ALGEBRAIC TOPOLOGY, EXERCISE SHEET 11, 11.12.2015

Exercise 1.

(1) Let X and Y be finite CW-complexes. Show that the product $X \times Y$ has the structure of a CW-complex as well.
Hint: every n-cell $e^n_\alpha \to X$ and every m-cell $e^m_\beta \to Y$ together determine an $(n+m)$-cell

$$e^{n+m}_{(\alpha,\beta)} \simeq e^n_\alpha \times e^m_\beta \longrightarrow X \times Y$$

Use this to construct a CW-complex Z, equipped with a continuous bijection $Z \to X \times Y$ and conclude that $X \times Y$ admits a CW-structure.

(2) Compute the homology of $\mathbb{C}P^n \times \mathbb{C}P^m$ for all m and n.

(3) Compute the homology of $\mathbb{C}P^\infty \times \mathbb{C}P^\infty$ using the fact that

$$\mathbb{C}P^\infty \times \mathbb{C}P^\infty = \operatorname*{colim}_n \mathbb{C}P^n \times \mathbb{C}P^n$$

Exercise 2.

(1) Recall that $\mathbb{R}P^n$ admits a CW-structure with one cell in each degree $\leq n$, obtained from the CW-structure on S^n with 2 cells in each degree $\leq n$. In particular, note that each

$$\big(\mathbb{R}P^n\big)^{(k)} / \big(\mathbb{R}P^n\big)^{(k-1)} \simeq S^k$$

can be identified with the quotient of an upper hemisphere by the equator. Use this to argue that each attaching map

$$\chi_k \colon S^k \longrightarrow \big(\mathbb{R}P^n\big)^{(k)} / \big(\mathbb{R}P^n\big)^{(k-1)} \simeq S^k$$

behaves on one half of S^k as the identity map, while on the other half of S^k it behaves as the antipodal map.

(2) Compute the homology of $\mathbb{R}P^n$ for any $0 \leq n \leq \infty$.

Exercise 3. Let M be the Möbius strip, obtained as a quotient of $[0,1] \times [-1,1]$ by identifying $(0,t) \sim (1,-t)$. The Möbius strip comes equipped with two inclusions of the circle

$$c \colon S^1 \longrightarrow M, \qquad \partial \colon S^1 \longrightarrow M,$$

where the image of c is the central circle $[0,1] \times \{0\}/\sim$ and the image of ∂ is the boundary circle $[0,1] \times \{\pm 1\}/\sim$.

(1) Give a CW-structure for the Möbius strip such that both maps c and ∂ are inclusions of CW-subcomplexes (draw a picture!).

(2) Show that c is the inclusion of a strong deformation retract. If $p \colon M \to S^1$ is the associated retraction, prove that

$$S^1 \xrightarrow{\partial} M \xrightarrow{p} S^1$$

provides a factorization of the map $\beta_2 \colon S^1 \to S^1$ defined as $\beta_2(e^{i\theta}) = e^{2i\theta}$ as the inclusion of a subcomplex, followed by a homotopy equivalence.

We will inductively define spaces $M_{(n)}$ by taking n Möbius strips and gluing the central circle of the k-th Möbius strip to the boundary circle of the $(k+1)$-st Möbius strip. The inclusion of the central circle of the n-th Möbius strip then provides an inclusion $c_{(n)}\colon S^1 \to M_{(n)}$.

More precisely, let $c_{(1)}\colon S^1 \to M_{(1)}$ be the inclusion of the central circle of the Möbius strip. Assuming we have defined $c_{(n)}\colon S^1 \to M_{(n)}$, we define $M_{(n+1)}$ as the pushout

$$\begin{tikzcd} S^1 \arrow[r, "\partial"] \arrow[d, "c_{(n)}"'] & M \arrow[d] \\ M_{(n)} \arrow[r] & M_{(n+1)} \end{tikzcd}$$

Let $c_{(n+1)}$ be the composite map $S^1 \xrightarrow{c} M \longrightarrow M_{(n+1)}$. In this way, we obtain a sequence of spaces $M_{(1)} \to M_{(2)} \to M_{(3)} \to \cdots$ with colimit $M_{(\infty)}$.

(3) Show that $M_{(n)}$ is a CW-complex for each $1 \leq n \leq \infty$ and that $M_{(m)} \to M_{(n)}$ is the inclusion of a CW-subcomplex for all $1 \leq m < n \leq \infty$.
(4) Show that each $c_{(n)}$ is the inclusion of a strong deformation retract and that the associated retractions $p_{(n)}$ fit into a commutative diagram

$$\begin{tikzcd} M_{(1)} \arrow[r] \arrow[d, "p_{(1)}"'] & M_{(2)} \arrow[r] \arrow[d, "p_{(2)}"] & \cdots \arrow[r] & M_{(n)} \arrow[r] \arrow[d, "p_{(n)}"'] & M_{(n+1)} \arrow[r] \arrow[d, "p_{(n+1)}"] & \cdots \\ S^1 \arrow[r, "\beta_2"'] & S^1 \arrow[r] & \cdots \arrow[r] & S^1 \arrow[r, "\beta_2"'] & S^1 \arrow[r] & \cdots \end{tikzcd}$$

Compute the homology groups of M_∞.