrow[d, "c_{(n)}"'] & M \arrow[d] \\ M_{(n)} \arrow[r] & M_{(n+1)} \end{tikzcd}$$

Let $c_{(n+1)}$ be the composite map $S^1 \xrightarrow{c} M \longrightarrow M_{(n+1)}$. In this way, we obtain a sequence of spaces $M_{(1)} \to M_{(2)} \to M_{(3)} \to \cdots$ with colimit $M_{(\infty)}$.

(iii) Show that $M_{(n)}$ is a CW-complex for each $1 \leq n \leq \infty$ and that $M_{(m)} \to M_{(n)}$ is the inclusion of a CW-subcomplex for all $1 \leq m < n \leq \infty$.
(iv) Each $c_{(n)}$ is the inclusion of a strong deformation retract and the associated retractions $p_{(n)}$ fit into a commutative diagram

$$\begin{tikzcd} M_{(1)} \arrow[r] \arrow[d, "p_{(1)}"'] & M_{(2)} \arrow[r] \arrow[d, "p_{(2)}"] & \cdots \arrow[r] & M_{(n)} \arrow[r] \arrow[d, "p_{(n)}"'] & M_{(n+1)} \arrow[r] \arrow[d, "p_{(n+1)}"] & \cdots \\ S^1 \arrow[r, "\beta_2"'] & S^1 \arrow[r] & \cdots \arrow[r] & S^1 \arrow[r, "\beta_2"'] & S^1 \arrow[r] & \cdots \end{tikzcd}$$

Prove that $M_{(\infty)} = \operatorname{colim}_{i \geq 1} M_{(i)}$ is a $K(G,1)$ and identify the group G.