ALGEBRAIC TOPOLOGY, EXERCISE SHEET 5, 23.10.2015

Exercise 1 (Naturality of the connecting homomorphism)**.**

(1) Show that the (algebraic) connecting homomorphism associated to a short exact sequence of chain complexes is natural. In other words, show that for every commutative diagram of abelian groups

$$\begin{tikzcd}
0 \arrow[r] & C' \arrow[r, "i"] \arrow[d, "f'"] & C \arrow[r, "p"] \arrow[d, "f"] & C'' \arrow[r] \arrow[d, "f''"] & 0 \\
0 \arrow[r] & D' \arrow[r, "j"] & D \arrow[r, "q"] & D'' \arrow[r] & 0
\end{tikzcd}$$

in which the rows are exact, the diagram

$$\begin{tikzcd}
H_n(C'') \arrow[r, "\delta_n"] \arrow[d, "f''_*"'] & H_{n-1}(C') \arrow[d, "f'_*"] \\
H_n(C'') \arrow[r, "\delta_n"] & H_{n-1}(C')
\end{tikzcd}$$

commutes for every $n \geq 1$.

(2) Define a category of short exact sequences of chain complexes of abelian groups and a category of long exact sequences of abelian groups. Show that homology defines a functor between these categories.

Exercise 2 (Homology long exact sequence of a triple)**.** A triple of spaces (X, A, B) is a space X together with subspaces $B \subseteq A \subseteq X$.

(1) Show that for each triple of spaces (X, A, B) there is a long exact sequence

$$\ldots \to H_n(A, B) \to H_n(X, B) \to H_n(X, A) \to H_{n-1}(A, B) \to \ldots . \to H_0(X, A).$$

(2) Give a definition of a morphism of triples with the property that each such morphism $(X, A, B) \to (X', A', B')$ induces a commutative diagram of pairs of spaces:

$$\begin{tikzcd}
(A, B) \arrow[r] \arrow[d] & (X, B) \arrow[r] \arrow[d] & (X, A) \arrow[d] \\
(A', B') \arrow[r] & (X', B') \arrow[r] & (X', A')
\end{tikzcd}$$

(3) Show that associated to each morphism of triples $(X, A, B) \to (X', A', B')$ we have a commutative diagram with exact rows as in:

$$\begin{tikzcd}
\cdots \arrow[r] & H_n(A, B) \arrow[r] \arrow[d] & H_n(X, B) \arrow[r] \arrow[d] & H_n(X, A) \arrow[r] \arrow[d] & H_{n-1}(A, B) \arrow[r] \arrow[d] & \cdots \\
\cdots \arrow[r] & H_n(A', B') \arrow[r] & H_n(X', B') \arrow[r] & H_n(X', A') \arrow[r] & H_{n-1}(A', B') \arrow[r] & \cdots
\end{tikzcd}$$

We define the *suspension* of a chain complex $(K., \partial_K)$ as the chain complex $SK.$ such that $SK_n := K_{n-1}$ for every $n > 0$ and $SK_0 = 0$. The n-th differential of $SK.$ is $-\partial_{K,n-1}$ for $n > 0$, the 0-th differential is just the trivial map. It follows that $H_n(SK) = H_{n-1}(K)$.

Exercise 3 (Mapping cone)**.** Suppose that $(K, \partial_K), (L, \partial_L)$ are chain complexes and a chain map $f: K \to L$ is given. We define a new chain complex C_f by letting $C_{f,n} := L_n \oplus K_{n-1}$ for $n > 0$ and $C_{f,0} = L_0$. The boundary operator of C_f is defined by the matrix

$$\begin{pmatrix} \partial_L & f \\ 0 & -\partial_K \end{pmatrix}$$

In other words the boundary of an element $(l, k) \in C_{f,n} = L_n \oplus K_{n-1}$ is $\partial((l, k)) = (\partial_L(l) + f(k), -\partial_K(k)) \in L_{n-1} \oplus K_{n-2} = C_{f,n-1}$.

(1) Show that C_f is indeed a chain complex.
(2) Show that there is a exact sequence of chain complexes

$$0 \longrightarrow L \xrightarrow{\binom{0}{1}} C_f \xrightarrow{(0\,1)} SK \longrightarrow 0$$

and that the n-th connecting homomorphism of the associated long exact sequence $\delta_n: H_n(SK) \simeq H_{n-1}(K) \to H_{n-1}(L)$ is just $H_{n-1}(f)$.
(3) Show that if C_f is contractible then f is a chain homotopy equivalence.

Exercise 4 (Prism operator)**.** Recall from exercise sheet 1 that given a convex space $X \subseteq \mathbb{R}^m$, each $n+1$-tuple of points $(a_0, \ldots, a_n) \in X^{n+1}$ determines an n-simplex $[a_0, \ldots, a_n]: \Delta^n \to X$ whose image is the convex hull of $a_0, \ldots, a_n$. The boundary of this simplex is

$$\partial([a_0, \ldots, a_n]) = \sum_{i=0}^{n} (-1)^i [a_0, \ldots, \hat{a_i}, \ldots, a_n]$$

where $[a_0, \ldots, \hat{a_i}, \ldots, a_n]$ is the n-simplex associated to the sequence $(a_0, \ldots, a_n)$ with a_i removed.

Consider the prism $\Delta^n \times [0, 1] \subset \mathbb{R}^{n+1} \times [0, 1] \subset \mathbb{R}^{n+2}$, where we identify the n-simplex Δ^n with the convex hull of the $n+1$-basis vectors $u_0, \ldots, u_n \in \mathbb{R}^{n+1}$. If we define $v_i = (u_i, 0)$ and $w_i = (u_i, 1)$, then $\Delta^n \times [0, 1]$ is the convex hull of $\{v_0, \ldots, v_n, w_0, \ldots, w_n\}$.

(1) Show that $\Delta^n \times [0, 1] = \bigcup_{i=0}^{n} S_i$ where S_i is the $(n+1)$-simplex $[v_0, \ldots, v_i, w_i, \ldots, w_n]$, for every $0 \le i \le n$.
(2) Given an homotopy $F: I \times X \to Y$ let us define P_n as

$$\begin{aligned} P_n: C_n(X) &\longrightarrow C_{n+1}(Y) \\ \sigma \quad &\longmapsto \sum_{i=0}^{n} (-1)^i F_*(\sigma \times \mathrm{id})_*(S_i) \end{aligned} .$$

Show that the P_n's define a chain homotopy from $f = F|_{\{0\} \times X}$ to $g = F|_{\{1\} \times X}$.