ALGEBRAIC TOPOLOGY, EXERCISE SHEET 2, 02.10.2015

In this exercise sheet we formalize the 'naturality' of morphisms as already mentioned in the lectures. Let $\mathcal{C}$ and $\mathcal{D}$ be categories and $F, G : \mathcal{C} \to \mathcal{D}$ functors. A *natural transformation* $\alpha\colon F \to G$ from F to G is a family of morphisms $\alpha_X\colon F(X) \to G(X)$ in $\mathcal{D}$ indexed by objects X of $\mathcal{C}$ satisfying the (so-called) naturality condition

$$\alpha_Y \circ F(f) = G(f) \circ \alpha_X$$

for each morphism $f\colon X \to Y$ in $\mathcal{C}$. Put more diagrammatically, this amounts to saying that the following diagram commutes:

$$\begin{tikzcd} X \arrow[d, "f"'] & & F(X) \arrow[r, "\alpha_X"] \arrow[d, "F(f)"'] & G(X) \arrow[d, "G(f)"] \\ Y & & F(Y) \arrow[r, "\alpha_Y"'] & G(Y) \end{tikzcd}$$

Exercise 1 (Lemma 4 of Lecture 2)**.** Let G be a group and let $[G, G]$ be the subgroup of G *generated* by the commutators in G.

(1) The subgroup $[G, G]$ is normal and the quotient group $G^{ab} = G/[G, G]$ is abelian. The subgroup $[G, G]$ is the *commutator subgroup of* G and the quotient group $G^{ab} = G/[G, G]$ is called the *abelianization* of G.

(2) The pair (G^{ab}, q) consisting of the abelianization G^{ab} and the canonical group homomorphism $q\colon G \to G^{ab}$ has the following universal property: Given a further pair (A, r) consisting of an abelian group A and a group homomorphism $r\colon G \to A$ then there is unique group homomorphism $g\colon G^{ab} \to A$ such that $g \circ q = r$.

(3) Let Gr denote the category of groups and group homomorphisms. The abelianization defines a functor $(-)^{ab}\colon \mathsf{Gr} \to \mathsf{Ab}$.

(4) Let $U\colon \mathsf{Ab} \to \mathsf{Gr}$ be the functor that includes the abelian groups into all groups. Show that the canonical maps $G \to G^{ab}$ determine a natural transformation from the identity functor on Gr to the functor $U \circ (-)^{ab}\colon \mathsf{Gr} \to \mathsf{Gr}$.

Exercise 2.

(1) Show that for every topological space X we have an isomorphism $\epsilon_X\colon H_0(X) \to \mathbb{Z}\pi_0(X)$, where $\mathbb{Z}\pi_0(X)$ is the free abelian group generated by the set $\pi_0(X)$.
Hint: have a look at the path-connected case as discussed in the lecture.

(2) Show that the isomorphisms constructed in (1) constitute a natural isomorphism $\epsilon\colon H_0 \to \mathbb{Z}\pi_0$ of functors $\mathsf{Top} \to \mathsf{Ab}$.

(3) Show that the Hurewicz isomorphisms $\tilde{h}_{(X,x_0)} : \pi_1^{ab}(X, x_0) \to H_1(X)$ assemble into a natural transformation $\tilde{h}\colon \pi_1(-)^{ab} = (-)^{ab} \circ \pi_1 \to H_1$ of functors $\mathsf{Top}_* \to \mathsf{Ab}$.

(4) Find more examples of natural transformations in the lecture notes and exercises. Do you know other examples from algebra?

Exercise 3.

(1) Show that the map $p\colon \mathbb{R} \to S^1$ given by $p(x) = e^{ix}$ has the following *unique path lifting property*: given a path on the circle $\gamma\colon [0,1] \to S^1$ and a point $x \in \mathbb{R}$ such that $\gamma(0) = p(x)$, there is a unique (continuous) path $\tilde{\gamma}\colon [0,1] \to \mathbb{R}$ with $\tilde{\gamma}(0) = x$ and $p(\tilde{\gamma}(t)) = \gamma(t)$.

$$\begin{tikzcd} \{0\} \arrow[r, "x"] \arrow[d] & \mathbb{R} \arrow[d, "p"] \\ {[0,1]} \arrow[r, "\gamma"'] \arrow[ur, dashed, "\exists!\tilde{\gamma}"] & S^1 \end{tikzcd}$$

Hint: p maps each interval in $\mathbb{R}$ of diameter $< 2\pi$ homeomorphically to a subspace of S^1.

(2) Use (1) to argue that $\pi_1(S^1, 1) \cong \mathbb{Z}$.

(3) Let X be a bouquet of two circles (i.e. the space ∞) and let x be the point where both circles meet. Convince yourself that $\pi_1(X, x)$ is a free group on two generators a and b, i.e. each element in $\pi_1(X, x)$ can be written uniquely as a word in a, a^{-1}, b and b^{-1}.
Hint: each loop in X decomposes into loops that take value in only one of the two circles.

(4) Describe the first homology group of the bouquet of two circles. How does it differ from the first homotopy group?

In the remaining exercises we recall some notions from point-set topology that will be useful in the upcoming lectures. A *quotient map* $q\colon X \to Y$ is a surjection between topological spaces satisfying that a subset $U \subseteq Y$ is open if and only if $q^{-1}(U)$ is open in X. If $\sim$ is an equivalence relation on the underlying set of a topological space X then we can endow the set of equivalence classes $X/_\sim$ with the unique topology such that the canonical surjection $q\colon X \to X/_\sim$ is a quotient map. Note that any quotient map can be obtained this way. We continue to use this notation in the next exercise.

Exercise 4. Show that for any continuous map $f\colon X \to Z$ which satisfies $f(x) = f(x')$ for all $x, x' \in X$ with $x \sim x'$ there is a unique continuous map $h\colon X/_\sim \to Z$ such that $h \circ q = f$. More diagrammatically:

$$\begin{tikzcd} X \arrow[r, "q"] \arrow[d, "f"'] & X/_\sim \arrow[dl, dashed, "\exists!h"] \\ Z & \end{tikzcd}$$

Recall that a map of spaces is *open* if it sends open subsets to open subsets. Similarly, a map of spaces is *closed* if it sends closed subsets to closed subsets.

Exercise 5.

(1) Show that a continuous surjection which is open or closed is a quotient map.

(2) Let X and Y be compact Hausdorff spaces and let $f\colon X \to Y$ be a continuous surjection. Show that f is a quotient map.

(3) Let X be a compact Hausdorff space and let $f\colon X \to Y$ be a closed continuous surjection. Show that Y is a compact Hausdorff space.
Hint: use that every compact Hausdorff space X is normal, i.e., that for every two disjoint closed subsets Z, Z' of X there are disjoint open subsets U, U' in X with $Z \subseteq U, Z' \subseteq U'$.

(4) Using part (2) show that the sphere S^n is homeomorphic to the quotient D^n/S^{n-1} of the disk with respect to its boundary.