LECTURE 6: EXCISION PROPERTY AND MAYER-VIETORIS SEQUENCE

In this lecture we will state the important excision property of singular homology which is one of the key features of singular homology allowing for calculations. While the proof of this excision property will only be given in the next lecture, we will here focus on some consequences and applications. In particular, we will deduce the important Mayer-Vietoris sequence and then calculate the homology groups of all spheres. A convenient reformulation is obtained in terms of reduced homology groups.

Here is the important excision theorem.

Theorem 1. *Let $U \subset A \subset X$ be subspaces such that the closure $\bar{U}$ of U lies in the interior A° of A. Then the inclusion $(X\backslash U, A\backslash U) \to (X, A)$ induces isomorphisms on relative homology groups:*

$$H_n(X\backslash U, A\backslash U) \xrightarrow{\cong} H_n(X, A), \qquad n \geq 0$$

This theorem can be equivalently reformulated as follows.

Theorem 2. *Let X be a space and let us consider subspaces $X_1,\, X_2 \subseteq X$ such that $X_1^\circ \cup X_2^\circ = X$. Then the inclusion $(X_1, X_1 \cap X_2) \to (X, X_2)$ induces isomorphisms on all relative homology groups:*

$$H_n(X_1, X_1 \cap X_2) \xrightarrow{\cong} H_n(X, X_2), \qquad n \geq 0$$

Lemma 3. *Theorem 1 and Theorem 2 are equivalent.*

Proof. Exercise. (Hint: consider the assignments $A = X_2$ and $U = X\backslash X_1$.) □

An important consequence of the second formulation of the theorem is given by the so-called Mayer-Vietoris sequence. This result is obtained by specializing the following algebraic fact to a certain topological situation.

Lemma 4. *(Algebraic Mayer-Vietoris sequence) Let us consider the following commutative diagram of abelian groups in which the rows are exact and all the f''_n are isomorphisms:*

$$\begin{tikzcd}
\ldots \arrow[r] & C''_{n+1} \arrow[r, "\delta_{n+1}"] \arrow[d, "f''_{n+1}"'] & C'_n \arrow[r, "i_n"] \arrow[d, "f'_n"'] & C_n \arrow[r, "p_n"] \arrow[d, "f_n"] & C''_n \arrow[r, "\delta_n"] \arrow[d, "f''_n"] & C'_{n-1} \arrow[r] \arrow[d, "f'_{n-1}"] & \ldots \\
\ldots \arrow[r] & D''_{n+1} \arrow[r, "\delta'_{n+1}"'] & D'_n \arrow[r, "j_n"'] & D_n \arrow[r, "q_n"'] & D''_n \arrow[r, "\delta'_n"'] & D'_{n-1} \arrow[r] & \ldots
\end{tikzcd}$$

Then there is an exact sequence in which $\Delta_n = \delta_n \circ f''^{-1}_n \circ q_n \colon D_n \to C'_{n-1}$:

$$\ldots \longrightarrow C'_n \xrightarrow{(i_n, f'_n)} C_n \oplus D'_n \xrightarrow{f_n - j_n} D_n \xrightarrow{\Delta_n} C'_{n-1} \longrightarrow \ldots$$

Proof. Let us give a proof of the exactness at C'_{n-1}. The relation $(i_{n-1}, f'_{n-1}) \circ \Delta_n = 0$ is immediate since we calculate for both coordinates:

$$i_{n-1} \circ \Delta_n = i_{n-1} \circ \delta_n \circ f''^{-1}_n \circ q_n = 0 \quad \text{and} \quad f'_{n-1} \circ \Delta_n = f'_{n-1} \circ \delta_n \circ f''^{-1}_n \circ q_n = \delta'_n \circ q_n = 0$$

Conversely, let us assume that we have an element c'_{n-1} such that $i_{n-1}(c'_{n-1}) = 0 = f'_{n-1}(c'_{n-1})$. Exactness of the upper row at C'_{n-1}implies that there is an element $c''_n \in C''_n$ with $\delta_n(c''_n) = c'_{n-1}$. But

$$0 \quad = \quad f'_{n-1}(c'_{n-1}) \quad = \quad f'_{n-1}(\delta_n(c''_n)) \quad = \quad \delta'_n(f''_n(c''_n))$$

shows that $f''_n(c''_n)$ lies in the kernel of δ'_n. Finally, exactness of the lower row at D''_n implies that there is a $d_n \in D_n$ such that $q_n(d_n) = f''_n(c''_n)$. But for this d_n we calculate

$$\Delta_n(d_n) \quad = \quad \delta_n \circ f''^{-1}_n \circ q_n(d_n) \quad = \quad \delta_n \circ f''^{-1}_n \circ f''_n(c''_n) \quad = \quad \delta_n(c''_n) \quad = \quad c'_{n-1}$$

as intended. Thus c'_{n-1} lies in the image of Δ_n. The remaining two cases are left as an exercise and can be established by similar diagram chases. □

The choice of the sign in the lemma was arbitrary. There are further choices and any of these would be equally good and lead to a similar statement. The only constraint was to reexpress the commutativity of the square induced by the inclusion as the vanishing of the composition of two homomorphisms with the given domains and targets.

