ALGEBRAIC TOPOLOGY, EXERCISE SHEET 7, 13.11.2015

Exercise 1. Let $U \subseteq A \subseteq X$ be subspaces of X such that $\overline{U} \subseteq A^\circ$ and let $i\colon C'(X) \to C(X)$ be the inclusion of the corresponding subcomplex of small chains. Recall from Lecture 7 that there exists a chain map $\mathrm{bs}^X\colon C(X) \to C(X)$, as well as a chain homotopy $R^X\colon C_\bullet(X) \to C_{\bullet+1}(X)$ between bs^X and the identity such that

- after applying bs^X sufficiently many times, one obtains a small chain.
- if $\alpha \in C_n(X)$ is a small chain, then $R^X(\alpha)$ is a small chain as well.

Use this to prove that the inclusion $i\colon C'(X) \to C(X)$ is a chain homotopy equivalence, along the following lines:

(1) show that for each k, there is a chain homotopy $R^X_k\colon C_\bullet(X) \to C_{\bullet+1}(X)$ between the k-fold composition $(\mathrm{bs}^X)^k = \mathrm{bs}^X \circ \cdots \circ \mathrm{bs}^X$ and the identity. Also prove that R^X_k preserves small chains.

(2) for each simplex $\sigma\colon \Delta^n \to X$, define $h(\sigma) := R^X_{\phi(\sigma)}(\sigma) \in C_{n+1}(X)$, where $\phi(\sigma)$ is the smallest k such that $\big(\mathrm{bs}^X\big)^k(\sigma)$ is a small chain. Show that h extends to a well-defined map of graded abelian groups $C_\bullet(X) \to C_{\bullet+1}(X)$.

(3) prove that there is a map of chain complexes $\rho\colon C(X) \to C(X)$ such that h is a chain homotopy between ρ and the identity map.

(4) prove that ρ takes values in the subcomplex $C'(X)$ of small chains and prove that the resulting map $\rho\colon C(X) \to C'(X)$ provides a homotopy inverse to the inclusion i.

Exercise 2. Compute the homology groups of the two-dimensional real projective space $\mathbb{R}P^2$.

Hint: recall that $\mathbb{R}P^2$ can be obtained as the quotient of the square $[0,1]\times[0,1]$ by the relations $(s,0) \sim (1-s,1)$ and $(0,t) \sim (1,1-t)$.

Exercise 3 (Colimits of sequences of abelian groups)**.** Given a sequence of abelian groups and group homomorphisms

$$A_0 \xrightarrow{f_0} A_1 \xrightarrow{f_1} A_2 \xrightarrow{f_2} \cdots \xrightarrow{f_{n-1}} A_n \xrightarrow{f_n} \cdots$$

let us define $f_{ji}\colon A_i \to A_j$ to be the composition $f_{j-1} \circ f_{j-2} \circ \cdots \circ f_i\colon A_i \to A_j$ for $i < j$. We define a new abelian group $\mathrm{colim}_n A_n$ which is called the *colimit of the above sequence.* Elements of this abelian group are equivalence classes $[x,n]$ of pairs (x,n) with $n \in \mathbb{N}$ and $x \in A_n$. Two such pairs (x,n), (y,m) are equivalent if there exists $k \geq n,m$ such that $f_{kn}(x) = f_{km}(y)$.

(1) Check that there is a well-defined group structure on $\mathrm{colim}_n A_n$ given by:

$$[x,n] + [x,m] = [f_{kn}(x) + f_{km}(y), k] \qquad \text{where} \qquad k \geq m,n$$

Moreover, for every $n \in \mathbb{N}$, the assignments $\iota_j\colon A_j \to \mathrm{colim}_n A_n\colon a \mapsto [a,j]$ define group homomorphisms which satisfy $\iota_j \circ f_{ji} = \iota_i$.

(2) Check that for any abelian group B and any family of group homomorphisms $\beta_n\colon A_n \to B$ such that for $i < j$ we have $\beta_j \circ f_{ji} = \beta_i$, there exists a unique homomorphism $\beta\colon \mathrm{colim}_n A_n \to B$ such that $\beta \circ \iota_n = \beta_n$ for every $n \in \mathbb{N}$.

In the next exercise we consider a sequence of chain complexes as in

$$C_{\bullet,0} \xrightarrow{f_{\bullet,0}} C_{\bullet,1} \xrightarrow{f_{\bullet,1}} C_{\bullet,2} \xrightarrow{f_{\bullet,2}} \cdots \xrightarrow{f_{\bullet,n-1}} C_{\bullet,n} \xrightarrow{f_{\bullet,n}} \cdots$$

and let us write $f_{\bullet,ji}\colon C_{\bullet,i} \to C_{\bullet,j}$ for the morphism of chain complexes obtained by composition.

Exercise 4 (Colimits of sequences of chain complexes)**.**

(1) Define homomorphisms $\partial_n\colon \operatorname{colim}_m C_{n,m} \to \operatorname{colim}_m C_{n-1,m}$ for all $n > 0$ using the universal property of the colimit of abelian groups and check that these maps define a new chain complex $\operatorname{colim}_m C_{\bullet,m}$:

$$\cdots \xrightarrow{\partial_{n+1}} \operatorname{colim}_m C_{n,m} \xrightarrow{\partial_n} \operatorname{colim}_m C_{n-1,m} \xrightarrow{\partial_{n-1}} \cdots \xrightarrow{\partial_1} \operatorname{colim}_m C_{0,m}$$

(2) Show that for every fixed $m \in \mathbb{N}$ the collection $\{\iota_{n,m}\colon C_{n,m} \to \operatorname{colim}_m C_{n,m}\}_{n\in\mathbb{N}}$ defines a map of chain complexes $\iota_k\colon C_{\bullet,k} \to \operatorname{colim}_m C_{\bullet,m}$ and check that for $i < j$ we have $\iota_j \circ f_{\bullet,ji} = \iota_i$.

(3) Prove that there is a (natural) isomorphism:

$$\operatorname*{colim}_m H_n(C_{\bullet,m}) \cong H_n(\operatorname*{colim}_m C_{\bullet,m})$$

Exercise 5 (Local homology groups)**.** Let X be the subspace of Δ_3 formed by the union of all the six edges (in other words, if we express the points of Δ_3 in baricentric coordinates, X is the subspace of points that have at least two baricentric coordinates equal to 0).

Consider the cone CX; we can think of it as the subspace of Δ_3 formed by the union of all the line segments joining a point in X to the baricenter of Δ_3.

(1) Compute all the local homology groups $H_n(CX, CX - \{x\})$ for every point $x \in CX$.
(2) Compute all the local homology groups $H_n(X, X - \{x\})$ for every point in X.
(3) Use the computations above to find subspaces $A \subseteq CX$ such that $f(A) \subseteq A$ for every homeomorphism $f\colon CX \to CX$.