ALGEBRAIC TOPOLOGY, EXERCISE SHEET 3, 09.10.2015

Exercise 1. Conclude the proof of Proposition 9 from Lecture 3 by showing:

(1) the connecting homomorphism $\delta_n \colon H_n(C'') \to H_{n-1}(C')$ is well-defined and a homomorphism of groups.

(2) the resulting long sequence of homology groups is exact at $H_n(C)$ and at $H_n(C')$.

Exercise 2.

(1) Show that the category Ch has coproducts. In detail, given a set I and chain complexes $C^i \in \mathsf{Ch}$, $i \in I$, then there is a chain complex C such that for all $D \in \mathsf{Ch}$ there is an isomorphism natural in D:

$$\hom_{\mathsf{Ch}}(C, D) \to \prod_{i \in I} \hom_{\mathsf{Ch}}(C^i, D)$$

Any chain complex C with this universal property is called a coproduct of the chain complexes C^i and will be denoted $\bigoplus_{i \in I} C^i$.
Hint: recall the corresponding statement for abelian groups first, this gives a hint how to define C.

(2) Try to justify why it is reasonable to call any chain complex C constructed in (1) *the* coproduct of the C^i as opposed to *a* coproduct of the C^i. This will also justify why we use the same notation $\bigoplus_{i \in I} C^i$ for them.

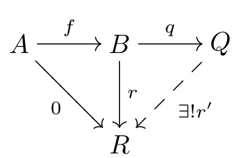
Exercise 3 (Homology is additive)**.**

(1) Given chain complexes $C^i \in \mathsf{Ch}$, $i \in I$, and $n \in \mathbb{Z}$ then there is a (natural) isomorphism of abelian groups $\bigoplus_{i \in I} H_n\left(C^i\right) \to H_n\left(\bigoplus_{i \in I} C^i\right)$.

(2) Let X be a topological space and X_i, $i \in I$ be its path components. Show that for every $n \geq 0$ there is a natural isomorphism $\bigoplus_{i \in I} H_n(X_i) \to H_n(X)$.

Exercise 4 (Universal property of the cokernel)**.**

(1) Given a homomorphism of abelian group $f \colon A \to B$, let $Q = B/f(A)$ be the quotient group and $q \colon B \to Q$ the canonical homomorphism. Show that $q \circ f = 0$ and that the pair (Q, q) has the following universal property: for every further such pair (R, r) consisting of an abelian group R and a homomorphism $r \colon B \to D$ with $r \circ f = 0$ there exist a unique homomorphism $r' \colon Q \to R$ such that $r' \circ q = r$. More diagrammatically:



Any such pair (Q, q) with this universal property is referred to as the *cokernel of* f.

(2) Let $f \colon C' \to C$ be a map of chain complexes and let $q_n \colon C_n \to C''_n = C_n/f_n(C'_n)$ be the (levelwise) quotient map. Use the previous point to show that there is a unique way to turn the $(C''_n)_{n \geq 0}$ into a chain complex such that the $(q_n)_{n \geq 0}$ assemble into a chain

map $q\colon C \to C''$. Moreover, if f is an inclusion (a levelwise injective map), then the sequence $0 \to C' \xrightarrow{f} C \xrightarrow{q} C'' \to 0$ is exact.

(3) Define the notion of the cokernel of a morphism of chain complexes. Why does it make sense to speak of 'the' cokernel? In the notation of (2) show that (C'', q) is the cokernel of f.

Exercise 5 (Five lemma)**.** Let us consider the following commutative diagram of abelian groups with exact rows:

$$\begin{array}{ccccccccc}
A_1 & \xrightarrow{a_1} & A_2 & \xrightarrow{a_2} & A_3 & \xrightarrow{a_3} & A_4 & \xrightarrow{a_4} & A_5 \\
\downarrow{\scriptstyle f_1} & & \downarrow{\scriptstyle f_2} & & \downarrow{\scriptstyle f_3} & & \downarrow{\scriptstyle f_4} & & \downarrow{\scriptstyle f_5} \\
B_1 & \xrightarrow{b_1} & B_2 & \xrightarrow{b_2} & B_3 & \xrightarrow{b_3} & B_4 & \xrightarrow{b_4} & B_5
\end{array}$$

Use the technique of 'diagram chasing' to show that:

(1) if f_2 and f_4 are surjective and f_5 is injective then f_3 is surjective.
(2) if f_2 and f_4 are injective and f_1 is surjective then f_3 is injective.
(3) if f_1, f_2, f_4, and f_5 are isomorphisms then so is f_3.

Exercise 6 (Snake lemma)**.** Let us consider the following commutative diagram of abelian group with exact rows:

$$\begin{array}{ccccccccc}
 & & A_0 & \xrightarrow{f_0} & B_0 & \xrightarrow{g_0} & C_0 & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle a} & & \downarrow{\scriptstyle b} & & \downarrow{\scriptstyle c} & & \\
0 & \longrightarrow & A_1 & \xrightarrow{f_1} & B_1 & \xrightarrow{g_1} & C_1 & &
\end{array}.$$

Show that there is an exact sequence of abelian groups

$$\ker(a) \xrightarrow{\tilde{f}} \ker(b) \xrightarrow{\tilde{g}} \ker(c) \longrightarrow \operatorname{coker}(a) \xrightarrow{\hat{f}} \operatorname{coker}(b) \xrightarrow{\hat{g}} \operatorname{coker}(c)$$

and furthermore:

(1) $\tilde{f}$ is injective if and only if f_0 is injective.
(2) $\hat{g}$ is surjective if and only if g_1 is surjective.

Try to solve this exercise both by 'diagram chasing' and as a corollary of Proposition 9, Lecture 3.