We want to apply this to the following topological situation. Let $X = X_1^\circ \cup X_2^\circ$ for two subspaces $X_1, X_2 \subset X$ and let us consider the inclusion $\iota\colon (X_1, X_1 \cap X_2) \to (X, X_2)$ obtained from the following commutative square

$$\begin{tikzcd} X_1 \cap X_2 \arrow[r, "j_1"] \arrow[d, "j_2"'] & X_1 \arrow[d, "i_1"] \\ X_2 \arrow[r, "i_2"'] & X. \end{tikzcd}$$

The naturality of the long exact sequence in homology with respect to morphisms of pairs implies that we have the following commutative ladder with exact rows:

$$\begin{tikzcd} \ldots \arrow[r] & H_{n+1}(X_1, X_1 \cap X_2) \arrow[r, "\delta_{n+1}"] \arrow[d, "\iota_*", "\cong"'] & H_n(X_1 \cap X_2) \arrow[r, "j_{1*}"] \arrow[d, "j_{2*}"'] & H_n(X_1) \arrow[r] \arrow[d, "i_{1*}"] & H_n(X_1, X_1 \cap X_2) \arrow[r] \arrow[d, "\iota_*", "\cong"'] & \ldots \\ \ldots \arrow[r] & H_{n+1}(X, X_2) \arrow[r, "\delta'_{n+1}"'] & H_n(X_2) \arrow[r, "i_{2*}"'] & H_n(X) \arrow[r] & H_n(X, X_2) \arrow[r] & \ldots \end{tikzcd}$$

Since we are in the situation of Theorem 2 we know that all induced maps ι_* are isomorphisms. Thus, if we denote by $\Delta_n\colon H_n(X) \to H_{n-1}(X_1 \cap X_2)$ the homomorphism

$$\Delta_n\colon H_n(X) \to H_n(X, X_2) \xrightarrow{\iota_*^{-1}} H_n(X_1, X_1 \cap X_2) \to H_{n-1}(X_1 \cap X_2),$$

then the algebraic Mayer-Vietoris sequence (Lemma 4) specializes to the following result.

Theorem 5. *In the above situation we have an exact sequence, the* ***Mayer-Vietoris sequence****:*

$$\ldots \longrightarrow H_n(X_1 \cap X_2) \xrightarrow{(j_{1*}, j_{2*})} H_n(X_1) \oplus H_n(X_2) \xrightarrow{i_{1*} - i_{2*}} H_n(X) \xrightarrow{\Delta_n} H_{n-1}(X_1 \cap X_2) \longrightarrow \ldots$$

This theorem allows for inductive calculations of homology groups. We will illustrate this by the calculation of the homology groups of spheres. Besides being interesting for its own sake, this will provide the basis for a large class of examples to be studied in a later lecture.

Proposition 6. *The singular homology groups of the spheres are as follows:*

$$H_n(S^0) \cong \begin{cases} \mathbb{Z}\oplus\mathbb{Z} & , \quad n=0 \\ 0 & , \quad \textit{otherwise} \end{cases} \qquad \textit{and} \qquad H_n(S^m) \cong \begin{cases} \mathbb{Z} & , \quad n=0,\, m \\ 0 & , \quad \textit{otherwise} \end{cases} \qquad \textit{if} \quad m>0$$

Proof. Since S^0 is just a disjoint union of two points we already calculated its homology groups. Moreover, since all the remaining spheres are path connected we know already all the zeroth homology groups. Let us calculate the homology group of S^m for $m \geq 1$. In these cases let us denote by $X_1 \subset X = S^m$ the subspace obtained by removing the 'north pole' $NP \in S^m$. Similarly, let $X_2 \subset S^m$ be obtained by removing the 'south pole' $SP \in S^m$. We are in the situation of the Mayer-Vietoris sequence since the interiors of X_1 and X_2 cover X. Moreover, the intersection $X_1 \cap X_2$ is homotopy equivalent to S^{m-1} in these cases and the subspaces X_1, X_2 are contractible. Thus, the homotopy invariance of singular homology implies that $H_*(X_i)$ is trivial in positive dimensions and that $H_*(X_1 \cap X_2) \cong H_*(S^{m-1})$.

