ALGEBRAIC TOPOLOGY, EXERCISE SHEET 1, 18.09.2015

For the first exercise recall that

- a subset $C \subseteq \mathbb{R}^n$ is *convex* if for every pair of points $a, b \in C$ the segment between them lies in C, i.e., $ta + (1-t)b \in C$ for every $0 \leq t \leq 1$.
- a *convex combination* of points $p_0, p_1, \ldots, p_m \in \mathbb{R}^n$ is a sum of the form:

$$x = \sum_{i=0}^{m} t_i p_i \qquad t_0 + \ldots + t_m = 1,\ t_i \geq 0 \tag{$\star$}$$

  The subset of all such convex combinations is denoted by $\mathrm{C}(p_0, \ldots, p_m) \subset \mathbb{R}^n$.
- the *convex hull* $[p_0, \ldots, p_m]$ of points $p_0, \ldots, p_m \in \mathbb{R}^n$ is the *smallest* convex subset of $\mathbb{R}^n$ containing these points (i.e., every convex set containing $p_0, \ldots, p_m$ also contains $[p_0, \ldots, p_m]$).

Exercise 1. The aim of this exercise is to show that Δ^m is up to homeomorphism the subspace of $\mathbb{R}^n$ given by the convex hull of $m+1$ affinely independent points.

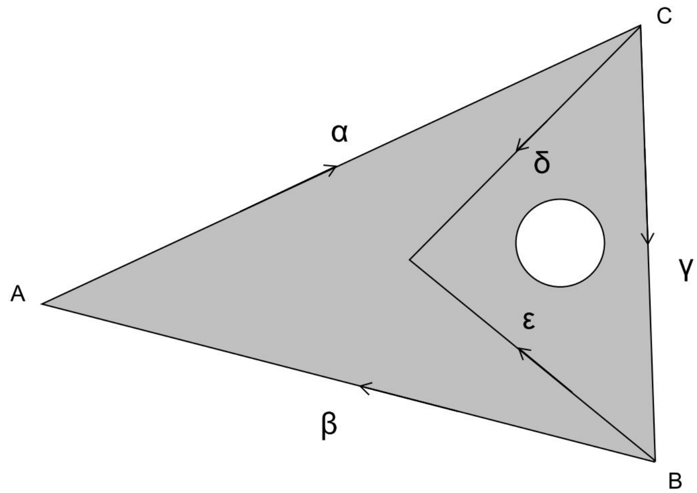
(1) For $m+1$ points $p_0, p_1, \ldots, p_m \in \mathbb{R}^n$ we have $\mathrm{C}(p_0, \ldots, p_m) = [p_0, \ldots, p_m]$. [Hint: to prove that $\mathrm{C}(p_0, \ldots, p_m) \subseteq [p_0, \ldots, p_m]$ you can try to reason by induction on the number of points.]
(2) Every convex combination of $p_0, \ldots, p_m$ has a unique expression of the form $(\star)$ if and only if $p_0, \ldots, p_m$ are affinely independent, i.e., the vectors $p_1 - p_0, p_2 - p_0, \ldots, p_m - p_0$ are linearly independent.
(3) Given affinely independent points $p_0, \ldots, p_m \in \mathbb{R}^n$ and $q_0, \ldots, q_m \in \mathbb{R}^k$ then $[p_0, \ldots, p_m]$ and $[q_0, \ldots, q_m]$ are homeomorphic.

Exercise 2. Recall the definition of a free abelian group generated by a set as given in the lecture.

(1) Prove that for every set S there exists a free abelian group $(F(S), i_S)$ generated by S.
(2) Show that if $(F(S), i_S)$ and $F(S)', i'_S)$ are free abelian groups generated by a set S then there is a unique isomorphism of groups $g\colon F(S) \to F'(S)$ such that $g \circ i_S = i'_S$.
(3) Show that the assignment $S \mapsto F(S)$ can be extended to a functor from Set (the category of sets and maps) to Ab (the category of abelian groups and group homomorphisms).
(4) Use the previous results to show that we have a functor $C_n\colon \mathsf{Top} \to \mathsf{Ab}$ sending a space to its n-th singular chain group. Moreover, the inclusion of a subspace $i\colon A \to X$ induces an injection $C_n(i)\colon C_n(A) \to C_n(X)$.

Exercise 3. Let $*$ be the one-point space. Calculate $H_n(*)$ for every $n \geq 0$.

Exercise 4. Let us consider the singular 1-chains $c_1 = \alpha + \beta + \gamma$ and $c_2 = \gamma + \epsilon - \delta$ in the following subspace X of $\mathbb{R}^2$ (see next page!). Show that $c_1,\ c_2 \in Z_1(X)$ and that they represent the same element in $H_1(X)$.



Exercise 5. Let A be a subspace of a space X and let $i\colon A \to X$ be the inclusion map. We say that A is a *deformation retract* of X if there is a function $r\colon X \to A$ such that $ri = 1_A$ and ir is homotopic to 1_X.

(1) Let $I = [0,1]$ be the interval. Show that $\{(x,y) \in I \times I\colon \quad x = 0 \text{ or } x = 1 \text{ or } y = 0\}$ is a deformation retract of $I \times I$.

(2) Let $X = \{(x,y) \in \mathbb{R}^2\colon \quad x^2 + y^2 \leq 36, \quad (x-3)^2 + y^2 \geq 1, \quad (x+3)^2 + y^2 \geq 1\}$ be a disc with two 'holes'. Convince yourself that the subspace

$$A = \{(x,y) \in X\colon \quad (x-3)^2 + y^2 = 2 \quad \text{or} \quad (x+3)^2 + y^2 = 2 \quad \text{or} \quad (y = 0,\ -1 \leq x \leq 1)\}$$

is a deformation retract of X.

(3) Let Y and Z be subspaces of a space X such that $Z \subseteq Y$. Show that if Y is a deformation retract of X then Z is a deformation retract of X if and only if Z is a deformation retract of Y.