## TOPOLOGICAL K-THEORY, EXERCISE SHEET 8, 26.03.2015

**Exercise 1.** Show that any continuous map from the closed $n$-disk to itself has a fixed point.

**Hint:** if there is a continuous map $f\colon D^n \to D^n$ without fixed point, show that the inclusion of $S^{n-1}$ into $D^n$ admits a retraction. Use $K$-theory to prove that such a retraction cannot exist.

**Exercise 2.**

(1) Let $\mathrm{GL}_n(\mathbb{C}) \to V_k(\mathbb{C}^n)$ be the map sending an $n\times n$ matrix to its first $k$ columns, viewed as a $k$-tuple of linearly independent vectors in $\mathbb{C}^n$. Recall that this map establishes a homeomorphism between the Stiefel variety $V_k(\mathbb{C}^n)$ and the homogeneous space $\mathrm{GL}_n(\mathbb{C})/\mathrm{GL}_{n-k}(\mathbb{C})$, where $A \in \mathrm{GL}_{n-k}(\mathbb{C})$ acts by right multiplication by the matrix

$$\begin{pmatrix} 1_{k\times k} & 0 \\ 0 & A \end{pmatrix}.$$

Taking the further quotient by the remaining action of $\mathrm{GL}_k(\mathbb{C})$ produces the Grassman variety $G_k(\mathbb{C}^n)$.

(2) Recall the canonical inclusion $i_n\colon G_k(\mathbb{C}^n) \subseteq G_k(\mathbb{C}^{n+1})$ sending a plane in $\mathbb{C}^n$ to the corresponding plane in $\mathbb{C}^n \times \{0\} \subseteq \mathbb{C}^{n+1}$. Show that this inclusion is induced by the map

$$i_n\colon \mathrm{GL}_n(\mathbb{C}) \longrightarrow \mathrm{GL}_{n+1}(\mathbb{C});\ A \longmapsto \begin{pmatrix} A & 0 \\ 0 & 1 \end{pmatrix}$$

by passing to the respective quotients.

(3) Show that the map

$$j_n\colon \mathrm{GL}_n(\mathbb{C}) \longrightarrow \mathrm{GL}_{n+1}(\mathbb{C});\ A \longmapsto \begin{pmatrix} 1 & 0 \\ 0 & A \end{pmatrix}$$

induces a map $j_n\colon G_k(\mathbb{C}^n) \to G_{k+1}(\mathbb{C}^{n+1})$. Furthermore, show that the maps $j_n$ fit in commutative squares

$$\begin{tikzcd} G_k(\mathbb{C}^n) \arrow[r, "j_n"] \arrow[d, "i_n"'] & G_{k+1}(\mathbb{C}^{n+1}) \arrow[d, "i_{n+1}"] \\ G_k(\mathbb{C}^{n+1}) \arrow[r, "j_{n+1}"'] & G_{k+1}(\mathbb{C}^{n+2}) \end{tikzcd}$$

Conclude that the family of $j_n$ induce a map $j\colon G_k(\mathbb{C}^\infty) \to G_{k+1}(\mathbb{C}^\infty)$.

(4) Prove that $j^*\gamma_{k+1} \simeq \gamma_k \oplus \mathbb{C}$. Conclude that for a compact Hausdorff space $X$, the map

$$j_*\colon [X, G_k(\mathbb{C}^\infty)] \longrightarrow [X, G_{k+1}(\mathbb{C}^\infty)]$$

agrees with the map

$$t_k\colon \mathrm{Vect}^k(X) \longrightarrow \mathrm{Vect}^{k+1}(X);\ E \longmapsto E \oplus \mathbb{C}$$

**Exercise 3.** Let $X$ be a compact Hausdorff space. Show that the map

$$\operatorname{colim}\big(\mathrm{Vect}^1(X) \xrightarrow{t_1} \mathrm{Vect}^2(X) \xrightarrow{t_2} \cdots\big) \longrightarrow \widehat{K}(X);\ \ [E] \longmapsto [E] - [\mathbb{C}^{\mathrm{rk}(E)}]$$

induces an isomorphism of abelian groups.