Let us begin by applying the Mayer-Vietoris sequence in the case of $m=1$. We know already that $H_1(S^1)\cong\mathbb{Z}$ since the same is true for the fundamental group of S^1. For $H_n(S^1), n \geq 2$, the relevant part of the Mayer-Vietoris sequence is given by

$$\ldots \to H_n(X_1)\oplus H_n(X_2) \to H_n(S^1) \to H_{n-1}(*\sqcup *) \to \ldots.$$

But since both $H_n(X_1)\oplus H_n(X_2)$ and $H_{n-1}(*\sqcup *)$ are zero the same is true for $H_n(S^1), n\geq 2$.

We now proceed by induction: consider $m \geq 2$ and assume that the calculations of the homology of S^k, $k \leq m-1$, are already done. For the calculation of $H_1(S^m)$ we consider the following part of the Mayer-Vietoris sequence:

$$\ldots \to H_1(X_1)\oplus H_1(X_2) \to H_1(S^m) \to H_0(S^{m-1}) \to H_0(X_1)\oplus H_0(X_2)$$

The path connectedness of S^{m-1}, X_1, and X_2 shows us that the last map in this sequence is isomorphic to $\mathbb{Z}\to\mathbb{Z}\oplus\mathbb{Z}\colon c\mapsto(c,c)$. Since this map is injective and the X_i are contractible we conclude that $H_1(S^m)\cong 0$. For $n\geq 2$ the interesting part of the Mayer-Vietoris sequence is:

$$\ldots \to H_n(X_1)\oplus H_n(X_2) \to H_n(S^m) \to H_{n-1}(S^{m-1}) \to H_{n-1}(X_1)\oplus H_{n-1}(X_2) \to \ldots$$

But the contractibility of the X_i implies that the outer groups are trivial so that we obtain an isomorphism $H_n(S^m)\cong H_{n-1}(S^{m-1})$ in this range. The inductive assumption allows us to conclude the proof. □

These calculations show us that for spheres the homology groups are trivial 'above the dimension', which turns out to be an important feature of singular homology theory. Let us mention the following immediate consequences.

Corollary 7.
(1) *For $m \neq n$ the spheres S^m and S^n are not homotopy equivalent.*
(2) *For $m \geq 0$ the sphere S^m is not contractible.*

Proof. This follows immediately from Proposition 6 and the homotopy invariance of singular homology. □

All vector spaces of the form $\mathbb{R}^n$ are contractible so that they are all homotopy equivalent. However, we can now use our above calculations to show that these vector spaces are not homeomorphic unless they have the same dimensions.

Corollary 8. *(Invariance of dimension) For $m \neq n$ the spaces $\mathbb{R}^m$ and $\mathbb{R}^n$ are not homeomorphic.*

Proof. Let us assume we are given a homeomorphism $f\colon \mathbb{R}^m \to \mathbb{R}^n$ and let us choose an arbitrary $x_0 \in \mathbb{R}^m$. We obtain an induced homeomorphism $f\colon \mathbb{R}^m \backslash \{x_0\} \cong \mathbb{R}^n \backslash \{f(x_0)\}$. If either of the dimensions happens to be zero then this must also be the case for the other one. So, let us assume that both of them are different from zero. Then the domain and the target of this restricted homeomorphism is homotopy equivalent to a sphere of the respective dimension, and we thus get a homotopy equivalence $S^{m-1} \simeq S^{n-1}$. By Corollary 7, this can only be the case if $m = n$. □

The corresponding result in the 'differentiable category' is much easier. A diffeomorphism $f\colon \mathbb{R}^m \to \mathbb{R}^n$ induces an isomorphism at the level of tangent spaces $df_{x_0}\colon T_{x_0}\mathbb{R}^m \cong T_{f(x_0)}\mathbb{R}^n$. Thus, the dimensions of these vector spaces have to coincide so that we get $m = n$.

There are many further classical applications. For the time being, we will content ourselves with the following one. Let us denote by $D^n \subset \mathbb{R}^n$ the closed ball of radius 1 centered at the origin.

Proposition 9. *(Brouwer fixed point theorem) Every continuous map $f\colon D^n \to D^n$ has a fixed point.*

Proof. Let us assume that f has no fixed points at all. For each $x \in D^n$ there is thus a unique ray from $f(x)$ through x. Each such ray has a unique point of intersection with the boundary S^{n-1} of D^n which we denote by $r(x)$. We leave it as an exercise to the reader to check that the assignment $x \mapsto r(x)$ defines a continuous retraction $r\colon D^n \to S^{n-1}$. Moreover, we leave it to the reader to show that this contradicts the calculations in Proposition 6. □

In many cases it is convenient to consider a minor variant of singular homology given by *reduced* singular homology. By definition, the k-th **reduced homology group** $\tilde{H}_k(X)$ of a space X is the k-th homology group of the following augmented chain complex $\tilde{C}(X)$:

$$\ldots \to C_2(X) \stackrel{\partial}{\to} C_1(X) \stackrel{\partial}{\to} C_0(X) \stackrel{\epsilon}{\to} \mathbb{Z}$$

This chain complex differs from the usual singular chain complex by the fact that in degree -1 there is an additional copy of the integers. The map

$$\epsilon\colon C_0(X) \to \mathbb{Z}\colon \quad \sum_{i=1}^{k} n_i x_i \mapsto \sum_{i=1}^{k} n_i$$

is the augmentation map which already played a role in the identification of H_0. We leave it as an exercise to show that from this definition we obtain isomorphisms:

$$H_k(X) \cong \begin{cases} \tilde{H}_k(X) & , \quad k > 0 \\ \mathbb{Z} \oplus \tilde{H}_0(X) & , \quad k = 0 \end{cases}$$

Thus, in positive dimensions the notions coincide while in dimension zero the reduced homology group is obtained from the unreduced one by splitting off a copy of the integers. Let us give two examples which follow immediately from our earlier calculations.

Example 10. (1) If X is a contractible space, then $\tilde{H}_k(X) \cong 0$ for all $k \geq 0$.
(2) For the spheres we have $\tilde{H}_n(S^n) \cong \mathbb{Z}$ and $\tilde{H}_k(S^n) \cong 0$ for all $k \neq n$.

Exercise 11. For a pointed space (X, x_0) there is an isomorphism $\tilde{H}_i(X) \cong H_i(X, x_0)$ which is natural with respect to pointed maps, i.e., maps sending base points to base points.

Exercise 12. Let (X, A) be a pair of spaces. Then there is a long exact sequence

$$\cdots \to H_2(X,A) \to \tilde{H}_1(A) \to \tilde{H}_1(X) \to H_1(X,A) \to \tilde{H}_0(A) \to \tilde{H}_0(X) \to H_0(X,A) \to 0.$$

Corollary 13. *For each $n \geq 1$ we have isomorphisms*

$$H_{k+1}(D^{n+1}, S^n) \cong \tilde{H}_k(S^n) \cong H_k(S^n, *) \cong \begin{cases} \mathbb{Z} & , \quad k = n \\ 0 & , \quad \textit{otherwise.} \end{cases}$$

Proof. We already know that this description is correct for the homology of the sphere so it remains to discuss the case of (D^{n+1}, S^n). But this follows immediately from the long exact sequence in reduced homology. □

A generator of any of the groups $H_n(D^n, S^{n-1}) \cong \mathbb{Z}$ is called a **fundamental class** or **orientation class**. Note that these classes are only well-defined up to a sign. Sometimes it is convenient to make explicit, compatible choices for these orientation classes in all dimensions, and we will be a bit more specific when we discuss the degree maps $S^n \to S^n$.

Related to this notion are the *local* homology groups of manifolds. Let M be a topological manifold of dimension n and let $x_0 \in M$. Then the k-th **local homology group** of M at x_0 is $H_k(M, M - \{x_0\})$. By definition of a manifold, we can find an open neighborhood $x_0 \in V$ and a homeomorphism $V \cong \mathbb{R}^n$ sending x_0 to $0 \in \mathbb{R}^n$. An application of excision implies that we have isomorphisms

$$H_k(V, V - \{x_0\}) \xrightarrow{\cong} H_k(M, M - \{x_0\}).$$

Using the homeomorphism we obtain a further isomorphism $H_k(V, V - \{x_0\}) \cong H_k(\mathbb{R}^n, \mathbb{R}^n - \{0\})$ and the reader will easily show that this group is isomorphic to $\tilde{H}_{k-1}(S^{n-1})$. Thus, the we have

$$H_k(M, M - \{x_0\}) \cong \begin{cases} \mathbb{Z} & , \quad k = n \\ 0 & , \quad \text{otherwise,} \end{cases}$$

and any generator

$$\omega = \omega_{x_0} \in H_n(M, M - \{x_0\})$$

is a **local orientation class** of M at x_0. Also these local orientation classes are only well-defined up to a sign, and one can show that a manifold M is orientable if and only if local orientation classes $\omega_{x_0}, x_0 \in M$, can be chosen in a compatible way (which we do not want to make precise).